Pontryagin Maximum Principle

Theorem: Suppose $\mathcal{U} \subset \mathbb{R}^n$ is closed & compact, $(\vec{y}^\circ, \vec{u}^\circ)$ minimizes

$$J(\vec{y}, \vec{u}) = \int_0^T f(t, \vec{y}, \vec{u})\, dt$$

subject to $\dot{\vec{y}} = \vec{g}(t, \vec{y}, \vec{u})$ & (a) or (b). Then $\exists \vec{p}^\circ$ such that $\dot{p}_i = -H_{y_i}$, $1 \le i \le n$, where

$$H = -f + \sum_{k=1}^{n} p_k g_k$$

& $\vec{u}^\circ(t)$ maximizes $H(t, \vec{y}, \vec{v}, \vec{p})$ for $\vec{v} \in \mathcal{U}$.
In case (b), add $\vec{p}(T) = \vec{0}$. If T is variable, add $H|_{t=T} = 0$.
If T has no explicit dependence on t, H = const.

Ex. (modification of previous example)
Minimize $\int_0^T dt = T$

subject to $\dot{y} = u$, $y(0) = 1$, $y(T) = 0$, $T > 0$, AND $|u| \le 1$

$H = -1 + pu$. again, $H = 0$

$$\begin{cases} \dot{p} = -H_y = 0 \\ \dot{y} = H_p = u \end{cases}$$

$p = c$

$c \ne 0$ maximize $H = -1 + cv$ s.t. $|v| \le 1$

If $c > 0$, $u = 1$, $\max H = -1 + c$

If $c < 0$, $u = -1$, $\max H = -1 - c$

$u = \frac{1}{c}$

If $c > 0$, $u = \frac{1}{c} = 1 \to c = 1$

If $c < 0$, $u = \frac{1}{c} = -1 \to c = -1$

$c = 1$: $\dot{y} = 1 \to y = t + d \to y = t + 1$ $y(T) = 0 \to T = -1$ ✗

$c = -1$: $\dot{y} = -1 \to y = -t + d \to y = -t + 1$ $y(T) = 0 \to T = 1$

Optimal values: $u = -1$, $T = 1$, $y = -t + 1$

Ex Rocket Car

Minimize $\int_0^T dt = T$

subject to $\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = u \end{cases}$ $\quad \vec{y}(0) = \begin{pmatrix} y_{10} \\ y_{20} \end{pmatrix} \quad \vec{y}(T) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad |u| \le 1 \quad T > 0$

$H = -1 + p_1 y_2 + p_2 u \qquad H = \text{const.} \quad H|_{t=T} = 0 \rightarrow 0 = H = -1 + p_1 y_2 + p_2 u$

$\begin{cases} \dot{p}_1 = -H_{y_1} = 0 \\ \dot{p}_2 = -H_{y_2} = -p_1 \end{cases} \longrightarrow \begin{matrix} p_1 = c \\ \dot{p}_2 = -c \rightarrow p_2 = -ct + d \end{matrix}$

If $p_2 = 0 \ \forall t \rightarrow p_1 = 0 \ \forall t \longrightarrow H = -1$ ※

$p_2 = at + b$, $p_2 \ne 0 \ \forall 0 \longleftarrow ?$

maximize H for $|u| \le 1$

when $p_2 > 0$, $u = 1$ maximizes H

when $p_2 < 0$, $u = -1$ maximizes H

$u(t) = \text{sgn}(p_2(t)) = \text{sgn}(at + b)$

Either $u = 1 \ \forall t$, $u = -1 \ \forall t$, or u switches sign once on $[0, T]$

Thus at most one switching time t_s.

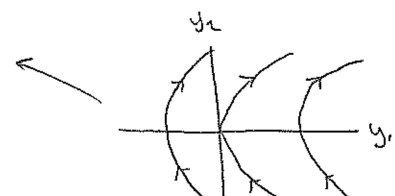
2015 11 25

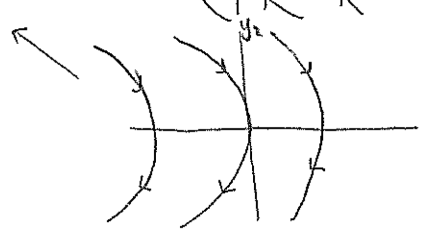
$u = 1$: $y_2 = t + c_2 \longrightarrow y_1 = t^2/2 + c_2 t + c_1$

$dy_1/dy_2 = y_2 \longrightarrow y_1 = y_2^2/2 + \alpha$

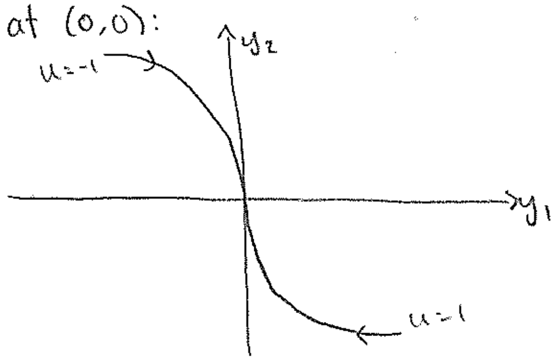
$u = -1$: $y_2 = -t + \bar{c}_2 \longrightarrow y_1 = -t^2/2 + \bar{c}_2 t + \bar{c}_1$

$dy_1/dy_2 = -y_2 \longrightarrow y_1 = -y_2^2/2 + \beta$





End at (0,0):

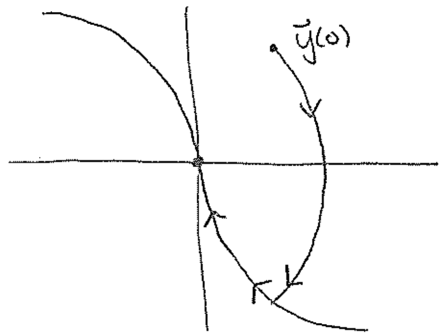


Switching curve

$$y_1 = \begin{cases} \dfrac{-y_2^2}{2}, & y_2 \ge 0 \\ \dfrac{y_2^2}{2}, & y_2 \le 0 \end{cases}$$

If $\begin{pmatrix} y_{10} \\ y_{20} \end{pmatrix}$ is on the switch curve then $u=1$ or $u=-1$ on $[0,T]$.
If $\vec{y}(0)$ is above the switching curve, $u=-1$ until the switching is met then change to $u=1$

$$u(t) = \begin{cases} -1 & 0 \le t < t_s \\ 1 & t_s < t \le T \end{cases}$$



If $\vec{y}(0)$ is below the switching curve, $u=1$ until the switching curve is met then it changes to $u=-1$

$$u(t) = \begin{cases} 1 & 0 \le t < t_s \\ -1 & t_s < t \le T \end{cases}$$

Special case: $\vec{y}(0) = \begin{pmatrix} w_0 \\ 0 \end{pmatrix}$, $w_0 > 0$

$t_s = \sqrt{w_0}$, $T = 2\sqrt{w_0}$

For $0 \le t \le t_s$: $y_1 = -t^2/2 + w_0$, $y_2 = -t$

For $t_s \le t \le T$: $y_1 = t^2/2 - 2\sqrt{w_0} + 2w_0$, $y_2 = t - 2\sqrt{w_0}$

Ex. Optimal Control of a Simple Harmonic Oscillator

Minimize $\int_0^t T\,dt = T$ ← $\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = -y_1 + u \end{cases}$

subject to $\vec{y}(0) = \begin{pmatrix} y_{10} \\ y_{20} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\vec{y}(T) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $|u(t)| \le 1$

$H = -1 + p_1 y_2 + p_2(-y_1 + u)$ $\begin{cases} H \text{ const.} \\ H|_{t=T} = 0 \end{cases} \rightarrow H = 0$

$\dot{p}_1 = -H_{y_1} = p_2$

$\dot{p}_2 = -H_{y_2} = -p_1$

$\rightarrow \ddot{p}_1 = -\dot{p}_1 = -p_2 \longrightarrow p_2(t) = A\cos(t+\delta)$

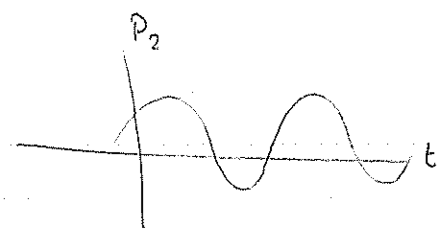
$\rightarrow \ddot{p}_2 + p_2 = 0$

$$u = \underset{|v| \le 1}{\operatorname{argmax}} \left(-1 + p_1 y_2 - p_2 y + p_2 v\right) = \operatorname{sgn}(p_2)$$

If $p_2 \equiv 0 \Rightarrow p_1 \equiv 0 \Rightarrow H = -1$ ※

Thus $A \neq 0$.

Thus switching times are π apart →



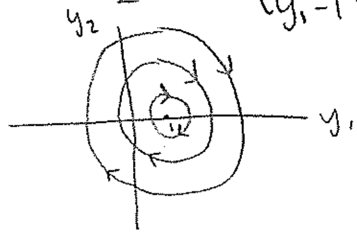
$u = 1$: $\ddot{y}_2 = -\dot{y}_1 = -y_2 \rightarrow \ddot{y}_2 + y_2 = 0$

$y_2 = A_1\cos(t+\delta)$, $y_1 = 1 - \dot{y}_2 = 1 + A_1 \sin(t+\delta)$

$\dfrac{dy_1}{dy_2} = \dfrac{y_2}{-y_1+1} \rightarrow (-y_1+1)\,dy_1 = y_2\,dy_2$

$-\dfrac{y_1^2}{2} + y_1 + c = \dfrac{y_2^2}{2} \longrightarrow y_1^2 + y_2^2 - 2y_1 = 2c$

$(y_1 - 1)^2 + y_2^2 = c_1$

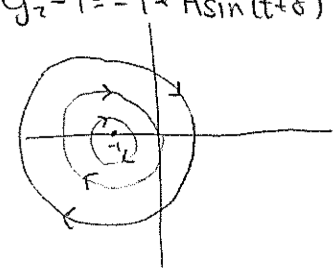


$u = -1$: $\ddot{y}_2 + y_2 = 0$, $y_2 = \bar{A}\cos(t+\bar{\delta})$, $y_1 = -\dot{y}_2 - 1 = -1 + \bar{A}\sin(t+\delta)$

$\dfrac{dy_1}{dy_2} = \dfrac{y_2}{-y_1-1}$

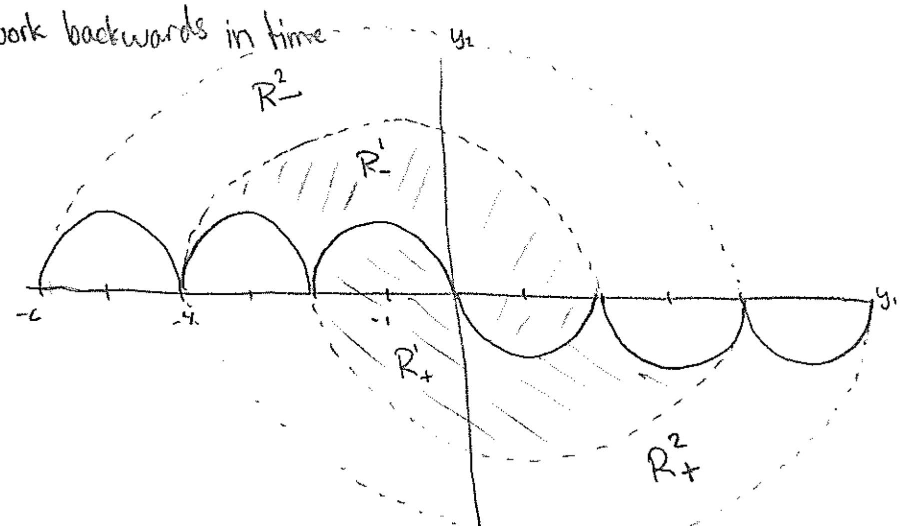
$-(y_1+1)\,dy_1 = y_2\,dy_2$ $y_1^2 + y_2^2 + 2y_1 = 2c$

$-\dfrac{y_1^2}{2} - y_1 + C = \dfrac{y_2^2}{2} \longrightarrow (y_1+1)^2 + y_2^2 = c_2$



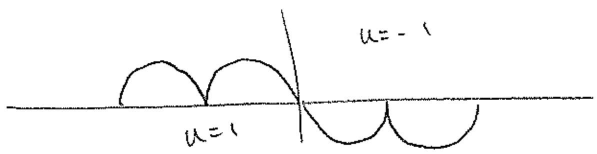
2015 11 27

work backwards in time



In R_-^1, start with $u=-1$, switch to $u=1$ when the switching curve is met.
In R_+^1, start with $u=1$, switch to $u=-1$ when the switching curve is met.
Etc.

Switching curve



$u=-1$ above switching curve or on the switching curve with $y_1<0$
$u=1$ below switching curve or on the switching curve with $y_1>0$

Special case: $\begin{pmatrix} y_{10} \\ y_{20} \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$

$$u(t) = \begin{cases} 1, & 0<t<\pi \\ -1, & \pi<t\leq 2\pi \\ 1, & 2\pi<t\leq 3\pi \end{cases}$$

$$(y_1(t), y_2(t)) = \begin{cases} (1+5\cos t, -5\sin t) & 0\leq t\leq \pi \\ (-1+3\cos t, -3\sin t) & \pi\leq t\leq 2\pi \\ (1+\cos t, -\sin t) & 2\pi\leq t\leq 3\pi \end{cases}$$