Riemann Mapping Theorem

Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, simply connected region. Then there exists a conformal bijection $w(z)$ from Ω to the unit disk $\{z; |z|<1\}$ such that $w(\alpha)=0$, for any $\alpha \in \Omega$ specified in advance.

$\{z; |z|<1\}$

holomorphic functions

Def A mapping is conformal in an open set Ω if $w \in H(\Omega)$ is holomorphic and $w'(z) \neq 0$ for all $z \in \Omega$.

Consider

$$G = \frac{1}{2\pi} \ln |w(z)|.$$

For $z \in \partial\Omega$, we have $|w(z)| = 1$, so $G = 0$ on $\partial\Omega$.
Let $\alpha = \xi + i\eta$, $z = x + iy$. Then

$$w(z) = (z-\alpha)H(z).$$

The only singularity of G is where $w(z)=0$.

$$G = \frac{1}{2\pi}\ln|w(z)| = \frac{1}{2\pi}\ln|(z-\alpha)H(z)|$$

$$= \frac{1}{2\pi}\ln r + \frac{1}{2\pi}\ln|H(z)|.$$

$H(z)$ is analytic in all Ω, and since

$$h = \frac{1}{2\pi}\mathrm{Re}(\ln(H(z))) \implies \Delta h = 0.$$

Construction of conformal maps (preliminary)

Suppose we require a conformal map w from the upper half plane

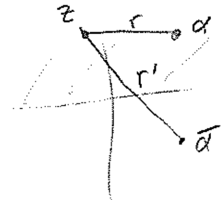
$$H^* = \{z;\ \mathrm{Im}(z) > 0\}$$

onto the open unit disk $D(0;1)$.
Note that

$$H^* = \{z;\ |z-\alpha| < |z-\bar{\alpha}|\}.$$

It is now clear that we should take

$$h(z) = \frac{z-\alpha}{z-\bar{\alpha}}.$$



That this maps to the unit disk is easy to see. Also ω is conformal in H^*, since

$$\omega'(z) = \frac{(z-\bar\alpha)-(z-\alpha)}{(z-\bar\alpha)^2} = \frac{2\,\mathrm{Im}(\alpha)}{(z-\bar\alpha)^2} \neq 0.$$

Note

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$$G = \frac{1}{2\pi}\ln\left|\frac{z-\alpha}{z-\bar\alpha}\right| = \frac{1}{2\pi}\ln r - \frac{1}{2\pi}\ln r',$$

as we got earlier.

The extended complex plane is $\widetilde{\mathbb{C}} = \mathbb{C}\cup\{\infty\}$.
We have stereographic projection from $\hat{\mathbb{C}}$ to $\Sigma := \{\underbrace{(x,y,w)}_{\in\mathbb{R}^3};\ x^2+y^2+(w-\tfrac12)^2 = \tfrac12{}^2\}$ by

$$x+iy \longleftrightarrow \left(\frac{2r\cos\theta}{1+r^2},\ \frac{2r\sin\theta}{1+r^2},\ \frac{r^2}{1+r^2}\right)$$

$$\infty \longleftrightarrow (0,0,1) \qquad \text{(North pole).}$$

Any circle in $\mathbb{C}$ is given by $|z-a|=r$. However, working ~~and~~ in $\widetilde{\mathbb{C}}$ and adjoining $\{\infty\}$ to any line, then circles and straight lines in $\hat{\mathbb{C}}$ correspond to circles on the Riemann sphere.

Theorem (Image point representation of circlines)
Let $\alpha,\beta\in\mathbb{C}$, $\alpha\neq\beta$. (Note: α,β are inverses wrt $|z-a|=r$ if $(\alpha-a)\overline{(\beta-a)}=r^2$.) For any $\lambda>0$, the equation

$$\left|\frac{z-\alpha}{z-\beta}\right| = \lambda$$

represents a circline with inverse (image) points α and β, and every circline can be so represented.

ex unit circle $|z|=1$
any points α and $\frac{1}{\bar\alpha}$ are image points wrt unit circle
Thus the equation for this circle can be written

$$\left|\frac{z-\alpha}{z-1/\bar\alpha}\right| = \left|\frac{1-\alpha}{1-1/\bar\alpha}\right| \qquad \text{(since 1 lies on the circle)}$$

ie $$\left|\frac{z-\alpha}{\bar\alpha z-1}\right| = \left|\frac{1-\alpha}{1-\bar\alpha}\right| = 1 \quad \text{since } |1-\alpha| = |1-\bar\alpha|$$

Möbius (fractional) transformation

Def) A Möbius transformation is a mapping

$$f(z) = \frac{az+b}{cz+d} \qquad \begin{pmatrix} a,b,c,d \in \mathbb{C} \\ ad-bc \neq 0 \end{pmatrix}$$

$$f: \tilde{\mathbb{C}} \to \tilde{\mathbb{C}} \quad \text{with } f(-d/c) := \infty \text{ and } f(\infty) := a/c$$

This is 1-1 with inverse

$$f^{-1}(w) = \frac{dw-b}{a-cw}$$

which is also a Möbius transformation.
These form a group under composition
We can use these to build up other mappings:

$z \mapsto ze^{i\varphi}$ (φ real) (anticlockwise rotation through φ)
$z \mapsto Rz$ $(R>0)$ (stretch by factor R)
$z \mapsto z+a$ $(a \in \mathbb{C})$ (translation)
$z \mapsto 1/z$ (inversion)

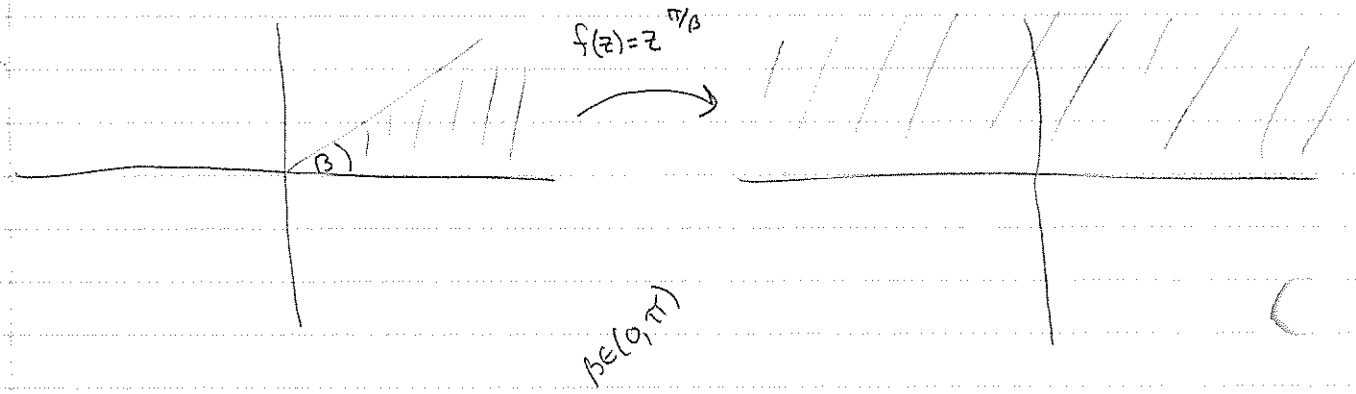
Note for $f(z) = \frac{az+b}{cz+d}$,

$$f'(z) = \frac{ad-bc}{(cz+d)^2} \neq 0 \quad \forall z \neq -\frac{d}{c}$$

ie f is conformal on $\mathbb{C} \setminus \{-d/c\}$.

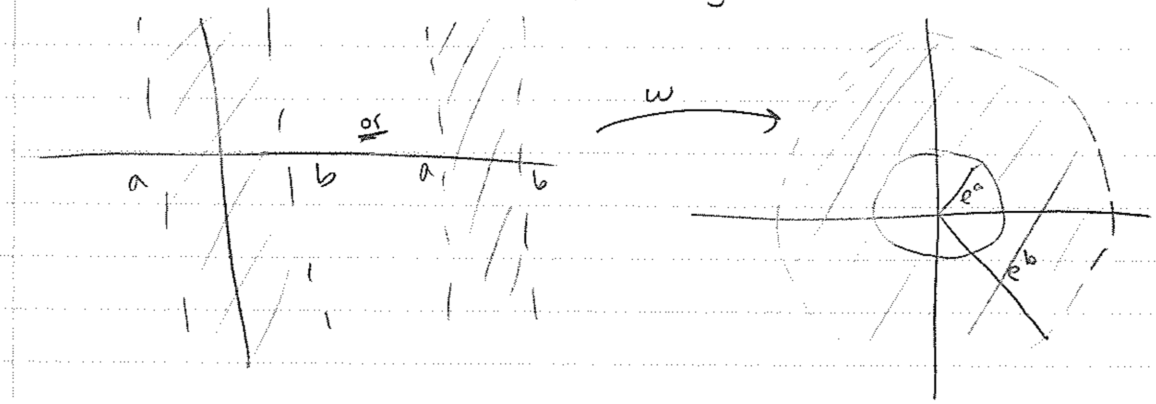
Powers: z^n $(n=2,3,\dots)$
This is conformal except at the origin

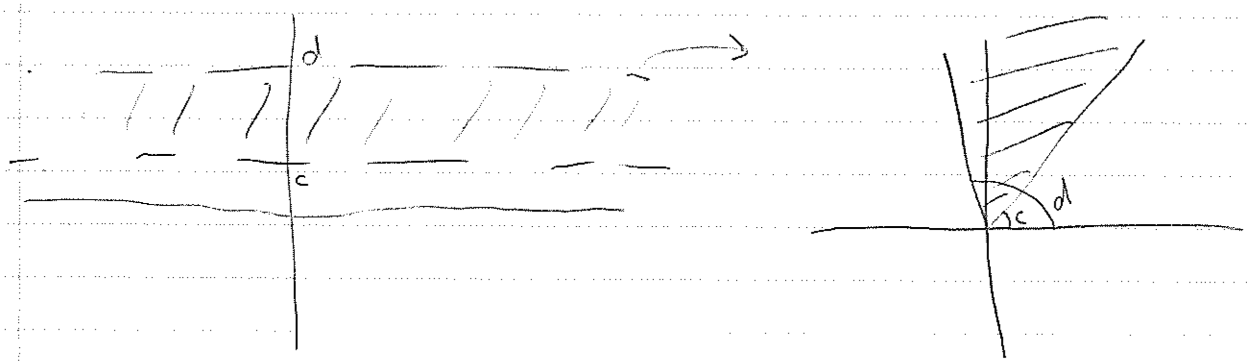
ex.



Exponentials $w(z) = e^z = Re^{i\varphi}$

where $R = e^x$, $\varphi = y$





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Joukowski transformation

$$w(z) = \frac{1}{2}\left(z + \frac{1}{z}\right) \qquad (*)$$

$$w+1 = \frac{1}{2}\left(z + \frac{1}{z} + 2\right) = \frac{1}{2}\frac{(z+1)^2}{z}$$

$$w-1 = \frac{1}{2}\left(z + \frac{1}{z} - 2\right) = \frac{1}{2}\frac{(z-1)^2}{z}$$

$$\Rightarrow \frac{w+1}{w-1} = \left(\frac{z+1}{z-1}\right)^2$$

$$w'(z) = \frac{1}{2}\left(1 - \frac{1}{z^2}\right) = \frac{1}{2}\frac{(z-1)(z+1)}{z^2}$$

$\therefore$ $(*)$ is holomorphic except at 0 and ∞, and conformal except at $z = \pm 1$.

If $z \mapsto w(z) = u + iv$ then

$$u = \frac{1}{2}\left(r + \frac{1}{r}\right)\cos\theta,$$

$$v = \frac{1}{2}\left(r - \frac{1}{r}\right)\sin\theta.$$

Thus the image of a circle $|z| = \rho$ in the z-plane becomes the ellipse

$$\frac{u^2}{\frac{1}{2}\left(\rho + \frac{1}{\rho}\right)^2} + \frac{v^2}{\frac{1}{2}\left(\rho - \frac{1}{\rho}\right)^2} = 1$$

in the w-plane, and the image of the half-line $\arg z = \mu$ is the hyperbola

$$\frac{u^2}{\cos^2\mu} - \frac{v^2}{\sin^2\mu} = 1$$

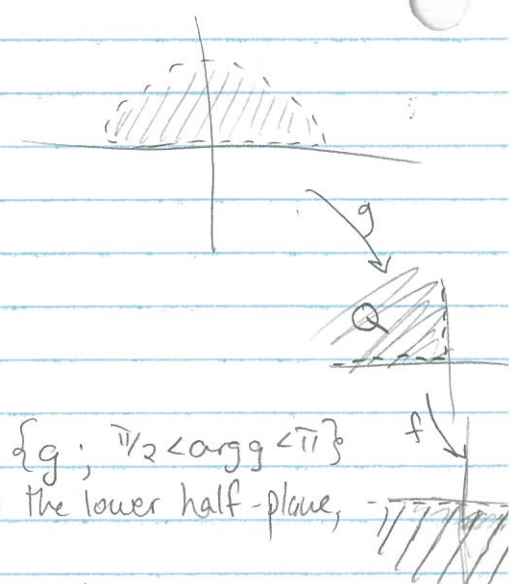
in the w-plane

ex Find a conformal map of the semi-circular domain $\Omega = \{z;\ \mathrm{im}(z) > 0, |z| < 1\}$ onto the unit disk.

Note that

$= \arg(z-1) - \arg(z+1)$

arg is θ $$\Omega = \left\{z;\ \frac{\pi}{2} < \arg\left(\frac{z-1}{z+1}\right) < \pi\right\}$$



Define

$$g(z) = \frac{z-1}{z+1} = u + iv$$

$\therefore \Omega$ is mapped conformally to the quadrant $\{g;\ \pi/2 < \arg g < \pi\}$

Let $f(g) = g^2$. Then Q is mapped conformally to the lower half-plane, $\{f;\ \pi < \arg f < 2\pi\}$

The lower half plane can be characterized as ~~the set~~

$$\{f;\ |f+i| < |f-i|\}$$

and thus

$$w(f) = \frac{f+i}{f-i}$$

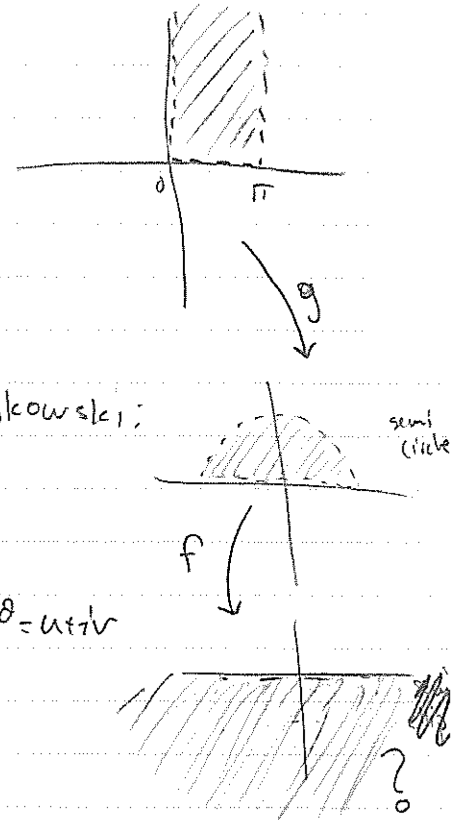
is mapped to $\{w;\ |w| < 1\}$. Thus a suitable conformal map is

$$F(z) = \frac{\left(\frac{z-1}{z+1}\right)^2 + i}{\left(\frac{z-1}{z+1}\right)^2 - i} = \frac{(z-1)^2 + i(z+1)^2}{(z-1)^2 - i(z+1)^2} : \Omega \to B_1(0)$$

ex. Find a conformal map of the semi-infinite strip

$$H = \{z \,;\ \operatorname{im}(z) > 0,\ 0 < \operatorname{Re}(z) < \pi\}$$

onto a half-plane.



Let $g(z) = e^{iz}$. Then $|g(z)| = e^{-\operatorname{im}(z)}$
and $\arg g = \operatorname{Re} z \pmod{2\pi}$.
H is mapped conformally onto the half-disk.

$$G = \{g \,;\ 0<|g|<1,\ 0<\arg g<\pi\}$$

We can proceed as in prev. ex., or use Joukowski:

$$f(g) = \frac{1}{2}\left(g+\frac{1}{g}\right),$$

which is conformal in G.
Each of the ~~[illegible]~~ semicircular arcs $g = re^{i\theta} = u+iv$
$(0<r<1,\ 0<\theta<\pi)$ is mapped to an elliptic arc

$$\frac{u^2}{\left(\frac{1}{2}\left(r+\frac{1}{r}\right)\right)^2} + \frac{v^2}{\left(\frac{1}{2}\left(r-\frac{1}{r}\right)\right)^2} = 1, \qquad v<0$$

and the conjugate? transformation is $w(z) = \frac{1}{2}(e^{iz}+e^{-iz}) = \cos z$
composition?

ex $D = \{(x,y);\ 0<y<d\}$

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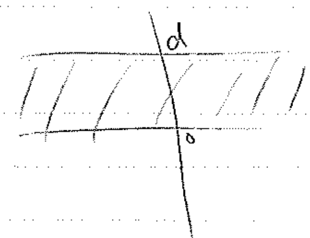


First map D to the upper half plane:

$$w_1(z) = e^{(\pi/d)z} \qquad (z = \xi + i\eta)$$

$$= e^{(\pi/d)\xi} + e^{i(\pi/d)\eta}$$

We next map the half-plane to the unit disk. The ~~[illegible]~~ map D→disk is

$$w_2(z) = \frac{w_1(z) - e^{(\pi/d)\alpha}}{w_1(z) - e^{(\pi/d)\bar{\alpha}}} = \frac{e^{(\pi/d)z} - e^{(\pi/d)\alpha}}{e^{(\pi/d)z} - e^{(\pi/d)\bar{\alpha}}}$$

$$\therefore\ G = \frac{1}{2\pi}\ln|w(z)| = \frac{1}{2\pi}\ln\left(\left|\frac{e^{(\pi/d)z} - e^{(\pi/d)\alpha}}{e^{(\pi/d)z} - e^{(\pi/d)\bar{\alpha}}}\right|\right)$$

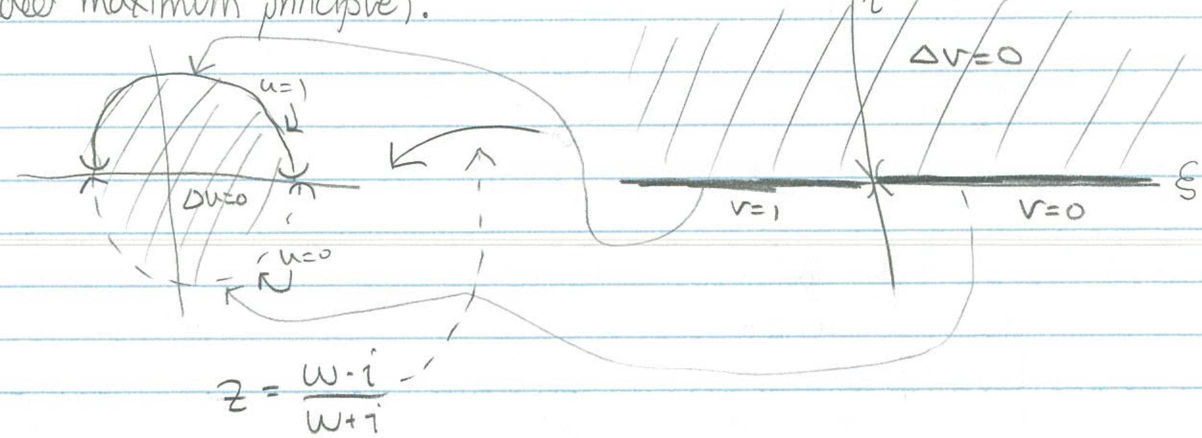
check arithmetic

$$\overset{?}{=}\frac{1}{2\pi}\ln\left(\left|\frac{e^{\frac{2\pi}{d}\xi} + e^{\frac{2\pi}{d}x} - 2e^{\frac{\pi}{d}(\xi+x)}\cos(\eta-y)}{e^{\frac{2\pi}{d}\xi} + e^{\frac{2\pi}{d}x} - 2e^{\frac{\pi}{d}(\xi+x)}\cos(\eta+y)}\right|\right)$$

ex Solve

$$\begin{cases} \Delta u = 0 & \rho < 1 \\ u(1,\theta) = 1 & 0 < \theta < \pi \\ u(1,\theta) = 0 & \pi < \theta < 2\pi \end{cases}.$$

We know this problem has at most one bounded solution (by the extended maximum principle).



$$z = \frac{w-i}{w+i}$$

is a conformal map from the upper half plane to the unit disk.

Consider $w = \xi$. This mapped to

$$z = \frac{\xi - i}{\xi + i} = \frac{\xi^2 - 1 - 2\xi i}{\xi^2 + 1}.$$

Therefore we see $\xi > 0$ is mapped to the lower circle boundary and $\xi < 0$ is mapped to the upper circle boundary.

In the upper half plane, it is clear that a solution is

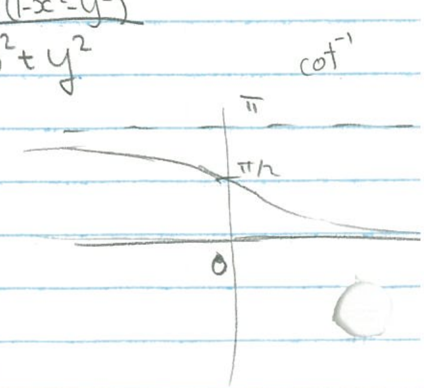
$$v = \frac{\theta}{\pi} = \frac{1}{\pi} \cot^{-1}\left(\frac{\xi}{\eta}\right).$$

Inverting $z = \frac{w-i}{w+i}$ gives $w = i\left(\frac{1+z}{1-z}\right)$, ie

$$\xi + i\eta = \frac{i((1+x) + iy)}{(1-x) - iy} = \frac{-2y + i(1-x^2-y^2)}{(1-x)^2 + y^2}$$

Therefore

$$u(x,y) = \frac{1}{\pi} \cot^{-1}\left(\frac{-2y}{1-x^2-y^2}\right)$$



From the graph of $\cot^{-1}$, we see

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = \begin{cases} 1 & \text{for } (x_0,y_0) \in \partial D,\ y_0 > 0 \\ 0 & \quad \text{"} \quad y_0 < 0 \end{cases}$$

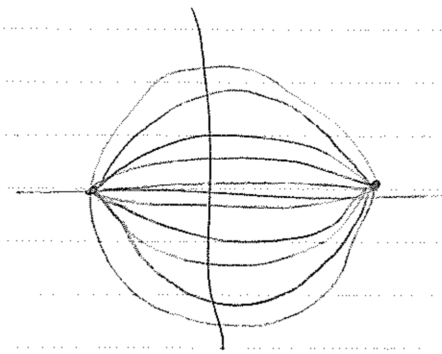
The level curves are given by

$$\frac{-2y}{1-x^2-y^2} = \frac{1}{c} \qquad \text{(c arbitrary constant)}$$

$$\Rightarrow 1-x^2-y^2 = -2cy$$

$$\Rightarrow x^2+(y-c)^2 = 1+c^2$$

All these circles pass through $(\pm 1, 0)$:



Alternatively, see chap 2 pg 30-31 and use the Poisson integral formula

new topic

Fourier Series and Eigenvalues of the Laplacian

$$(1) — u_{tt} = c^2 \Delta u \qquad (2) — u_t = k\Delta u$$

in a bounded domain $\Omega \subset \mathbb{R}^3$.

Let $u(x,y,z,t) = T(t)v(x,y,z)$. Then

$$(1) \to -\lambda = \frac{T''}{c^2 T} = \frac{\Delta v}{v}, \qquad -\lambda = \frac{T'}{kT} = \frac{\Delta v}{v} \leftarrow (2)$$

Both problems lead to the eigenvalue problem

$$\Delta v + \lambda v = 0 \quad \text{in } \Omega$$

with some Dirichlet, Neumann, or Robin condition on $\partial\Omega$.

If the problem has eigenvalues λ_n (all positive) and associated eigenfunctions $v_n(x)$, then solutions of (1) 2013 11 04

$$u(x,t) = \sum_{m,n} \left(a_n \cos(\sqrt{\lambda_n}\, ct) + b_n \sin(\sqrt{\lambda_n}\, ct)\right) v_m(x)$$

and for (2)

$$u(x,t) = \sum_n a_n e^{-\lambda_n kt} v_n(x).$$