Further, if $\bar{\xi} \in \mathbb{R}^{n+1}$, $|\bar{\xi} - x| = |\bar{\xi} - x^*|$, and therefore 2013.11.16

$$K(\bar{\xi} - x) = K(\bar{\xi} - x^*).$$

$\therefore$ $G(\xi, x) = K(\xi - x) - K(\xi - x^*)$ is the Green's function on $\Omega (= \mathbb{R}^n)^+$.

To calculate the Poisson kernel, we need to differentiate $G(\xi, x)$ in the negative ξ_n direction:

for $n \geq 3$, $$\frac{\partial}{\partial \xi_n} K(x - \xi) = \frac{(\xi_n - x_n)}{S} |\xi - x|^{-n}.$$

So the Poisson kernel is

$$-\frac{\partial}{\partial \xi_n} G(\xi, x)\Big|_{\xi_n = 0} = \frac{2x_n}{S} |\bar{\xi} - x|^{-n}$$

and we obtain the solution to the Dirichlet problem for $\Omega = (\mathbb{R}^n)^+$:

$$(*) \quad u(x) = \frac{2x_n}{S} \int_{\mathbb{R}^{n-1}} \frac{g(\bar{\xi})}{|\bar{\xi} - x|^n} \, d\bar{\xi}$$

Assumptions about boundary data:

- assume g is bounded (to guarantee convergence),
- since our uniqueness theorems require $u \in C^2(\Omega) \cap C(\bar{\Omega})$, it is reasonable to require continuity of g.

Theorem: If $g(\bar{x})$ is bounded and continuous for $\bar{x} \in \mathbb{R}^{n-1}$, then the function $(*)$ is C^∞ and harmonic in $(\mathbb{R}^n)^+$ and extends continuously to $\overline{(\mathbb{R}^n)^+}$ such that $u(\bar{x}) = g(\bar{x})$ on $\partial(\mathbb{R}^n)^+$

new example

Dirichlet problem on a sphere

ex #2

ex. Let $\Omega = B(0; a) = \{x \in \mathbb{R}^n; |x| < a\}$ and for $\xi \in \Omega$, we define

$$\xi^* = \frac{a^2}{|\xi|^2} \cdot \xi$$

to be the image of ξ in $\partial\Omega$ (if is not $\in \partial\Omega$).

For boundary points $|x| = a$,

$$|x - \xi^*|^2 = (x - \xi^*) \cdot (x - \xi^*)$$
$$= \underbrace{|x|^2}_{a^2} - 2x \cdot \xi^* + |\xi^*|^2$$

$$= a^2 - \frac{2a^2}{|\xi|^2}\sum_{i=1}^{n} x_i \xi_i + \frac{a^4}{|\xi|^2}$$

$$= \frac{a^2}{|\xi|^2}\left(|\xi|^2 - 2\sum_{i=1}^{n} x_i \xi_i + a^2\right)$$

$$= \frac{a^2}{|\xi|^2}|x - \xi|^2.$$

This implies

$$\frac{|x - \xi^*|}{|x - \xi|} = \frac{a}{|\xi|} \quad \text{for } |x| = a.$$

Notice that for $x \in \partial\Omega$ (ie $|x| = a$), we have

$$K(x - \xi) = \frac{1}{2\pi}\log|x - \xi| = \frac{1}{2\pi}\log\frac{|\xi|}{a}|x - \xi^*|$$

$$= \frac{1}{2\pi}\log|x - \xi^*| + \frac{1}{2\pi}\log\left(\frac{|\xi|}{a}\right), \qquad n = 2$$

$$K(x - \xi) = \left(\frac{|\xi|}{a}\right)^{2-n} K(x - \xi^*). \qquad n \geq 3$$

Thus, ~~for~~ we define Green's function for $x, \xi \in \Omega$ as

$$G(x, \xi) = \frac{1}{2\pi}\left(\log|x - \xi| - \log\frac{|\xi|}{a}|x - \xi^*|\right), \qquad n = 2$$

$$G(x, \xi) = K(x - \xi) - \left(\frac{|\xi|}{a}\right)^{2-n} K(x - \xi^*). \qquad n \geq 3$$

By construction, $G(x, \xi) = 0$ if $x \in \partial\Omega$.

$\therefore$ G is the Green's function for Ω.

If we calculate the Poisson kernel on $\partial\Omega$, we can use the Poisson integral formula to show existence of a solution to the Dirichlet problem in a sphere.

Given G, to show existence of a solution to the Dirichlet problem in a ball: 2013 10 18

$$\frac{\partial G}{\partial x_i}(x, \xi) = \frac{1}{S}\left(|x - \xi|^{-n}(x_i - \xi_i) - \left(\frac{a}{|\xi|}\right)^{n-2}|x - \xi^*|^{-n}(x_i - \xi_i^*)\right).$$

Let $|x| = a$ and using

$$\frac{|x - \xi^*|}{|x - \xi|} = \frac{a}{|\xi|},$$

we get (arithmetic)

$$\frac{\partial G}{\partial x_i} = \frac{x_i}{S}\left(1 - \frac{|\xi|^2}{a^2}\right) \quad \text{for } x \in \partial\Omega$$

The exterior unit normal on $\partial\Omega$ is $n = x/a$. Therefore the Poisson kernel is

$$\left.\frac{\partial G}{\partial n}\right|_{\partial\Omega} = n \cdot \nabla G = \frac{a^2 - |\xi|^2}{aS|x-\xi|^n} \quad \text{for } x \in \partial\Omega$$

and the solution to the Dirichlet problem for $\Omega = B(0;a)$ is

$$(*) \quad u(\xi) = \frac{a^2 - |\xi|^2}{aS}\int_{|x|=a} \frac{g(x)}{|x-\xi|^n}\, dS_x.$$

Theorem: If $g(x)$ is continuous on $\partial\Omega = \{x;\ |x| = a\}$ then the function defined by $(*)$ above is C^∞ and harmonic in Ω, and extends continuously to $\bar{\Omega}$ such that $u(\xi) = g(\xi)$ for $|\xi| = a$.

Corollary (Smoothness): If $u \in C^2(\Omega)$ is harmonic then $u \in C^\infty(\Omega)$ and extends continuously to $\bar{\Omega}$ such that $u(\xi) = g(\xi)$ for $|\xi| = a$.

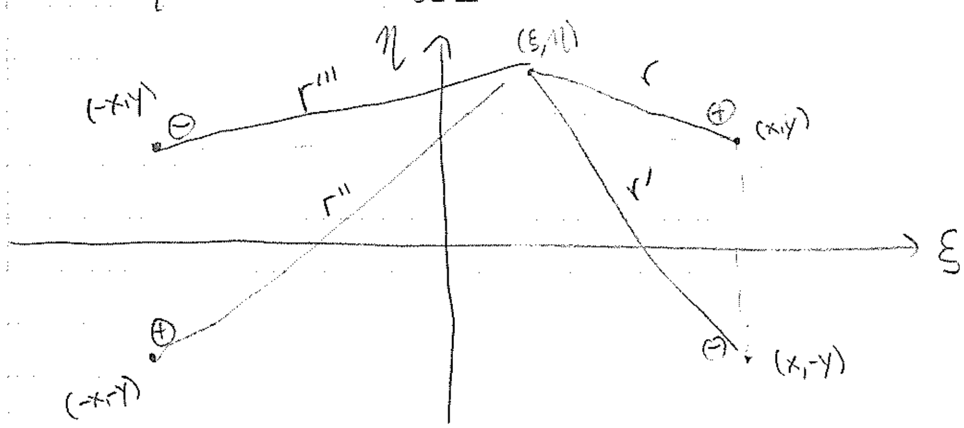
Corollary 2 (Harnack inequality): If $u \in C^2(\Omega)$ is harmonic and non-negative and Ω_1 is a bounded domain st $\Omega_1 \subset \Omega$ then there exists a constant C that depends only on Ω_1 such that

$$\sup_{x \in \Omega_1} u(x) \leq C \inf_{x \in \Omega_1} u(x).$$

ex #3

ex Consider $\Omega = \{(x,y) \in \mathbb{R}^2;\ x, y > 0\}$. Solve

$$\begin{cases} \Delta u = F & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \end{cases}$$



$(*)$ —— $G = \frac{1}{2\pi}\ln r - \frac{1}{2\pi} r' + \frac{1}{2\pi}\ln r'' - \frac{1}{2\pi}\ln r'''$

$$= \frac{1}{4\pi}\ln\underbrace{\left(\left(\frac{(\xi-x)^2+(\eta-y)^2}{(\xi-x)^2+(\eta+y)^2}\right)\cdot\left(\frac{(\xi+x)^2+(\eta+y)^2}{(\xi+x)^2+(\eta-y)^2}\right)\right)}_{(\star)}$$

Along the x-axis, we use $(*)$ to calculate

$$\frac{\partial G}{\partial \underline{n}} = -\frac{\partial G}{\partial \eta}\bigg|_{\eta=0} = \frac{y}{\pi}\left(\frac{1}{(\xi-x)^2+y^2} - \frac{1}{(\xi+x)^2+y^2}\right)$$

Along the y-axis, we use $(*)$ to calculate

$$\frac{\partial G}{\partial \underline{n}} = -\frac{\partial G}{\partial \xi}\bigg|_{\xi=0} = \frac{x}{\pi}\left(\frac{1}{x^2+(\eta-y)^2} - \frac{1}{x^2+(\eta+y)^2}\right).$$

So the solution to the BVP is

$$u(x,y) = \frac{y}{\pi}\int_0^\infty f(\xi)\left(\frac{1}{(\xi-x)^2+y^2} - \frac{1}{(\xi+x)^2+y^2}\right)d\xi$$
$$+ \frac{x}{\pi}\int_0^\infty f(\eta)\left(\frac{1}{x^2+(\eta-y)^2} - \frac{1}{x^2+(\eta+y)^2}\right)d\eta$$
$$+ \frac{1}{4\pi}\int_0^\infty\int_0^\infty F(\xi,\eta)\ln((\star))\,d\xi\,d\eta.$$

The method of Green's functions can be applied to other canonical elliptic DEs.

Helmholtz equation
$$\begin{cases} \Delta u + u = F & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \end{cases}$$

Using Green's second identity,

$$\int_\Omega (u\Delta G - G\Delta u)\,d\underline{x} = \int_{\partial\Omega}\left(u\frac{\partial G}{\partial \underline{n}} - G\frac{\partial u}{\partial \underline{n}}\right)dS.$$

We impose the requirement

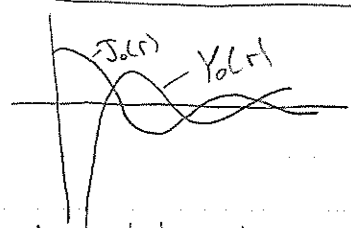
$$\begin{cases} \Delta G + G = \delta(r) & \text{in } \Omega \\ G = 0 & \text{on } \partial\Omega. \end{cases}$$

Substituting into Green's second identity:

$$\int_\Omega \left(u(\delta(r) - \cancel{G}) - G(F - \cancel{u})\right)d\underline{x} = \int_{\partial\Omega} f\frac{\partial G}{\partial \underline{n}}\,dS$$

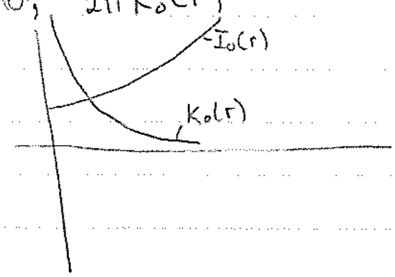
$$\Longrightarrow u(\underline{x}) = \int_\Omega GF\,d\underline{x} + \int_{\partial\Omega} f\frac{\partial G}{\partial \underline{n}}\,dS$$

Fundamental Solution for Helmholtz Equation



$$L[u] = \Delta u + u$$

A fundamental solution is $\frac{1}{4} Y_0(r)$.
For $L[u] = \Delta u - u$, the fundamental solution is the modified Bessel function of the second kind, $-\frac{1}{2\pi} K_0(r)$



2013.10.21

recall from last day

For $L[u] = \Delta u + u$,
$$\begin{cases} G = \frac{1}{4} Y_0(r) + h \\ h \text{ regular: } \Delta h + h = 0, \quad h = -\frac{1}{4} Y_0(r) \text{ on } \partial\Omega. \end{cases}$$
For $L[u] = \Delta u - u$,
$$\begin{cases} G = -\frac{1}{2\pi} K_0(r) + h \\ h \text{ regular: } \Delta h - h = 0, \quad h = \frac{1}{2\pi} K_0(r) \text{ on } \partial\Omega \end{cases}$$

for Dirichlet boundary condition

Boundary conditions of the second kind (Neumann bdy condition)

$$\begin{cases} \Delta u = F & \text{on } \Omega \\ \frac{\partial u}{\partial n} = f & \text{on } \partial\Omega \end{cases}$$

We require
$$\begin{cases} \Delta G = \delta(r) & \text{in } \Omega \\ \frac{\partial G}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

From Green's second identity,
$$u(\underline{x}) = -\int_{\partial\Omega} G f \, dS + \int_{\Omega} G F \, d\underline{x}.$$

are the signs right?

So

$$G = \frac{1}{2\pi}\ln(r) + h$$

where h is regular and

$$\begin{cases} \Delta h = 0 & \text{in } \Omega \\ \frac{\partial h}{\partial \underline{n}} = -\frac{1}{2\pi}\frac{\partial}{\partial \underline{n}}(\ln r) & \text{on } \partial\Omega. \end{cases}$$

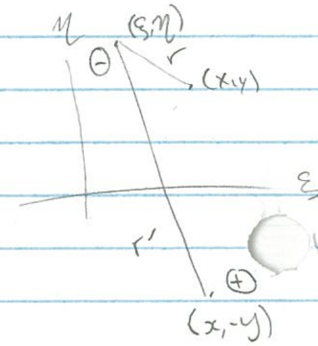
ex Solve

$$\begin{cases} \Delta u = F(x,y) & \text{in } \Omega = \{(x,y); y>0\} \\ \frac{\partial u}{\partial y}(x,0) = f(x) \end{cases}$$

normal is $-y$, but just multiply $\frac{\partial u}{\partial -y} = -f$ by -1

Well,

$$\frac{\partial G}{\partial \underline{n}} = 0 \Rightarrow -\frac{\partial G}{\partial \eta}\bigg|_{\eta=0} = 0 \quad \text{ie} \quad \frac{\partial G}{\partial \eta} = 0$$



Let $G = \frac{1}{2\pi}\ln(r) + \frac{1}{2\pi}\ln(r')$ (even function in η).

~~Hence~~ Since

$$(*) \quad G = \frac{1}{4\pi}\ln\left(\left((\xi-x)^2+(\eta-y)^2\right)\left((\xi-x)^2+(\eta+y)^2\right)\right),$$

$$\Rightarrow G = \frac{1}{4\pi}\ln\left((\xi-x)^2+y^2\right) \quad \text{for } \eta = 0.$$

Hence

$$\frac{\partial u}{\partial \underline{n}} = -\frac{\partial u}{\partial \eta}(\xi,\eta)\bigg|_{\eta=0} = -f(\xi).$$

Therefore

$$u(\underline{x}) = \frac{1}{4\pi}\int_0^\infty\int_{-\infty}^\infty F(\xi,\eta)\ln\left(\text{from } (*)\right)\,d\underline{x}$$

$$+\frac{1}{2\pi}\int_{-\infty}^\infty f(\xi)\ln\left((\xi-x)^2+y^2\right)d\xi.$$

Modified Green's Functions

$$\begin{cases} \Delta u = F & \text{in } \Omega \quad (\text{bounded } \Omega) \\ \frac{\partial u}{\partial n} = f & \text{on } \partial\Omega \end{cases}$$

note the prev. ex is not in a bounded domain

Notice if we seek

$$(*) \text{---} \begin{cases} \Delta G = \delta(r) & \text{in } \Omega \\ \frac{\partial G}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$$

then

$$1 = \int_\Omega \delta(r)\, d\underline{x} = \int_\Omega \Delta G\, d\underline{x} = \int_{\partial\Omega} \frac{\partial G}{\partial \underline{n}}\, dS = 0,$$

a contradiction. Thus we look for a modified function set

$$\begin{cases} \Delta G = \delta(r) & \text{in } \Omega \\ \frac{\partial G}{\partial n} = \ell & \text{on } \partial\Omega \quad (\ell \text{ constant}). \end{cases}$$

Now we see

$$1 = \cdots = \int_{\partial\Omega} \frac{\partial G}{\partial \underline{n}}\, ds = \ell \int_{\partial\Omega} ds = \ell L$$

where L is the length of $\partial\Omega$. Hence $\ell = 1/L$. By Green's second identity,

$$u(x) - \int_\Omega G F\, d\underline{x} = \frac{1}{L}\int_{\partial\Omega} u\, ds - \int_{\partial\Omega} G f\, ds$$

$$\Rightarrow u(x) = \int_\Omega G F\, d\underline{x} - \int_{\partial\Omega} G f\, ds + \frac{1}{L}\int_{\partial\Omega} u\, ds.$$

It's clear that a Green's function can be shifted by a constant and still satisfy

$$\begin{cases} \Delta G = \delta(r) \\ \frac{\partial G}{\partial n} = \ell. \end{cases}$$

Therefore

$$u(x) = \int_\Omega (G+a) F\, d\underline{x} - \int_{\partial\Omega} (G+a) f\, dS + \frac{1}{L}\int_{\partial\Omega} u\, dS$$

$$= \int_\Omega G F\, d\underline{x} - \int_{\partial\Omega} G f\, ds + \frac{1}{L}\int_{\partial\Omega} u\, ds$$

$\int_\Omega a \Delta u = \int_{\partial\Omega} a \frac{\partial u}{\partial n}$
$\shortparallel$
$\int_\Omega a F = \int_{\partial\Omega} a f$

by div thm