ex If u is unbounded, consider $\Omega = \{(x,y);\ y>0\}$ and $u=y$.
Then

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has $u=y$ and $u=0$ as solutions.

ex Find the unique solution to

$$\begin{cases} \Delta u = 0 & y>0 \\ u(x,0) = 0 & x>0 \\ u(x,0) = 1 & x<0 \\ u \text{ bounded} & \end{cases}$$

We look for $u = u(\theta)$. Recall

$$\Delta u = u_{rr} + \tfrac{1}{r}u_r + \tfrac{1}{r^2}u_{\theta\theta} = 0 = \tfrac{1}{r^2}u_{\theta\theta}.$$

Hence $u_{\theta\theta}=0$. Integrate twice to get

$$u = c_1\theta + c_2.$$

When $\theta=0$, $u(r,0)=0$ and so $c_2=0$.
When $\theta=\pi$, $u(r,\pi)=1$ implies $c_1 = 1/\pi$.
Therefore $u = \theta/\pi$. Note $0\le u\le 1$ so we have indeed found the (unique bounded) solution.

Green's Functions

The fundamental solution $K(\underline{x})$ for the laplace operator is a distribution that satisfies

$$\Delta K(\underline{x}) = \delta(\underline{x}). \qquad \text{—— } (*)$$

We note that Δ is symmetric in the variables $x_1, \dots, x_n$ and so is $\delta(\underline{x})$ (in the sense that the value of $\delta(\underline{x})$ depends only on $r=|\underline{x}|$).
Since $\delta(\underline{x})=0$ for $\underline{x}\neq 0$, we require (1) K to be harmonic for $r>0$.
For radially symmetric functions, Laplace's equation simplifies to

$$\Psi'' + \frac{(n-1)}{r}\Psi' = 0 \qquad \text{for } r>0 \qquad \text{—— } (2)$$

The general solution (by separation of variables) is

$$\Psi(r) = \begin{cases} c_1 + c_2 \ln r & n=2 \\ c_1 + c_2 r^{2-n} & n\geq 3. \end{cases}$$

For any $v \in C_0^\infty(\mathbb{R}^n)$, we show that

$$\int_{\mathbb{R}^n} \Psi(|\underline{x}|)\,\Delta v(\underline{x})\,d\underline{x} = v(\underline{0}). \qquad (3)$$

Let $\Omega_\varepsilon = B(0,R) \setminus \overline{B(0,\varepsilon)}$, where $\text{supp}(v) \subset B(0,R)$ and $\varepsilon > 0$.
Since Ψ is integrable at $x=0$,

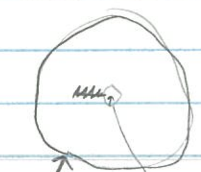
$$\int_{\Omega_\varepsilon} \Psi \Delta v \, d\underline{x} \longrightarrow \int_{\Omega} \Psi \Delta v \, d\underline{x}.$$

Further, Ψ is harmonic in Ω_ε. By Green's second identity,

$$\int_{\Omega_\varepsilon} \Psi \Delta v \, d\underline{x} = \int_{\partial\Omega_\varepsilon} \left(\Psi \frac{\partial v}{\partial \underline{n}} - v \frac{\partial \Psi}{\partial \underline{n}} \right) d\underline{S}$$

$$= \int_{|\underline{x}|=\varepsilon} \left(\Psi \frac{\partial v}{\partial \underline{n}} - v \frac{\partial \Psi}{\partial \underline{n}} \right) d\underline{S}$$

since $v=0$ on



$\underline{n}$ points out of Ω, ie towards the origin

But $\underline{n} = \frac{-\underline{x}}{\|\underline{x}\|}$, so this becomes

$$= \int_{|\underline{x}|=\varepsilon} \Psi \frac{\partial v}{\partial \underline{n}} \, d\underline{S} - \int_{|\underline{x}|=\varepsilon} v \frac{\partial \Psi}{\partial \underline{n}} \, d\underline{S}$$

divergence thm

$$= \left(-\Psi(\varepsilon) \int_{|\underline{x}|<\varepsilon} \Delta v \, d\underline{x} \right) - \left(\int_{|\underline{x}|=\varepsilon} v C^* \varepsilon^{1-n} \, d\underline{S} \right)$$

where $C^* = \begin{cases} -c_2 & n=2 \\ (n-2)c_2 & n\geq 3 \end{cases}$

$$= \underbrace{\Psi(\varepsilon)\,O(\varepsilon^n)}_{\xrightarrow{\varepsilon\to 0} 0} - C^* \varepsilon^{1-n} \int_{|\underline{x}|=\varepsilon} v \, d\underline{S}$$

surface area of ball of radius 1

$$= -C^* S \cdot \underbrace{M_v(\underline{0}; \varepsilon)}$$

2013 10 09

$$\longrightarrow -C^* S v(\underline{0})$$

since v is continuous at $\underline{0}$.

Choosing c_2 such that $C^* S = -1$ we find that (3) is satisfied.
Therefore a fundamental solution for the Laplace operator is

definition of K is here

$$K(\underline{x}) = \begin{cases} \frac{1}{2\pi} \ln(r) & n=2, \\ \frac{1}{(2-n)S} r^{2-n} & n \geq 3. \end{cases}$$

?

$\langle a,b \rangle = \int ab$

Def] We say that u is a distribution solution of

$$Lu = f$$

if $\langle u, L^* v \rangle = \langle f, v \rangle$ for all $v \in C_0^\infty(\Omega)$, where L^* is the adjoint of L.

If $Lu=f$ is an inhomogeneous DE (with constant coefficients), we consider the first the case

$$Lu = \delta. \qquad \text{—} \quad (1)$$

A solution $u = F$ of (1) is a fundamental solution of L, ie

$$\langle F, L^* v\rangle = \int_{\mathbb{R}^n} F(\underline{x})\, L^* v(\underline{x})\, \underline{dx}$$

$$\parallel$$

$$\langle \delta, v\rangle = \int_{\mathbb{R}^n} \delta(\underline{x}) v(\underline{x})\, d\underline{x} = v(\underline{0}).$$

It is clear that a fundamental solution is not unique since we can always add any solution $Lu=0$ without affecting (1).
If f has compact support, we can take the convolution with F, to get

$$u(\underline{x}) = F * f\,(\underline{x}) = \int_{\mathbb{R}^n} F(\underline{x}-\underline{\xi}) f(\underline{\xi})\, d\underline{\xi}.$$

This is a solution of $Lu=f$ since

$$L(u(\underline{x})) = L(F*f(\underline{x}))$$

isn't it supposed to be
$(L(u))(\underline{x}) = (L(F*f))(x)$

$$= \int_{\mathbb{R}^n} L\big(F(\underline{x}-\underline{\xi}) f(\underline{\xi})\big)\, \underline{d\xi}$$

$$= \int_{\mathbb{R}^n} L(F(\underline{x}-\underline{\xi}))\, f(\underline{\xi})\, d\underline{\xi}$$

$$= \int_{\mathbb{R}^n} \delta(\underline{x}-\underline{\xi}) f(\underline{\xi})\, d\underline{\xi} = f(\underline{x}).$$

K fund. sol. for $\Delta u = f$

Thus, in particular for $L = \Delta$,

$$u(\underline{x}) = \int_{\mathbb{R}^n} K(\underline{x}-\underline{\xi})\, \Delta u(\underline{\xi})\, d\underline{\xi}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$.

Representation Theorem:

Let Ω be a /bounded smooth domain in $\mathbb{R}^n$ with $u \in C^2(\overline{\Omega})$, and $x \in \Omega$. Then

$$u(x) = \int_{\Omega} K(x-\xi)\Delta u(\xi)\,d\xi + \int_{\partial\Omega}\left(u(\xi)\frac{\partial K}{\partial n_\xi}(x-\xi) - K(x-\xi)\frac{\partial u}{\partial n_\xi}\right)dS$$

Proof: Let $x \in \Omega$ and let $0 < \varepsilon < \operatorname{dist}(x, \partial\Omega)$. If $\Omega_\varepsilon = \Omega \setminus \overline{B(x;\varepsilon)}$, then by Green's second identity,

$$\int_{\Omega_\varepsilon} K(x-\xi)\Delta u(\xi)\,d\xi = \int_{\partial\Omega_\varepsilon}\left(K(x-\xi)\frac{\partial u}{\partial n_\xi} - u(\xi)\frac{\partial K(x-\xi)}{\partial n_\xi}\right)dS_\xi$$

where $\partial\Omega_\varepsilon = \partial\Omega \cup \partial B(x;\varepsilon)$. As $\varepsilon \to 0$, since u is continuous, $\int_{\Omega_\varepsilon} \to \int_{\Omega}$.
Now by the argument from start of lecture, we have on the RHS,

$$\int_{\partial B(x;\varepsilon)} \longrightarrow u(x).$$

■

In what sense does

$$u(x) = \int_{\Omega} K(x-\xi) f(\xi)\,d\xi \quad \text{---} \quad (*)$$

define a solution of Poisson's equation $\Delta u = f$.
If f is integrable, we have that u is at least a distributional solution.

($\clubsuit$) It can be shown that for a <u>bounded</u> domain Ω, and $f \in L^1(\Omega)$, we have that u (see $(*)$) is a C^∞ function and harmonic on $\mathbb{R}^n \setminus \overline{\Omega}$. 2013 10 11
For f bounded on Ω, then $u \in C^1(\mathbb{R}^n)$ (in addition to previous assumption ($\clubsuit$)), for $f \in C^1(\overline{\Omega})$ we have $u \in C^2(\Omega)$ (assuming ($\clubsuit$)).

ex To solve

$$\begin{cases} \Delta w = f & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases}$$

let u be defined by $(*)$ and let $w = u + v$, where

$$\begin{cases} \Delta v = 0 & \text{on } \Omega, \\ v = -u & \text{on } \partial\Omega. \end{cases}$$

So we focus on Laplace's equation for which the representation theorem takes the form:

~~Midterm up to end of today, 25th~~

This follows from since $\Delta u = 0$ in Ω.

double layer potential   single layer potential

Theorem: Let Ω be a smooth, bounded domain in $\mathbb{R}^n$, and let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be harmonic in Ω, and let $\underline{x} \in \Omega$. Then

$$u(\underline{x}) = \int_{\partial\Omega} \left(u(\underline{\xi}) \frac{\partial K}{\partial n_\xi}(\underline{x} - \underline{\xi}) - K(\underline{x} - \underline{\xi}) \frac{\partial u}{\partial n_\xi}(\underline{\xi}) \right) d\underline{S}_\xi .$$

This suggests that we may be able to find a solution to the Dirichlet problem as a double layer potential, and a solution to the Neumann problem as a single layer potential.
But to do this we have to modify the fundamental solution $K(\underline{x} - \underline{\xi})$.
Let $h(\underline{x})$ be any harmonic function in Ω, and define

$$G(\underline{x}, \underline{\xi}) = K(\underline{x} - \underline{\xi}) + h(\underline{x}).$$

Using Green's second identity, we have

$$\int_\Omega h \Delta u \, d\underline{x} + \int_{\partial\Omega} \left(u \frac{\partial h}{\partial \underline{n}} - h \frac{\partial u}{\partial n} \right) d\underline{S} = 0.$$

Then the representation theorem (previous page) generalizes to

$$(**) \qquad u(\underline{x}) = \int_\Omega G(\underline{x}; \underline{\xi}) \Delta u \, d\underline{\xi} + \int_{\partial\Omega} \left(u(\underline{\xi}) \frac{\partial G}{\partial n_\xi}(\underline{x}, \underline{\xi}) - G(\underline{x}; \underline{\xi}) \frac{\partial u}{\partial n}(\underline{\xi}) \right) d\underline{S}_\xi .$$

Any $G(\underline{x}; \underline{\xi})$ satisfying $(**)$ for $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is a fundamental solution in Ω with pole at $\underline{x}$. Thus if we can find $h_\xi(\underline{x})$ which is harmonic in Ω and satisfies

$$h_\xi(\underline{x}) = -K(\underline{x} - \underline{\xi}) \qquad \forall \underline{x} \in \partial\Omega.$$

Then $G(\underline{x}; \underline{\xi}) = K(\underline{x} - \underline{\xi}) + h_\xi(\underline{x})$ is a fundamental solution with the property that $G(\underline{x}; \underline{\xi}) = 0 \ \forall \underline{x} \in \partial\Omega$.
This is the Green's function. (Difficult to construct in general.)
To construct this, we have to solve

$$\Delta h_\xi(\underline{x}) = 0 \quad \text{in } \Omega \ni \underline{x},$$
$$h_\xi(\underline{x}) = -K(\underline{x} - \underline{\xi}) \quad \text{on } \partial\Omega \ni \underline{x}.$$

The "physical" argument from electrostatics:
Suppose $\partial\Omega$ is a perfectly conducting surface and a positive unit charge is placed at $\underline{\xi} \in \Omega$, then negative charges are induced on $\partial\Omega$, and the potential function for the resultant electric field has the stated properties of the Green's function.

Summary:

- Let $\Omega \subset \mathbb{R}^n$ be smooth and bounded, and $u \in C^2(\bar{\Omega})$, if $x \in \Omega$ then

$$u(x) = \int_\Omega K(x-\xi)\Delta u(\xi)\,d\xi + \int_{\partial\Omega} \overbrace{\left(u(\xi)\frac{\partial K}{\partial n_\xi} - K\frac{\partial u}{\partial n}\right)}^{(\dagger)} dS$$ ← rep. thm

- If u satisfies $\Delta u = 0$, this gives for $u \in C^2(\Omega) \cap C^1(\Omega)$:

$$u(x) = \int_{\partial\Omega}\left(u(\xi)\frac{\partial K}{\partial n_\xi}(x-\xi) - K(x-\xi)\frac{\partial u}{\partial n}\right) dS$$

- Construct $G(x;\xi) = K(x-\xi) + h_\xi(x)$

$$u(x) = \int_\Omega G(x;\xi)\Delta u\,d\xi + \int_{\partial\Omega} (\dagger) \text{ with } K \mapsto G$$

Since by construction $G(x,\xi) = 0$ for $x \in \Omega$, $\xi \in \partial\Omega$,

$$\Rightarrow u(x) = \int_\Omega G(x;\xi)\Delta u\,d\xi + \int_{\partial\Omega} u(\xi)\frac{\partial G}{\partial n_\xi}(x;\xi)\,dS_\xi.$$

In particular, if

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

this gives the Poisson integral formula

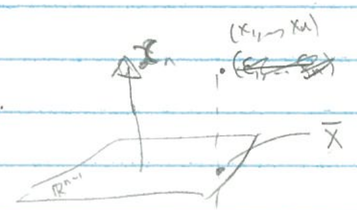
$$u(x) = \int_{\partial\Omega} \underbrace{\frac{\partial G}{\partial n_\xi}(x;\xi)}_{\text{Poisson kernel}} g(\xi)\,dS_\xi.$$

So if the Dirichlet problem has a solution $u \in C^2(\bar{\Omega})$, we can construct this from the Poisson integral formula (provided we can compute $G(x;\xi)$).

ex #1

ex. Consider $\Omega = (\mathbb{R}^n)^+ = \{(\xi_1, \ldots, \xi_n) \in \mathbb{R}^n : \xi_n > 0\}$ and let $K = (\xi - x^*)$ be the fundamental solution.

Notice that $K(\xi - x^*)$ is harmonic $\forall \xi \in \Omega$.



$\xi = (\xi_1, \ldots, \xi_{n-1}, -\xi_n)$

$x^* = (x_1, \ldots, x_{n-1}, -x_n)$

Recall,

$$K(x) = \begin{cases} \frac{1}{2\pi}\ln(r) & n = 2 \\ \frac{1}{(2-n)S} r^{2-n} & n \geq 3. \end{cases}$$

THE VARIABLES are wrong, see my review sheets

Further, if $\bar{\xi} \in \mathbb{R}^{n+1}$, $|\bar{\xi} - x| = |\bar{\xi} - x^*|$, and therefore 2013 11 16

$$K(\bar{\xi} - x) = K(\bar{\xi} - x^*).$$

$\therefore$ $G(\xi, x) = K(\xi - x) - K(\xi - x^*)$ is the Green's function on $\Omega (= \mathbb{R}^n)^+$).

To calculate the Poisson kernel, we need to differentiate $G(\xi, x)$ in the negative ξ_n direction:

$$\text{for } n \geq 3, \quad \frac{\partial}{\partial \xi_n} K(x - \xi) = \frac{(\xi_n - x_n)}{S} |\xi - x|^{-n}.$$

So the Poisson kernel is

$$-\frac{\partial}{\partial \xi_n} G(\xi, x)\Big|_{\xi_n = 0} = \frac{2x_n}{S} |\bar{\xi} - x|^{-n}$$

and we obtain the solution to the Dirichlet problem for $\Omega = (\mathbb{R}^n)^+$:

$$(*) \text{ ——— } u(x) = \frac{2x_n}{S} \int_{\mathbb{R}^{n-1}} \frac{g(\bar{\xi})}{|\bar{\xi} - x|^n} \, d\bar{\xi}$$

Assumptions about boundary data:

- assume g is bounded (to guarantee convergence),
- since our uniqueness theorems require $u \in C^2(\Omega) \cap C(\bar{\Omega})$, it is reasonable to require continuity of g.

Theorem: If $g(\bar{x})$ is bounded and continuous for $\bar{x} \in \mathbb{R}^{n-1}$, then the function $(*)$ is C^∞ and harmonic in $(\mathbb{R}^n)^+$ and extends continuously to $\overline{(\mathbb{R}^n)^+}$ such that $u(\bar{x}) = g(\bar{x})$ on $\partial(\mathbb{R}^n)^+$

new example

Dirichlet problem on a sphere

ex #2

ex. Let $\Omega = B(0; a) = \{x \in \mathbb{R}^n ; |x| < a\}$ and for $\xi \in \Omega$, we define

$$\xi^* = \frac{a^2}{|\xi|^2} \cdot \xi$$

to be the image of ξ in $\partial\Omega$ (if is not $\in \partial\Omega$).

For boundary points $|x| = a$,

$$|x - \xi^*|^2 = (x - \xi^*) \cdot (x - \xi^*)$$

$$= \underbrace{|x|^2}_{a^2} - 2x \cdot \xi^* + |\xi^*|^2$$