harmonic → satisfies laplace's eq.

a domain Ω is ~~a~~ open subset of $\mathbb{R}^n$ — path connected

BVP (1) has more than one solution if and only if BVP (2) has a non-zero solution.

Proof: Exercise.

here we start maximum principles

Maximum Principles and Mean Values:

Theorem (Mean Value Property):

means path connected

Let $u \in C^2(\Omega)$ harmonic in a connected domain Ω, let $\xi \in \Omega$, and pick $r > 0$ such that $B(\xi; r) \subseteq \Omega$. Then

$$(*) \quad u(\xi) = M_u(\xi; r) := \frac{1}{A(\xi;1)} \int_{|x|=1} u(\xi + r x)\, dS.$$

surface area of ball of radius 1

average/mean value of u on surface of ba[ll]

Proof: Using Green's identity with $v = 1$,

$$\int_\Omega \Delta u \, dx = \int_{\partial\Omega} \frac{\partial u}{\partial n} \, dS.$$

also works for volum[e]

Since $\Delta u = 0$ in Ω, we have

change of variable

$$0 = \int_{\partial B(\xi; r)} \frac{\partial u}{\partial n} \, dS = \int_{|x|=1} \frac{\partial u}{\partial r}(\xi + r x)\, dS$$

$$" \int_\Omega \Delta u \, dx "$$

$$= A \cdot \frac{\partial}{\partial r} \frac{1}{A} \int_{|x|=1} u(\xi + r x)\, dS$$

$$= A \cdot \frac{\partial}{\partial r} M_u(\xi; r).$$

Hence

$$\frac{\partial}{\partial r} M_u(\xi; r) = 0$$

which means $M(\xi; r)$ is independent of r. By continuity

$$M_u(\xi; r) \xrightarrow[r \to 0]{} u(\xi).$$

As $M_u(\xi; r)$ is constant wrt r, we must have $M_u(\xi; r) = u(\xi)$.

Def | A continuous function u that satisfies $(*)$ is said to have the mean value property (MVP).

The theorem says that harmonic functions have the MVP. Is it true that for $u \in C^2(\Omega)$, where u has the MVP, u is harmonic?

recall "domain Ω" = "open path connected subset of $\mathbb{R}^n$"

Replacing r by ρr and multiplying through by ρ^{n-1}, $(*)$ becomes

(?) $(*)$ — $u(\xi) = \dfrac{1}{V} \displaystyle\int_{|x|\le 1} u(\xi + r x)\, dx$

volume of ball of radius 1

once we integrate over $\rho \in [0,1]$

subharmonic case

Now consider $u \in C^2(\Omega)$ where u is <u>subharmonic</u> (that is, $\Delta u \geq 0$) in Ω. Then 20131002

$$\int_{\partial B_r} \frac{\partial u}{\partial r}\, dS = \int_{B_r} \Delta u \, dx \geq 0$$

$A(\xi, 1)$

$$\Rightarrow \int_{|x|=1} \frac{\partial u}{\partial r}(\xi + r x)\, dS = A \frac{\partial}{\partial r}\left(\frac{1}{A} \int_{|x|=1} u(\xi + r x)\, dS \right) \geq 0$$

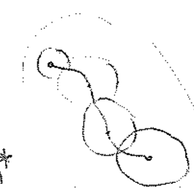
$$\Rightarrow \frac{\partial}{\partial r} M_u(\xi; r) \geq 0.$$

Thus $M_u(\xi; r)$ is an increasing function of of r, so that

$(**)$ — $u(\xi) \leq M_u(\xi; r) = \dfrac{1}{A} \displaystyle\int_{|x|=1} u(\xi + r x)$,

and using $(*)$ we have (the volume MVP)

$$u(\xi) \leq M_u(\xi; r) = \frac{1}{V(\xi, 1)} \int_{|x|\le 1} u(\xi + r x)\, dx.$$



Let $A^* = \sup_{\Omega} u$. ~~Then~~

Thus if $u(\xi) = A^*$ at an interior point ξ, it follows that $u(x) = A^*$ for all x in a ball of radius r about ξ. Using path-connectedness it follows that $u(\xi) = A^*$ for all $\xi \in \Omega$. (ie $\exists$ max in $\Omega \Rightarrow u$ constant)

Weak Maximum Principle:

If Ω is bounded and $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies $\underbrace{\Delta u \geq 0, \text{ in } \Omega}_{\text{ie } \Omega \text{ is subharmonic}}$ then for all $x \in \Omega$ we have

$$u(x) \leq \max_{\partial\Omega} u$$

Proof:

Step 1: If $v \in C^2(\Omega) \cap C(\bar{\Omega})$ and $\Delta v > 0$, then we show that for all $x \in \Omega$ we have

$$v(x) < \max_{\partial\Omega} v,$$

and that if the maximum occurs at $\underline{p} \in \overline{\Omega}$, then $\underline{p} \notin \Omega$.
If $\underline{p} \in \Omega$ then at max we have

$$v_x(\underline{p}) = v_y(\underline{p}) = 0$$

and

$$v_{xx}(\underline{p}) \le 0, \quad v_{yy}(\underline{p}) \le 0,$$

which imply $\Delta v(\underline{p}) \le 0$. Contradiction, so $\underline{p} \in \partial\Omega$ and

$$v(\underline{x}) < \max_{\partial\Omega} v$$

for $\underline{x} \in \Omega$.
Step 2: Let $v = u + \varepsilon\|\underline{x}\|^2$ where $\varepsilon > 0$. Then

$$\Delta v = \Delta u + 4\varepsilon \geq 4\varepsilon > 0.$$

Applying step 1, we get

$$v(\underline{x}) < \max_{\partial\Omega} v$$

$u \le v < \max_{\partial\Omega} v = \max_{\partial\Omega}(u + \varepsilon\|x\|^2) \le \max_{\partial\Omega} u + \max_{\partial\Omega} \varepsilon\|x\|^2 \to 0$

for $\underline{x} \in \Omega$. Hence

$$v(\underline{x}) < \max_{\partial\Omega} u + \varepsilon R^2$$

where R is the radius of a ball containing Ω. Then

$$u(\underline{x}) \le \max_{\partial\Omega} u + \varepsilon R^2$$

Letting $\varepsilon \to 0$ we get the desired result. ■

Corollary (Max/Min Principle for Harmonic Functions):
If $\Delta u = 0$ in Ω then for all $\underline{x} \in \Omega$ we have

Ω bounded!

$$\min_{\partial\Omega} u \le u(\underline{x}) \le \max_{\partial\Omega} u.$$

Proof: Apply the previous theorem to u and $-u$. ■

Corollary: If Ω is bounded and $u, v \in C^2(\Omega) \cup C(\overline{\Omega})$ both satisfy

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

Then $u = v$ on Ω.

note: Ω is always open

These two corollaries show that the Dirichlet problem
$$\begin{cases}\Delta u = f & \text{in } \Omega\\ u = g & \text{on } \partial\Omega\end{cases}$$
$\rightarrow$ will be well-posed <u>if</u> we can prove existance of a solution.

Theorem: Suppose u is continuous and has the MVP in Ω. Then u is harmonic in Ω.

Proof: Let $\underline{x}^* \in \Omega$ and let $\overline{B(\underline{x}^*;\rho)} \subseteq \Omega$. Since u is continuous on ~~the~~ $\partial B(\underline{x}^*;\rho)$. Consider the problem:

Find $v \in C^2(B(\underline{x}^*,\rho)) \cap C(\overline{B(\underline{x}^*,\rho)})$

s.t. $\begin{cases}\Delta v = 0 & \text{in } B(\underline{x}^*;\rho)\\ v = u & \text{on } \partial B(\underline{x}^*;\rho).\end{cases}$

Such a v is harmonic in $B(\underline{x}^*;\rho)$ and hence has the MVP in this ball. By construction, $v=u$ on $\partial B(\underline{x}^*;\rho)$. Note $u-v$ since u and v do $\rightarrow$ <u>has the MVP</u> in $B(\underline{x}^*,\rho)$, with $u-v=0$ on $\partial B(\underline{x}^*;\rho)$. Thus $u-v$ assumes it's maximum value on $\partial B(\underline{x}^*,\rho)$, ~~and so~~ by the maximum principle. Thus $u-v=0$ in $B(\underline{x}^*,\rho)$ and is harmonic in these area. ∎ (?)

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this is in course text $\rightarrow$ Note: If $\Delta u = F(\underline{x})$ in Ω (bounded (and open as always)), then
$$\min_{\partial\Omega} u - \frac{R^2}{4}\max_{\Omega}|F| \le u(\underline{x}) \le \max_{\partial\Omega} u + \frac{R^2}{4}\max_{\Omega}|F|$$
for all $\underline{x}\in\Omega$, where R is the radius of the smallest ball containing Ω.

Proof: Let $v = u + \frac{\|\underline{x}\|^2}{4}\max_{\Omega}|F|$. Then
$$\Delta v = \Delta u + \max_{\Omega}|F|$$
$$= F + \max_{\Omega}|F| \geq 0.$$
Applying the weak maximum principle to v, we have
$$v \le \max_{\partial\Omega} v$$
Therefore
$$u \le \underbrace{u + \frac{\|\underline{x}\|^2}{4}\max_{\Omega}|F|}_{v} \le \max_{\partial\Omega}\left(u + \frac{\|\underline{x}\|^2}{4}\max_{\Omega}|F|\right)$$

$(\nu) \leq \max_{\partial\Omega} u + \frac{R^2}{4} \max_{\Omega} |F|.$

Applying the result to $-u$ (ie $\Delta(-u) = -F$) gives

$$-u \leq \max_{\partial\Omega} -u + \frac{R^2}{4} \max_{\Omega} |F|$$

$$= -\min_{\partial\Omega} u + \frac{R^2}{4} \max_{\Omega} |F|.$$

Multiply by -1 and flip inequality to get the result. ■

It immediately follows that

$$(\dagger) — \quad |u| \leq \max_{\partial\Omega} |u| + \frac{R^2}{4} \max_{\Omega} |F|.$$

this is for $\Delta u = F$ in Ω, bounded/open/path connec and where $\Omega \subseteq B(R)$

We know have the following continuous dependence result.

Proposition: Consider the BVP

$$(*) — \begin{cases} \Delta u_1 = F_1 & \text{in } \Omega, \\ u_1 = f_1 & \text{on } \partial\Omega, \end{cases}$$

and the perturbed BVP

$$(**) — \begin{cases} \Delta u_2 = F_2 & \text{in } \Omega \\ u_2 = f_2 & \text{on } \partial\Omega, \end{cases}$$

where $|F_1 - F_2| \leq \varepsilon_1$ and $|f_1 - f_2| \leq \varepsilon$. Then

$$|u_1 - u_2| \leq \varepsilon + \frac{R^2}{4} \varepsilon_1.$$

if perturbations are sma

(ie. Solutions of the perturbed problem $(**)$ are close to solutions of the orginal problem $(*)$.)

Proof: $\begin{cases} \Delta(u_1 - u_2) = F_1 - F_2 & \text{in } \Omega \\ (u_1 - u_2) = f_1 - f_2 & \text{on } \partial\Omega \end{cases}$

From $(\dagger)$, we have

$$|u_1 - u_2| \leq \max_{\partial\Omega} |f_1 - f_2| + \frac{R^2}{4} \max_{\Omega} |F_1 - F_2| = \varepsilon + \frac{R^2}{4} \varepsilon_1.$$

ex. Consider

$$\begin{cases} \Delta u = -1 & r<1, \\ u=0 & r=1. \end{cases}$$

Show (i) $0 \le u \le 1/4$, and (ii) the exact solution.

Solution: Using (+) with $R=1$, we have

$$\max_{\Omega} |F| = 1$$

$$\Rightarrow \max_{\partial\Omega} u = \min_{\partial\Omega} u = 0.$$

Now actually using (+), we get $-1/4 \le u \le 1/4$. But notice that $\Delta(-u) = 1 \ge 0$. So by the maximum principle,

$$-u \le \max_{r=1} (-u) = 0$$

$$\Rightarrow u \ge 0$$

as desired by (i).

As for (ii), we look for $u = u(r)$. In this case we have

$$-1 = \Delta u = u_{rr} + \tfrac{1}{r} u_r.$$

Hence

$$\frac{\partial}{\partial r}(r u_r) = -r.$$

Integrating gives

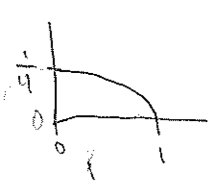
$$u_r = \frac{-r}{2} + \frac{c_1}{r}.$$

Again yields

$$u = \frac{-r^2}{4} + c_1 \ln(r) + c_2.$$

Since $\ln(r)$ is singular at 0, we set $c_1 = 0$. Now use the boundary condition to get $c_2 = 1/4$. Thus

$$u(r) = \frac{1-r^2}{4},$$



note $0 \le u \le 1/4$ is the best bound possible domain $\overset{def}{\Longrightarrow}$ open

$\Delta u \geq 0$

u must be subharmonic!

Theorem (Discontinuous Boundary Values):
If $u \in C^2(\Omega) \cap C(\overline{\Omega} \setminus \{p_1, \dots, p_n\})$ for $p_i \in \partial\Omega$, and Ω a bounded domain, and if u is bounded in Ω and $u \leq M$ on $\cancel{\partial\Omega}$ $\partial\Omega \setminus \{p_1, \dots, p_n\}$, then

$$u \leq M \text{ in } \Omega.$$

ex To show that bounded in Ω cannot be dropped, consider the half-disk Ω of radius 1 (in $\mathbb{R}^2$), and $u = (\sin(\theta))/r$. Easy check shows $\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$ (or appeal to complex analysis). We know $u = 0$ for $\theta = 0$ or $\theta = \pi$ $(r \neq 0)$, and $u = \sin\theta$ for $r = 1$. Therefore $u \leq 1$ on $\partial\Omega \setminus \{(0,0)\}$. But $u(r, \pi/2) = 1/r$, so as $r \to 0^+$, $u(r, \pi/2) \to \infty$

Proposition: The Boundary Value Problem 2013 10 07

$$\begin{cases} \Delta u = F & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \setminus \{p_1, \dots, p_n\} \\ u \text{ bounded in } \Omega \end{cases}$$

has at most one solution $u \in C^2(\Omega) \cap C(\overline{\Omega} \setminus \{p_1, p_2, \dots, p_n\})$.

Proof: If u_1, u_2 are solutions, let $v = u_1 - u_2$. Then

$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega \setminus \{p_1, \dots, p_n\}, \\ v \text{ bounded in } \Omega. \end{cases}$$

Using the extended maximum principle, $v \leq 0$ in Ω. The same argument applied to $-v$ gives $-v \leq 0$ in Ω. Thus $v \equiv 0 \Rightarrow u_1 = u_2$. ∎

Theorem (unbounded domains):
If $u \in C^2(\Omega) \cap C(\overline{\Omega})$, where Ω is an unbound (open) domain with boundary $\partial\Omega$, and

$$\Delta u \geq 0 \text{ in } \Omega \quad \text{(subharmonic)}$$
$$u \leq M \text{ on } \partial\Omega$$
$$u \text{ is bounded in } \Omega$$

then $u \leq M$ in Ω.

Corollary: The BVP $\begin{cases} \Delta u = F & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \\ u \text{ bounded in } \Omega \end{cases}$ has at most one solution, where Ω is unbounded.

ex If u is unbounded, consider $\Omega = \{(x,y);\ y>0\}$ and $u=y$.
Then

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has $u=y$ and $u=0$ as solutions.

ex Find the unique solution to

$$\begin{cases} \Delta u = 0 & y>0 \\ u(x,0) = 0 & x>0 \\ u(x,0) = 1 & x<0 \\ u \text{ bounded} & \end{cases}$$

We look for $u = u(\theta)$. Recall

$$\Delta u = u_{rr} + \tfrac{1}{r} u_r + \tfrac{1}{r^2} u_{\theta\theta} = 0 = \tfrac{1}{r^2} u_{\theta\theta}.$$

Hence $u_{\theta\theta} = 0$. Integrate twice to get

$$u = c_1\theta + c_2.$$

When $\theta = 0$, $u(r,0) = 0$ and so $c_2 = 0$.
When $\theta = \pi$, $u(r,\pi) = 1$ implies $c_1 = 1/\pi$.
Therefore $u = \theta/\pi$. Note $0 \le u \le 1$ so we have indeed found the (unique bounded) solution.

Green's Functions

The fundamental solution $K(\underline{x})$ for the laplace operator is a distribution that satisfies

$$\Delta K(\underline{x}) = \delta(\underline{x}). \quad \text{——} \quad (*)$$

We note that Δ is symmetric in the variables $x_1, \dots, x_n$ and so is $\delta(\underline{x})$ (in the sense that the value of $\delta(\underline{x})$ depends only on $r = |\underline{x}|$).
Since $\delta(\underline{x}) = 0$ for $\underline{x} \neq 0$, we require (1) K to be harmonic for $r>0$.
For radially symmetric functions, Laplace's equation simplifies to

$$\Psi'' + \frac{(n-1)}{r}\Psi' = 0 \quad \text{for } r>0 \quad \text{——} \quad (2)$$

The general solution (by separation of variables) is

$$\Psi(r) = \begin{cases} c_1 + c_2 \ln r & n=2 \\ c_1 + c_2 r^{2-n} & n \geq 3. \end{cases}$$