$\Delta = \nabla^2$

then introduce $\underline{v}$ such that $\underline{u} = P\underline{v}$. Now the original system becomes

$$\underline{v}_t + \Lambda \underline{v}_x = d(x,t,\underline{v})$$

where $d(x,t,\underline{v}) = P^{-1}(\underline{c}(x,t,\underline{u}) - P_t\underline{v} - AP_x\underline{v})$. Furthermore, the IC becomes

$$\underline{v}(x,0) - g(x) = P^{-1}f(x).$$

Well-posedness (1923) (Hadamard):
A problem is well-posed if:
(1) a solution exists;
(2) the solution is unique;
(3) the solution is stable wrt perturbations of the input data.

Consider the Dirichlet problem:
Find a function u such that

$$\begin{cases} \Delta u = 0 & \text{in } S = \{(x,y,z);\ x^2+y^2+z^2<1\} \\ u|_{\partial S} = 1 \end{cases}$$

Clearly any $u(x,y,z) = \begin{cases} \alpha x + \beta y + \gamma z & \text{in } S \\ 1 & \text{on } \partial S \end{cases}$

satisfies the BVP. However, if we require u to be continuous in $\bar{S} = S \cup \partial S$ then the only admissible solution is $u \equiv 1$

starting chapter 2 ...

Chapter 2: Elliptic Equations

Laplace's Equation: $\Delta u = 0$ in $\underline{\Omega} \subset \mathbb{R}^n$ — (1)
Poisson's Equation: $\Delta u = f$ in $\underline{\Omega} \subset \mathbb{R}^n$ — (2)

Solutions to (1) are known as harmonic functions. The Cauchy problems for (1) and (2) are not well-posed.

Dirichlet problem: $u = g$ on $\partial\underline{\Omega}$,
Neumann problem: $\partial u / \partial \underline{n} = h$ on $\partial\underline{\Omega}$ — as well as (1) or (2)
Robin problem: $\alpha u + \partial u / \partial \underline{n} = h$ on $\partial\underline{\Omega}$

(mixed)

constant

ex If we wish to solve

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

then we can write $u = u_1 + u_2$ where

$$\begin{cases} \Delta u_1 = f & \text{in } \Omega, \\ u_1 = 0 & \text{on } \partial\Omega, \end{cases}$$

$$\begin{cases} \Delta u_2 = 0 & \text{in } \Omega, \\ u_2 = g & \text{on } \partial\Omega. \end{cases}$$

$L[u]$

see next page for context of $L[u]$, $B[u]$

$B[u]$: BCs	Δu	$\Delta u - u$	$\Delta u + u$
$u = g$	yes	yes	no
$\partial u/\partial n = h$	yes up to additive constant	yes	no
$\partial u/\partial n + \alpha u = h$,			
$\hookrightarrow \alpha > 0$	yes	yes	no
$\hookrightarrow \alpha < 0$	no	no	no

Green's identities: 2013 09 30

Let $\Omega \subset \mathbb{R}^n$ be a smooth, bounded domain, and let $u, v \in C^2(\overline{\Omega})$. Then

$$\int_\Omega (v\Delta u + \nabla u \cdot \nabla v)\, d\underline{x} = \int_{\partial\Omega} v \frac{\partial u}{\partial n}\, d\underline{S}, \quad \text{(first identity)}$$

$$\int_\Omega (v\Delta u - u\Delta v)\, d\underline{x} = \int_{\partial\Omega} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) d\underline{S}. \quad \text{(second identity)}$$

note $\Delta = \nabla^2 = \nabla \cdot \nabla$

Note that the second follows from the first by switch u and v and subtracting. The first follows from the divergence theorem:

$$\int_\Omega \nabla \cdot \mathrm{v}\, d\underline{x} = \int_{\partial\Omega} \mathrm{v} \cdot \underline{n}\, d\underline{S}$$

v is $\mathbb{R}^n \to \mathbb{R}^n$!

by taking $\mathrm{v} = v\nabla u$.

Several useful consequences of these identities:

- setting $v = 1$ in the first yields

$$\int_\Omega \Delta u\, d\underline{x} = \int_{\partial\Omega} \frac{\partial u}{\partial \underline{n}}\, d\underline{S},$$

a solvability condition for the Neumann problem.

· setting $u=v$ in the first yields

$$(***) \quad — \quad \int_\Omega \left(u\Delta u + |\nabla u|^2\right) \underline{dx} = \int_{\partial\Omega} u \frac{\partial u}{\partial n} \underline{ds}$$

Using these identities, we can establish the following

Theorem:

(1) There exists at most one solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ of the Dirichlet problem

scope: this page

$$(*) \quad — \begin{cases} \Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

(2) Two solutions of the Neumann problem

$$(**) \quad — \begin{cases} \Delta u = f & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} = h & \text{on } \partial\Omega \end{cases}$$

must differ by a constant.

Proof: Suppose u_1, u_2 are two solutions of (*). Set $u = u_1 - u_2$. Then

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Substituting into (***) yields

$$\int_\Omega |\nabla u|^2 \underline{dx} = 0.$$

This implies $\nabla u = 0$, ie u is constant. Since $u=0$ on the boundary, we must have $u \equiv 0$ on $\overline{\Omega}$. Thus $u_1 = u_2$.

Part (2) is similar and is left as an exercise. ∎

see course text

Lemma: Consider

$$\begin{cases} L[u] = f & \text{in } \Omega, \\ B[u] = g & \text{on } \partial\Omega, \end{cases} \quad — \quad (1)$$

and associated homogeneous problem

$$\begin{cases} L[u] = 0 & \text{in } \Omega, \\ B[u] = 0 & \text{on } \partial\Omega. \end{cases} \quad — \quad (2)$$

Then the BVP (1) has at most one solution if and only if the BVP (2) has only the zero solution, and harmonic → satisfies laplace's eq.

a domain Ω is a open subset of $\mathbb{R}^n$ — path connected

BVP (1) has more than one solution if and only if BVP (2) has a non-zero solution.

Proof: Exercise.

here we start maximum principles

Maximum Principles and Mean Values:

Theorem (Mean Value Property):

Let $u \in C^2(\Omega)$ harmonic in a connected domain Ω, let $\xi \in \Omega$, and pick $r > 0$ such that $B(\xi; r) \subseteq \Omega$. Then

(means path connected)

$$(*) \qquad u(\xi) = M_u(\xi; r) := \frac{1}{A(\xi;1)} \int_{|x|=1} u(\xi + r x)\, dS.$$

surface area of ball of radius 1

average/mean value of u on surface of ba

Proof: Using Green's Identity with $v = 1$,

$$\int_{\Omega} \Delta u \, dx = \int_{\partial\Omega} \frac{\partial u}{\partial n} dS.$$

also works for volum

Since $\Delta u = 0$ in Ω, we have

$$0 = \int_{\partial B(\xi;r)} \frac{\partial u}{\partial n} dS = \int_{|x|=1} \frac{\partial u}{\partial r}(\xi + r x)\, dS$$

(change of variable)

$$\text{“} \int_{\Omega} \Delta u\, dx \text{”}$$

$$= A \cdot \frac{\partial}{\partial r} \frac{1}{A} \int_{|x|=1} u(\xi + r x)\, dS$$

$$= A \cdot \frac{\partial}{\partial r} M_u(\xi; r).$$

Hence

$$\frac{\partial}{\partial r} M_u(\xi; r) = 0$$

which means $M(\xi; r)$ is independent of r. By continuity

$$M_u(\xi; r) \xrightarrow[r \to 0]{} u(\xi).$$

As $M_u(\xi; r)$ is constant wrt r, we must have $M_u(\xi; r) = u(\xi)$.

Def | A continuous function u that satisfies $(*)$ is said to have the mean value property (MVP).

The theorem says that harmonic functions have the MVP. Is it true that for $u \in C^2(\Omega)$, where u has the MVP, u is harmonic?