now we turn to systems

## First Order Systems

From ODEs, recall that

$$y'' = f(x, y, y')$$

can be reduced to a first order system by setting $u_1 = y$, $u_2 = y'$:

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = \begin{pmatrix} u_2 \\ f(x, u_1, u_2) \end{pmatrix}.$$

ex Consider the wave equation

$$u_{xx} - u_{yy} = 0. \qquad \text{— } (*)$$

Let $u = (u_1, u_2)$ where $u_1 = u_x$, $u_2 = u_y$. Since $(u_1)_y = (u_2)_x$ and $(u_2)_y = u_{yy} = u_{xx} = (u_1)_x$. Therefore $(*)$ can be written as

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_y + \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

back to general

In order to define "characteristics" (i.e. base chars) for a first-order system, we consider the more general

$$A(x,y)\underline{u}_x + B(x,y)\underline{u}_y = \underline{c}(x,y,\underline{u})$$

where $\underline{u}$ and $\underline{c}$ are n-vectors, and $A$ and $B$ are $n \times n$-matrices. From experience, define a characteristic to be a curve $\gamma$ in $\mathbb{R}^2$ for which Cauchy data $\underline{u}|_\gamma$ does not determine $\underline{u}_x$ and $\underline{u}_y$ uniquely along $\gamma$.

(?) → ie if $\gamma : (x, \varphi(x))$ and if $\underline{u}(x, \varphi(x)) = \underline{f}(x)$ is the Cauchy data then

$$\begin{cases} (1)\ A\underline{u}_x + B\underline{u}_y = \underline{c} \\ (2)\ \underline{u}_x + \varphi'\underline{u}_y = \underline{f}' \end{cases}$$ ← diff. Cauchy data along $\gamma$

Now $(1) - A \times (2)$ gives

$$(B - \varphi' A)\underline{u}_y = \underline{c} - A\underline{f}'.$$

$\left(\frac{dy}{dx} = \varphi'(x)\right)$

definition of characteristic for systems

We will not be able to solve uniquely for $\underline{u}_y$ if

$$\det\left(B - \tfrac{dy}{dx}A\right) = 0. \quad \text{—— (characteristic equation)}$$

We define the principal part of the PDE system to be

$$A(x,y)\underline{u}_x + B(x,y)\underline{u}_y,$$

and the principal symbol to be the $n \times n$-matrix ← $(\xi = (\xi_1, \xi_2))$

$$\sigma(\xi) = \sigma(x,y;\xi) = A(x,y)\xi_1 + B(x,y)\xi_2$$

Hence $\gamma$ is characteristic at $(x,y)$ if and only if the principal symbol matrix is singular on the vector $\underline{\xi}$ normal to $\gamma$. 2013 09 27

Consider an IVP where $\gamma$ is the curve $t=0$. In this case the normal vector $\xi$ is $(0, \xi_2)$ and $\gamma$ is non-characteristic provided

$$\det(B(x,0)) \neq 0 \quad \forall x.$$

By continuity, this ~~gor~~ guarantees that

$$\det(B(x,t)) \neq 0$$

for $t$ sufficiently small. Therefore $B$ is invertible (for small $t$).
Multiplying through by $B^{-1}(x,t)$:

we will write $A$ for $\tilde{A}$

$$\underline{u}_t + \tilde{A}(x,t)\underline{u}_x = \tilde{C}(x,t,\underline{u}) \qquad (*)$$

near $t=0$, and the characteristic equation takes simpler form:

$$\det\left(I\frac{dx}{dt} - A\right) = 0.$$

So for $\gamma: (x(t),t)$ to be a characteristic, we should have

$$\frac{dx}{dt} = \lambda(x,t)$$

where $\lambda(x,t)$ is an eigenvalue of $A(x,t)$.
Thus we define for a first order $n\times n$ system of the form $(*)$ to be:

- <u>strictly hyperbolic</u> if the $n\times n$ matrix has $n$ real distinct eigenvalues (in this case, the associated eigenvectors are linearly independent);
- <u>hyperbolic</u> if $A$ has $n$ real eigenvalues and a basis of eigenvectors;
- <u>parabolic</u> if $A$ has $n$ real eigenvalues but less than $n$ linearly independent eigenvectors;
- <u>elliptic</u> if $A$ has no real ~~eigenvecl~~ eigenvalues.

For a hyperbolic system, we can use eigenvalues and eigenvectors to diagonalize $A$ to simplify the LHS of $(*)$. If

$$P = (\underline{\mu}_1, \ldots, \underline{\mu}_n) \qquad \leftarrow \text{(eigenvectors)}$$

and

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} =: \Lambda$$

$\Delta = \nabla^2$



then introduce $\underline{v}$ such that $\underline{u} = P\underline{v}$. Now the original system becomes

$$\underline{v}_t + \Lambda \underline{v}_x = \underline{d}(x,t,\underline{v})$$

where $\underline{d}(x,t,\underline{v}) = P^{-1}(\underline{c}(x,t,\underline{u}) - P_t\underline{v} - AP_x\underline{v})$. Furthermore, the IC becomes

$$\underline{v}(x,0) - \underline{g}(x) = P^{-1}\underline{f}(x).$$

Well-posedness (1923) (Hadamard):

A problem is well-posed if:

(1) a solution exists;

(2) the solution is unique;

(3) the solution is stable wrt perturbations of the input data.

Consider the Dirichlet problem:

Find a function $u$ such that

$$\begin{cases} \Delta u = 0 & \text{in } S = \{(x,y,z);\ x^2+y^2+z^2<1\} \\ u|_{\partial S} = 1 \end{cases}$$

Clearly any $u(x,y,z) = \begin{cases} \alpha x + \beta y + \gamma z & \text{in } S \\ 1 & \text{on } \partial S \end{cases}$

satisfies the BVP. However, if we require $u$ to be continuous in $\bar{S} = S \cup \partial S$ then the only admissible solution is $u \equiv 1$

starting chapter 2 [illegible]

## Chapter 2: Elliptic Equations

Laplace's Equation: $\Delta u = 0$ in $\Omega \subset \mathbb{R}^n$ — (1)

Poisson's Equation: $\Delta u = f$ in $\Omega \subset \mathbb{R}^n$ — (2)

Solutions to (1) are known as harmonic functions. The Cauchy problems for (1) and (2) are not well-posed.

Dirichlet problem: $u = g$ on $\partial\Omega$,

Neumann problem: $\partial u / \partial \underline{n} = h$ on $\partial\Omega$

Robin problem: $\alpha u + \partial u / \partial \underline{n} = h$, on $\partial\Omega$ } as well as (1) or (2)

constant (mixed)