we classify all forms of (*)

Classification of Second Order PDEs

Consider

(*) $a(x,y)u_{xx} + b(x,y)u_{xy} + c(x,y)u_{yy} = d(x,y,u,u_x,u_y)$.

$\gamma : (f(s),g(s))$

Given γ in the xy-plane, Cauchy data along γ is

$$u|_\gamma = h, \quad \frac{\partial u}{\partial \underline{n}}\Big|_\gamma = h_1. \qquad (\underline{n} \text{ is unit normal to } \gamma)$$

so from here, we just talk about ICs (ie Cauchy data)

If $\underline{\tau}$ is the unit tangent along γ then we can find $\partial u/\partial \underline{\tau}$ along γ by differentiating h. Thus we can find any directional derivative along γ by taking a linear combination of $\partial u/\partial \underline{n}$ and $\partial u/\partial \underline{\tau}$.

Thus the Cauchy data can also be expressed as

$$u|_\gamma = h, \quad \frac{\partial u}{\partial x}\Big|_\gamma = \varphi, \quad \frac{\partial u}{\partial y}\Big|_\gamma = \psi,$$

since $u(f(s),g(s)) = h(s)$ and diff.

provided $h'(s) = \varphi(s)f'(s) + \psi(s)g'(s)$, which we call compatibility condition. Note that for any function $v = v(x,y)$ we have by chain rule:

$$dv = v_x dx + v_y dy. \quad (*)$$

(chain rule) ~~$\frac{dv}{ds}$~~ $v(x(s),y(s))$

Substituting $v = u_x$ into (*), we have

$$\varphi' = u_{xx}f' + u_{xy}g'.$$

scope, all other (*)'s refer to PDE at the top

Substituting $v = u_y$ into (*), we have

$$\psi' = u_{xy}f' + u_{yy}g'.$$

Thus we have a linear system of 3 equations and 3 unknowns:

$$\begin{cases} f'u_{xx} + g'u_{xy} = \varphi', \\ f'u_{xy} + g'u_{yy} = \psi', \\ au_{xx} + bu_{xy} + cu_{yy} = d. \end{cases}$$

This can be solved uniquely for u_{xx}, u_{xy}, u_{yy} provided

$$\det \begin{bmatrix} f' & g' & 0 \\ 0 & f' & g' \\ a & b & c \end{bmatrix} \neq 0, \text{ or} \qquad (\star)$$

$$a(g')^2 - b(f')(g') + c(f')^2 \neq 0. \qquad (\star)$$

definition of non-characteristic (see top) for general (*)

This is called the non-characteristic condition.

Therefore a curve γ is characteristic if $(\star) = 0$.

~~for general [illegible]~~ for $\gamma = \{(f(s),g(s))\} \subseteq \mathbb{R}^2$

<u>Def</u> The <u>principal symbol</u> associated with the principal part of the PDE

$(*)$ $\xrightarrow{\text{principal part}}$ $a(x,y)u_{xx} + b(x,y)u_{xy} + c(x,y)u_{yy} = d(x,y,u,u_x,u_y)$

is defined to be $\quad \xi = (\xi_1, \xi_2)$

$$\sigma(\xi) = \sigma(x,y;\xi) = a(x,y)\xi_1^2 + b(x,y)\xi_1\xi_2 + c(x,y)\xi_2^2,$$

where $\xi = (\xi_1, \xi_2)$ is a vector defined at (x,y).

recall $\gamma: (f(s), g(s))$

Note that γ has tangent vector (f', g'). So if we take $(g', -f')$, then this is normal to γ. Therefore γ is characteristic for $(*)$ at (x,y) if and only if the principal symbol vanishes on its normal vector (ie $\sigma(x,y;(g',-f')) = 0$).

Writing the characteristic ~~equation~~ condition as $a\,dy^2 - b\,dx\,dy + c\,dx^2 = 0$ and solving for dy/dx, we obtain

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$$

equation for characteristic curve γ

Here we have three cases:

Def)

(1) $b^2 - 4ac > 0$: there are two characteristics, and the PDE $(*)$ is called <u>hyperbolic</u>;

(2) $b^2 - 4ac = 0$: There is one characteristic, and the PDE $(*)$ is called <u>parabolic</u>;

(3) $b^2 - 4ac < 0$: there are no characteristics, and the PDE $(*)$ is called <u>elliptic</u>.

ex. The 1-D wave equation

$$u_{xx} - u_{yy} = 0$$

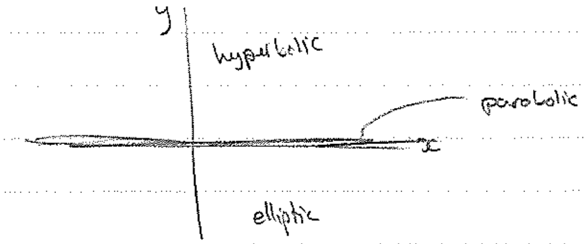
has $a=1$, $b=0$, $c=-1$, and so $b^2 - 4ac = 4 > 0$. Thus the equation is hyperbolic.

ex. The 1-D heat equation

$$u_{xx} - u_t = 0$$

has $a=1$, $b=0$, $c=0$, and so $b^2 - 4ac = 0$. Thus the equation is parabolic.

ex $u_{yy} - y u_{xx} = 0$ has $a = -y$, $b = 0$, $c = 1$. So $b^2 - 4ac = 4y$. Thus the equation is elliptic if $y < 0$, and hyperbolic if $y > 0$, and parabolic if $y = 0$.



For $y > 0$ we have

$$\frac{dy}{dx} = \pm \frac{1}{\sqrt{y}}$$

so the characteristic curves are $3x \pm 2y^{3/2} = \text{const}$.

By an appropriate change of coordinates

$$(x,y) \mapsto (\xi, \eta),$$

the equation

(**) $$a u_{xx} + b u_{xy} + c u_{yy} = d(x, y, u, u_x, u_y)$$

can be transformed so that its principal part has form

$$u_{\xi\xi} - u_{\eta\eta} + \text{lower order}, \text{ or}$$ hyperbolic

$$u_{\xi\xi} + u_{\eta\eta} + \text{lower order}, \text{ or}$$ parabolic

$$u_{\xi\xi} + \text{lower order}.$$ elliptic

why?

If (**) is hyperbolic, then characteristics are given by

$$\xi(x,y) = \text{const.}, \quad \eta(x,y) = \text{const.}$$

where ξ, η define non-degenerate coordinates,

$$\det \begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix} \neq 0.$$

reducing to canonical form steps: (for hyperbolic)

Recall that if 2013 09 25

$$a u_{xx} + b u_{xy} + c u_{yy} = d$$
$$u|_\gamma = g$$
$$u_x|_\gamma = f$$

is hyperbolic, then the characteristic curves are given by

$$\xi(x,y) = \text{const.}, \quad \eta(x,y) = \text{const},$$

where ξ, η are non-degenerate coordinates, ie

$$\det \begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix} \neq 0.$$

That is to say, $\nabla\xi = (\xi_x, \xi_y)$ and $\nabla\eta = (\eta_x, \eta_y)$ are <u>not</u> co-linear. Since the curves ξ, η are characteristics, the principal symbol vanishes on the normals to the curves.

$$(1) - \begin{cases} a(\xi_x)^2 + b(\xi_x)(\xi_y) + c(\xi_y)^2 = 0 \\ a(\eta_x)^2 + b(\eta_x)(\eta_y) + c(\eta_y)^2 = 0 \end{cases}$$

on that $\Downarrow$

By chain rule, (check!)

$$a u_{xx} + b u_{xy} + c u_{yy} = A u_{\xi\eta} + \text{lower order}$$

where

$$A = 2a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + 2c\xi_y\eta_y$$

It is easily verified that

$$A^2 = \underbrace{(b^2 - 4ac)}_{>0}\underbrace{(\xi_x\eta_y - \xi_y\eta_x)^2}_{\neq 0 \text{ as non-degenerate}}.$$

Thus we can divide through by A to get

$$(2) \text{ — } u_{\xi\eta} = \tilde{d}(\xi, \eta, u, u_\xi, u_\eta).$$

Finally, substituting $\bar{x} = \xi + \eta$, $\bar{y} = \xi - \eta$ into (2) yields

$$u_{\bar{x}\bar{x}} - u_{\bar{y}\bar{y}} = d(\bar{x}, \bar{y}, u, u_{\bar{x}}, u_{\bar{y}}).$$

If the PDE is parabolic or elliptic, reduction to canonical form is a bit trickier. - see A2C05

ex Consider

$$x u_{xx} + 2x^2 u_{xy} = u_x - 1.$$

Here we have $a = x$, $b = 2x^2$, $c = 0$, and so $b^2 - 4ac = 4x^2 > 0$ if $x \neq 0$. Thus the PDE is hyperbolic if $x \neq 0$

The characteristic curves are given by

$$\frac{dy}{dx} = \frac{2x^2 \pm \sqrt{4x^4}}{2x} = 2x \text{ or } 0.$$

Integrating gives

$$y = x^2 + C \quad \text{or} \quad y = C_1.$$

Let $\xi(x,y) = x^2 - y$ and let $\eta(x,y) = y$. Then

$$\xi_x = 2x, \quad \xi_y = -1, \quad \eta_x = 0, \quad \eta_y = 1.$$

Scrap

$u_\xi = u_{\bar x}\xi_{\bar x} + u_{\bar y}\xi_{\bar y}$

$= u_{\bar x}$

Therefore

$$u_x = u_\xi \cdot 2x \implies u_{xx} = 2u_\xi + 4x^2 u_{\xi\xi}$$
$$u_{xy} = -2x u_{\xi\xi} + 2x u_{\xi\eta}.$$

Substituting into our PDE yields

$$4x^3 u_{\xi\eta} = -1, \quad \text{or}$$

$$u_{\xi\eta} = \frac{-1}{4(\xi+\eta)^{3/2}}.$$

Integrate wrt η to get

arbitrary function

$$u_\xi = \frac{1}{2}(\xi+\eta)^{-1/2} + f(\xi)$$

and then wrt ξ to get

arbitrary

$$u = (\xi+\eta)^{1/2} + F(\xi) + G(\eta).$$

Now we can return to x,y and get

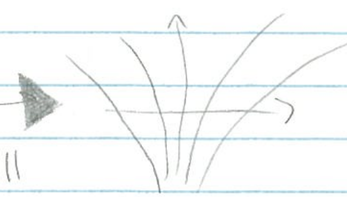
$$u(x,y) = x + F(x^2-y) + G(y).$$

ex Consider

$$x^2 u_{xx} + 2x u_{xy} + u_{yy} = u_y.$$

We have $a = x^2$, $b = 2x$, $c = 1$. So $b^2 - 4ac = 0$, and so the PDE is parabolic. The characteristics are given by

$$\frac{dy}{dx} = \frac{1}{x}.$$



Thus $y = \ln|x| + c$, ie $x = C^* e^y$ (or $C^* = xe^{-y}$).

Let $\xi(x,y) = xe^{-y}$ and $\eta(x,y) = y$ (we can see $y = \text{const.}$ will always be transverse to $xe^{-y} = \text{const.}$ in the picture). Using these coordinates and substituting for u_{xx}, u_{xy}, u_{yy}, u_y we get

$$u_{\eta\eta} = u_\eta \implies u_\eta - u = F(\xi).$$

Multiply through by an integrating factor to obtain (say $e^{-\eta}$) (check!)

$$u = -F(\xi) + G(\xi)e^\eta.$$

arbitrary functions