So $\underbrace{x^2+y^2}_{g} = c_1$ and $\underbrace{\tfrac{1}{2}u^2 + x+y}_{f} = c_2$.

So $\tfrac{1}{2}u^2+x+y = F(x^2+y^2) \Rightarrow u^2 = -2x-2y+F^*(x^2+y^2)$

Generalization to n-independent variables

$$\sum_{i=1}^{n} a_i(x_1, \ldots, x_n, u)\frac{\partial u}{\partial x_i} = c(x_1, \ldots, x_n, u)$$

The characteristic curves are (the integral curves) given by the system of n+1 ODEs in n+1 unknowns:

$$\frac{dx_i}{dt} = a_i(x_1, \ldots, x_n, u) \qquad i=1, \ldots, n$$

$$\frac{du}{dt} = c(x_1, \ldots, x_n, u)$$

This can be solved if we are given initial conditions on an (n-1)-dimensional manifold:

$$x_i = f_i(s_1, \ldots, s_{n-1}) \qquad i=1, \ldots, n$$

$$u = h(s_1, \ldots, s_{n-1})$$

This will generate an n-dimensional integral manifold M parameterized by $(s_1, \ldots, s_{n-1}, t)$. The solution $u(x_1, \ldots, x_n)$ is obtained by solving for $(s_1, \ldots, s_{n-1}, t)$ in terms of $(x_1, \ldots, x_n)$

Semi-linear DEs

$$\left(\sum_{|\alpha|=k} a_\alpha(\underline{x}) D^\alpha \{u\}\right) + a_0(x, u, Du, \ldots, D^{k-1}u) = 0$$

$\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$

$|\alpha| = \alpha_1 + \cdots + \alpha_n$

$\underline{x} = (x_1, \ldots, x_n)$

$\underline{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$

$\alpha! = (\alpha_1!)(\cdot)\cdots(\ )(\alpha_n!)$

$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}(\quad)\cdots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$ (composition) ex Consider (semi-linear)

$$a(x,y)u_x + b(x,y)u_y = c(x,y,u)$$

with

$$\Gamma: (f(s), g(s), h(s))$$

The characteristic equations are

$$\frac{dx}{dt} = a(x,y), \quad \frac{dy}{dt} = b(x,y), \quad \frac{du}{dt} = c(x,y). \qquad (\star)$$

The ICs are

$$x(s,0) = f(s), \quad y(s,0) = g(s), \quad u(s,0) = h(s).$$

The first two of ($\star$) form a system decoupled from u. Solve it to obtain $(x(t), y(t))$, the projected/base characteristic curves. Having obtained these, we can (in principle) integrate the last ODE to find u.

When can we solve for s and t in terms of x and y? The inverse function theorem says that this can be achieved provided the Jacobian 2013 09 16

$$\det\begin{bmatrix} x_s & y_s \\ x_t & y_t \end{bmatrix} \neq 0. \qquad (*)$$

Notice that this condition is independent of the behaviour of u. In particular, when $t=0$ and we are on Γ we get from $(*)$:

$$f'(s)\, b(f(s), g(s)) - g'(s)\, a(f(s), g(s)) \neq 0. \qquad (**)$$

Note LHS is

$$\underbrace{(f'(s), g'(s))}_{\text{normal}} \cdot (a(f(s), g(s)), b(f(s), g(s))).$$

Geometrically, this ensures that the projection of Γ in the xy-plane is a curve γ which is nowhere parallel to the vector field (a,b). (This is certainly true if Γ is non-characteristic.)

Note $(**)$ implies that $(*)$ holds at least for small t, by continuity.

ex Solve the IVP:

$$\begin{cases} u_x + 2u_y = u^2 \\ u(x,0) = h(x) \end{cases}$$

implies

$\Gamma: (s, 0, h(s))$

We find

$$\det\begin{bmatrix} x_s & y_s \\ x_t & y_t \end{bmatrix} = \det\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} = 2 \neq 0.$$

Characteristic equations are

$$\frac{dx}{dt} = 1 \qquad \frac{dy}{dt} = 2 \qquad \frac{du}{dt} = u^2.$$

Solving yields

$$x(s,t) = t + c_1(s), \quad y(s,t) = 2t + c_2(s)$$

$$h(s) = x(s,0) = c_1(s) \qquad y(s,0) = 0 = c_2(s) ???$$

$$\Rightarrow \quad x = t + s, \quad y = 2t$$

$$\Rightarrow \quad s = x - y/2, \quad t = y/2.$$

Integrating the last ODE yields

$$-\frac{1}{u} = t + c_3(s)$$

$$\Rightarrow \quad u = \frac{-1}{t + c_3(s)}.$$

The IC yields

$$h(s) = u(s,0) \Rightarrow c_3(s) = -1/h(s).$$

And so

$$u = \frac{h(s)}{1 - t\,h(s)}.$$

Thus, substituting for s and t,

$$u(x,y) = \frac{h(x - y/2)}{1 - y/2\, h(x - y/2)}.$$

In general, would get (extension to semi-linear equations in n variables)

$$x_i(s_1, \ldots, s_{n-1}, t) \qquad i = 1, \ldots, n$$

$$u(s_1, \ldots, s_{n-1}, t)$$

which can be solved to get $u(x_1, \ldots, x_n)$ provided

$$\det\left[\frac{\partial x_i}{\partial s_j}\right] \neq 0.$$

↳ or $\frac{\partial x_i}{\partial t}$ when $i = n$

ex (n=3) Solve

$$\begin{cases} u_x + x u_y - u_z = u \\ u(x,y,1) = x+y \end{cases}$$

Solution: Replace variables

$$\begin{cases} u_{x_1} + x_1 u_{x_2} - u_{x_3} = u \\ u(x_1,x_2,1) = x_1 + x_2 \end{cases}$$

The characteristic equations are

$$\frac{dx_1}{dt} = 1, \quad \frac{dx_2}{dt} = x_1, \quad \frac{dx_3}{dt} = -1, \quad \frac{du}{dt} = u.$$

The initial surface Γ is the hyperplane $x_3 = 1$, $u = x_1 + x_2$. $\{(x_1, x_2, 1, x_1+x_2) ; x_1, x_2 \in \mathbb{R}\}$

Note Γ is noncharacteristic since any curve on Γ must have $dx_3/dt = 0$ (yet $dx_3/dt = -1$ for characteristic equations).

Parameterize Γ by

$$x_1 = s_1, \quad x_2 = s_2, \quad x_3 = 1, \quad u = s_1 + s_2$$

Solving shows (we use IC, I think)

$$x_1 = t + s_1, \quad x_2 = \tfrac{1}{2}t^2 + s_1 t + s_2, \quad x_3 = -t + 1, \quad u = (s_1+s_2)e^t$$

It remains to solve for s_1, s_2, t in terms of x_1, x_2, x_3. Doing so lets us obtain

$$u(x_1,x_2,x_3) = \left(x_1 + x_2 + (x_3-1)\left(1 + x_1 + \tfrac{1}{2}(x_3-1)\right)\right)e^{1-x_3}.$$

(check - this expression may be wrong)

now we look at general case

$$a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} = c(x,y,u) \quad (*)$$

$$\Gamma: (f(s), g(s), h(s))$$

We solve

$$\frac{dx}{dt} = a(x,y,u), \quad \frac{dy}{dt} = b(x,y,u), \quad \frac{du}{dt} = c(x,y,u),$$

with

$$x(s,0) = f(s), \quad y(s,0) = g(s), \quad u(s,0) = h(s).$$

To be able to solve uniquely (locally) for s,t in terms of x,y, we require

$$\det \begin{bmatrix} x_s & y_s \\ x_t & y_t \end{bmatrix} \neq 0.$$

In particular, at $t=0$, we require

$$f'(s)\,b(f(s),g(s),h(s)) - g'(s)\,a(f(s),g(s),h(s)) \neq 0.$$

ex. Solve the Cauchy Problem

$$\begin{cases} uu_x + yu_y = x, \\ u(x,1) = 2x \end{cases}$$

Solution:
$$\frac{dx}{dt} = u, \quad \frac{dy}{dt} = y, \quad \frac{du}{dt} = x$$

$$\Gamma: (s, 1, 2s)$$

Check the ~~non-characteristic condition~~:

we are checking if we can solve for x,y in terms of s,t, we are not checking the non-characteristic condition

This was right →
$$\det\begin{bmatrix} x_s & y_s \\ x_t & y_t \end{bmatrix} = x_s y_t - y_s x_t = 1\cdot 1 - 0\cdot 2s = 1 \neq 0 \checkmark$$

Now integrating:
$$y = C(s)e^t \overset{IC}{\Longrightarrow} C(s) = 1, \quad y = e^t \qquad (*)$$

Re-examining the ODEs of x and u, we observe that they can be written as

$$\frac{d}{dt}(x+u) = x+u, \quad \frac{d}{dt}(x-u) = -(x-u),$$

and so

$$x+u = C_1(s)e^t, \quad x-u = C_2(s)e^{-t}. \qquad (**)$$

Using ICs gives

$$C_1(s) = 3s, \quad C_2(s) = -s.$$

Solving for x and u yields

$$x = \tfrac{3}{2}se^t - \tfrac{1}{2}se^{-t}, \quad u = \tfrac{3}{2}se^t + \tfrac{1}{2}se^{-t}$$

As $y = e^t$ $(*)$, we have $\frac{1}{y} = e^{-t}$. By $(**)$ we also find $s = \frac{1}{3}\frac{x+u}{y}$. 2013 09 18

Substituting into u yields

$$u = \frac{3}{2}\cdot\frac{1}{3}\frac{x+u}{y}\cdot y + \frac{1}{2}\cdot\frac{1}{3}\frac{x+u}{y}\cdot\frac{1}{y}$$

$$\Rightarrow u(x,y) = \frac{x(3y^2+1)}{3y^2-1}$$

which is defined except for $y = 1/\sqrt{3}$.

now we turn to semilinear

Let γ be a smooth curve in the xy-plane, $\gamma:(f(s),g(s))$, and let $h(s)$ be a smooth function defined along γ. Consider the Cauchy problem

$$(*)\quad \begin{cases} a(x,y)u_x + b(x,y)u_y = c(x,y,u), \\ u(f(s),g(s)) = h(s). \end{cases}$$

We now think of the base curve $\gamma:(f(s),g(s))$ as fixed, and the Cauchy data $u|_\gamma = h$ as a condition to be satisfied on γ.

$\gamma = \{(f(s),g(s)); s \in \mathbb{R}\}$

The Cauchy problem $(*)$ has a locally unique solution provided

$$(**)\quad \det\begin{bmatrix} f'(s) & g'(s) \\ a(f(s),g(s)) & b(f(s),g(s)) \end{bmatrix} \neq 0$$

(ie When the system $f'(s)u_x + g'(s)u_y = h'(s)$, $au_x + bu_y = c$ can be solved.)

definition of non-characteristic for $(*)$ — the curve in $\mathbb{R}^2$

We say γ is non-characteristic for $(*)$ if $(**)$ holds. In this case the Cauchy problem is well-posed.

Note: If γ is non-characteristic, then the Cauchy data $u|_\gamma = h$ allows us to determine $u_x|_\gamma$ and $u_y|_\gamma$.

~~We say γ is non-characteristic if $(*)$ and $(**)$ hold.~~ ← what is this sentence even?

$$\begin{cases} a(\gamma(s))\,u_x(\gamma(s)) + b(\gamma(s))\,u_y(\gamma(s)) = c(\gamma(s),h(s)) \\ f'(s)\,u_x(\gamma(s)) + g'(s)\,u_y(\gamma(s)) = h'(s) \end{cases}$$

$$\det\begin{bmatrix} a(\gamma(s)) & b(\gamma(s)) \\ f'(s) & g'(s) \end{bmatrix} \neq 0$$

$\Longrightarrow$ unique solutions for u_x and u_y

In fact, we can solve for all partial derivatives of u along γ (using the PDE and the initial condition). (check - exercise)

ex. $u_x + xu_y = u^2$

The base characteristics are given by

$$\frac{dx}{1} = \frac{dy}{x} \Longrightarrow y = \tfrac{1}{2}x^2 + c$$

Provided γ is nowhere tangent to the parabolas then the problem is well-posed and admits a unique solution for arbitrary smooth h, $(u|_\gamma = h)$.

Thus Cauchy Problem is well-posed for Cauchy data h perscribed on the y-axis:

$$u|_\gamma = u|_{x=0} = h.$$

We verify that all derivatives of u can be determined along the y-axis: We can determine $u_x|_{x=0}$ and $u_y|_{x=0}$ by solving

$$\begin{cases} au_x + bu_y = c \\ f'u_x + g'u_y = h' \end{cases},$$

To determine $u_{xx}|_{x=0}$, we differentiate the PDE wrt x:

(*) $$u_{xx} + u_y + xu_{xy} = 2uu_x.$$

We can determine $u_{xy}|_{x=0}$ and $u_{yy}|_{x=0}$ by simply differentiating $u_x|_{x=0}$ and $u_y|_{x=0}$ along the y-axis. Then from (*),

$$u_{xx} = \underbrace{2uu_x - u_y - xu_{xy}}_{\text{all known}}.$$

(still doing ex from front)

Now suppose γ is characteristic, say $\gamma: y = \frac{1}{2}x^2$. Then for Γ to be characteristic, we require

$$\frac{du}{u^2} = \frac{dx}{1} \Longrightarrow -\frac{1}{u} = x + c$$

The constant c is determined by picking a point over γ through which Γ passes.

e.g. if Γ passes through $(0,0,z_0)$ then $c = -1/z_0$ and Γ is given by $u = \frac{z_0}{1 - z_0 x}$

We can find the infinite family of solutions to this Cauchy problem by taking two independent first integrals:

$$\varphi(x,y,z) = y - \tfrac{1}{2}x^2 \quad , \quad \psi(x,y,z) = x + 1/z$$

Now the general solution can be written as $\varphi = f(\psi)$

i.e. $y - \frac{1}{2}x^2 = f(x + 1/z)$.

Along Γ we have $y - \frac{1}{2}x^2 = 0$ and $x + \frac{1}{z} = z_0$ (we are in the e.g.)

So the solution surface will pass through Γ, provided $f(1/z) = 0$.

With this restriction, there exists an infinite number of solutions.

So for a general m^{th} order PDE in $\mathbb{R}^n$,

$$(*) \qquad F(\underline{x}, D^\alpha u) = 0,$$

where $|\alpha| \le m$.

What is the Cauchy problem?

For first order PDEs, we considered $\gamma \subset \mathbb{R}^2$ and Cauchy data $u|_\gamma$ and found that this determines all derivatives of u along γ when γ is non-characteristic.

(cont. from above) When $n \geq 3$, we have to replace the curve γ by an initial hyper-surface $S \subseteq \mathbb{R}^n$, but if $m > 1$ what should we use for Cauchy data? 2013 09 20

What is non-characteristic S?

<u>Normal form</u>

If we consider the Cauchy problem as an IVP, then experience from ODEs suggests that "Cauchy data" for $(*)$ should consist of values of

$$u, \frac{\partial u}{\partial n}, \ldots, \frac{\partial^{m-1} u}{\partial n^{m-1}}$$

(where n is the unit normal on S)

on the hypersurface S.

(definition of non-characteristic)

Def The hypersurface S is non-characteristic for $(*)$ if the values of $u, \frac{\partial u}{\partial n}, \ldots, \frac{\partial^{m-1} u}{\partial n^{m-1}}$ on S determine all derivatives of u on S.

(S is in $\mathbb{R}^{n-1}$ ($\sim \mathbb{R}^2$))

This will be true if

$$F(\underline{x}, D^\alpha u) = 0, \quad |\alpha| \le m$$

can be expressed as

$$\frac{\partial^m u}{\partial n^m} = G(\underline{x}, D^\alpha u), \quad |\alpha| < m.$$

Most easily verified by a change of variables

$$\underline{x} \mapsto \tilde{\underline{x}}$$

such that

$$S \mapsto \{\text{hyperplane with } x_n = 0\}. \qquad \text{ie } S \cong \mathbb{R}^{n-1}$$

$$\underbrace{\{x \in \mathbb{R}^n ; x(n) = 0\}}_{\text{hyperplane}}$$

The PDE can be transformed to the normal form:

$$(1)\qquad \frac{\partial^m u}{\partial x_n^m} = G(\underline{x}, D^\alpha u), \quad |\alpha| \le m, \ \alpha \ne (0, -, 0, m),$$

with Cauchy data on $\{x \in \mathbb{R}^n, x_n = 0\} \cong \mathbb{R}^{n-1}$:

$$(2)\qquad u\big|_{x_n=0} = g_1, \ \ldots, \ \frac{\partial^{m-1} u}{\partial x_n^{m-1}} = g_m.$$

Clear "Cauchy" data determines all derivatives of u on S. To find $D^\alpha u$:

- if $\alpha_n = 0$, just differentiate g_1 on S;
- if $\alpha_n = 1$, just differentiate g_2 on S;

  ⋮

- if $\alpha_n = m-1$, " g_m on S;
- if $\alpha_n = m$, we use (2) and all of the above and substitute into the RHS of (1)
- if $\alpha_n > m$, first differentiate (1) wrt x_n and the proceed as above.

ex Consider the Cauchy problem

$$\begin{cases} u_{xx} + u u_{yy} - u_y = u^2, \\ u(x,0) = 1, \\ u_y(x,0) = x. \end{cases}$$

In normal form, the DE becomes

$$(*)\qquad u_{yy} = u + \frac{u_y - u_{xx}}{u}.$$

Assuming u is C^∞, we determine all derivatives of u on S, ie $y=0$. Using $u(x,0)=1$ we get $u_x(x,0)=0$, $u_{xx}(x,0)=0, \ldots$. Now use $u_y(x,0)=x$ to get $u_{yx}(x,0) = 1 = u_{xy}(x,0)$, and $u_{yxx}(x,0) = u_{yxxx}(x,0) = \cdots = 0$. To determine u_{yy} on S use $(*)$ to get

$$u_{yy}(x,0) = u(x,0) + \frac{u_y(x,0) - u_{xx}(x,0)}{u(x,0)} = 1 + x.$$

Further we get $u_{yyx}(x,0) = 1$ and so on...

Differentiating the normal form wrt y we obtain

$$u_{yyy} = u_y + \frac{u_{yy} - u_{xxy}}{u} - \frac{(u_y - u_{xx})}{u^2} u_y$$

and so along S we find $u_{yyy}(x,0) = ~~1+x-x^2~~$ $1+2x-x^2$ (check). We can proceed in this manner to obtain all derivatives of u on S.

different version in course notes

Theorem (Cauchy-Kowalevski): If g_j $(j \in \{1, \dots, m\})$ are real analytic functions in a neighbourhood of $\underline{0} \in \mathbb{R}^{n-1}$ and G is real analytic in a neighbourhood of $(\underline{0}, D^\alpha u)(0))$, then there exists a unique real analytic solution u of

$$\begin{cases} \dfrac{\partial^m u}{\partial x_n^m} = G(\underline{x}, D^\alpha u) & \text{where } |\alpha| \le m \text{ and } \alpha \ne (0,\dots,0,m) \\ u\big|_{x_n=0} = g_1, \dots, \dfrac{\partial^{m-1} u}{\partial x_n^{m-1}}\Big|_{x_n=0} = g_m \end{cases}$$

in a neighbourhood of $\underline{0} \in \mathbb{R}^n$.

Remark(s):

i) This requires the Cauchy data to be analytic (ie doesn't recognize other well-posed Cauchy problems (eg if $g_k \xrightarrow{\text{uniformly}} f$ with g_k's analytic and f continuous, then we still cannot conclude that the solutions u_k for data g_k will converge to a solution for data f)).

ii) This asserts uniqueness of a real analytic solution but does not preclude existence of non-analytic solutions.

iii) Not only does this require real analytic Cauchy data, but it also requires G to be real analytic. 2013 09 23

ex a linear DE

$$-u_x - iu_y + 2i(x+iy)u_z = F(x,y,z),$$

for $F \in C^\infty(\mathbb{R}^3)$, which admits no solutions in any open set, regardless of choice of Cauchy data. (see F. John, Ch8)