Parabolic equations!

Parabolic Equations and Transform Methods

Consider $\Omega \subseteq \mathbb{R}^n$ with constant heat conductivity k:

$$u_t = k \Delta u \quad \text{for } \underline{x} \in \Omega,\ t>0$$

governs diffusion (propagation) of heat.
Rescale $t \mapsto kt$.

I.C.: $u(\underline{x},0) = g(\underline{x})$
and some sort of BC.

Existence

$$\begin{cases} u_t = \Delta u, & \underline{x} \in \Omega,\ t>0, \\ u(\underline{x},0) = g(\underline{x}) & \underline{x} \in \Omega \\ u(\underline{x},t) = 0 & \underline{x} \in \partial\Omega \end{cases}$$

We assume $g \in C^2(\overline{\Omega})$ and $g=0$ on $\partial\Omega$.
Then we can write

$$g(\underline{x}) = \sum_{n=1}^{\infty} a_n \varphi_n(\underline{x})$$

where

$$a_n = \int_\Omega g(\underline{x})\, \varphi_n(\underline{x})\, d\underline{x},$$

and λ_n, φ_n are the eigenvalues, eigenvectors of the Laplacian on Ω with respect to Dirichlet BCs.
Write

$$u(\underline{x},t) = \sum_{n=1}^{\infty} u_n(t)\, \varphi_n(\underline{x}).$$

The coeffs are given by

$$u_n'(t) + \lambda_n u_n(t) = 0 \longrightarrow u_n(t) = A_n e^{-\lambda_n t}.$$

At $t=0$, $u_n(0) = A_n$, so

$$u(\underline{x},0)=\sum_{n=1}^{\infty} A_n \varphi_n(\underline{x})$$

Therefore

$$u(\underline{x},t)=\int_\Omega \sum_{n=1}^{\infty} e^{-\lambda_n t}\varphi_n(\underline{x})\varphi_n(\underline{y})g(\underline{y})d\underline{y}.$$

Formally, set

$$K(\underline{x},\underline{y},t)=\sum_{n=1}^{\infty} e^{-\lambda_n t}\varphi_n(\underline{x})\varphi_n(\underline{y}). \qquad \text{(heat kernel)}$$

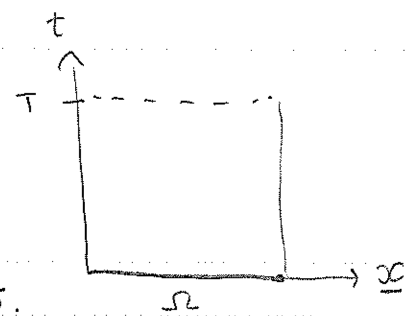
Then we can write

$$u(\underline{x},t)=\int_\Omega K(\underline{x},\underline{y},t)g(\underline{y})d\underline{y}.$$

N.B.: This method can be applied to more general problems: 2013 11 25

$$\begin{cases} u_t = \Delta u + f(x,t) & \\ u(\underline{x},0)=g(\underline{x}) & \underline{x}\in\overline{\Omega} \\ u(\underline{x},t)=h(\underline{x}) & \underline{x}\in\partial\Omega \end{cases}$$

<u>Maximum Principle</u>



Let

$$\Omega_T = \Omega\times(0,T),$$
$$C_T = \partial_p\Omega_T = \{(\underline{x},t)\in\overline{\Omega}_T;\ \underline{x}\in\partial\Omega \ \underline{\text{or}}\ t=0\}.$$

That is, C_T is the boundary of Ω_T except the top.

Weak Maximum Principle:

If $u\in C^{2;1}(\Omega_T)\cap C(\overline{\Omega}_T)$ satisfies

$$\Delta u \geq u_t \quad \text{in } \Omega_T$$

then u ~~att~~ achieves its maximum on the parabolic boundary C_T of Ω_T. That is,

$$u(\underline{x},t)\leq \max_{C_T} u \qquad \forall (\underline{x},t)\in\Omega_T \qquad (T>0).$$

in fact, $\forall(\underline{x},t)\in\overline{\Omega}_T$

Proof: Step 1: Assume $\overbrace{\Delta u > u_t}^{(\dagger)}$ in Ω_T.
For $0 < \tau < T$, consider $\Omega_\tau = \Omega \times (0, \tau)$,

$$C_\tau = \{(x,t) \in \bar{\Omega}_\tau ; x \in \partial\Omega \text{ or } t = 0\}.$$

If max of u on $\bar{\Omega}_\tau$ occurs at $x \in \Omega$ and $t = \tau$,
Then $u_t(x,t) \geq 0$ and $\Delta u(x,t) \leq 0$. Contradicting $(\dagger)$.
Similarly. u cannot attain an interior max since $\Rightarrow u_t = 0$ and $\Delta u(x,t) \leq 0$.
But

$$\max_{C_\tau} u \leq \max_{C_T} u$$

$$\Rightarrow u(x,t) \leq \max_{C_T} u.$$

Step 2: General case $\Delta u \geq u_t$ in Ω_T.
Let $v = u - kt$ for $k > 0$ (note $v \leq u$ on $\bar{\Omega}$).
And $\Delta v - v_t = \Delta u - u_t + k > 0$. By step 1 applied to v,

$$\max_{\bar{\Omega}_T} u = \max_{\bar{\Omega}_T} v + kt \leq \left(\max_{\bar{\Omega}_T} v\right) + kT$$

$$\leq \left(\max_{C_T} u\right) + kT.$$

As $k > 0$ was arbitrary, let $k \to 0^+$ to get

$$\max_{\bar{\Omega}_T} u \leq \max_{C_T} u.$$

$\blacksquare$

Uniqueness: If $u, v \in C^{2;1}(\Omega) \cap C(\bar{\Omega})$ are solutions to

$$\begin{cases} u_t = \Delta u + f(x,t) & x \in \Omega \quad t > 0 \\ u(x,0) = g(x) & x \in \bar{\Omega} \\ u(x,t) = h(x,t) & x \in \partial\Omega \end{cases}$$

then $u \equiv v$.
Proof: Apply above to $u - v$. $\blacksquare$

Continuous dependence (on initial boundary data): follows from weak max principle.

new topic(s)

Cauchy Problem:

Consider

$$\begin{cases} u_t = \Delta u & \text{for } t>0 \;\; \underline{x} \in \mathbb{R}^n \\ u(\underline{x},0) = g(\underline{x}) & \text{for } \;\; \underline{x} \in \mathbb{R}^n \end{cases}$$

Fourier Transform: For $f \in C_0^\infty(\mathbb{R}^n)$ we define its Fourier Transform to be

$$\hat{f}(\underline{\omega}) = \mathcal{F}\{f\} = \int_{\mathbb{R}^n} e^{i\underline{\omega}\cdot\underline{x}} f(\underline{x})\,d\underline{x}. \qquad \underline{\omega} \in \mathbb{R}^n$$

We can differentiate:

$$\frac{\partial}{\partial \omega_j} \hat{f}(\underline{\omega}) = \int_{\mathbb{R}^n} e^{i\underline{\omega}\cdot\underline{x}} (ix_j) f(\underline{x})\,d\underline{x}$$

which is also well-defined since $ix_j f(\underline{x})$ has compact support.
Iterating, we have $\hat{f} \in C^\infty$ and

(1) —— $$\left(\frac{\partial}{\partial \omega_j}\right)^k \hat{f}(\underline{\omega}) = \mathcal{F}\left\{(ix_j)^k f(\underline{x})\right\}.$$

Similarly IBP shows that

(2) — $$\mathcal{F}\left\{\frac{\partial^k f}{\partial x_j^k}\right\}(\underline{\omega}) = (-i\omega_j)^k \hat{f}(\underline{\omega})$$

Properties (1) and (2) make it very natural to consider the Fourier transform on a subspace S of $L^1(\mathbb{R}^n)$ known as the Schwarz class of functions.

not too important

$$S = \{u \in C^\infty(\mathbb{R}^n);\ \forall k \in \mathbb{N},\ \underline{\alpha} \in \mathbb{N}^n,\ |\underline{x}|^k |D^\alpha u(\underline{x})| \text{ bounded}\}$$

Lemma: 2013 11 26

(1) If $g \in L^1(\mathbb{R}^n)$ then $\hat{g}$ is bounded.
(2) If $g \in S$ then $\hat{g} \in S$.

We can define the inverse Fourier transform for $f \in L^1(\mathbb{R}^n)$ by

$$\check{f}(\underline{\omega}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\underline{x}\cdot\underline{\omega}} f(\underline{x})\,d\underline{x}, \qquad \underline{\omega} \in \mathbb{R}^n$$

see Yoshida for proof

Theorem [Fourier Inversion Theorem]:
If $f \in S$ then $(\check{\hat{f}}) = f$:

$$f(\underline{x}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\underline{x}\cdot\underline{\omega}} \hat{f}(\underline{\omega})\, d\underline{\omega}$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} e^{-i(\underline{x}-\underline{y})\cdot\underline{\omega}} f(\underline{y})\, d\underline{y}\, d\underline{\omega}.$$

Remark: This holds for more functions than those in S, but does not hold for general $L^1(\mathbb{R}^n)$ functions.

Pure initial value problem

Consider

$$\begin{cases} u_t = \Delta u \\ u(\underline{x},0) = g(\underline{x}) \end{cases}$$

where $g \in C_0^\infty(\mathbb{R}^n)$.

Formally

$$\mathcal{F}\{u_t\} = \hat{u}_t(\underline{\omega},t) = \int_{\mathbb{R}^n} e^{i\underline{\omega}\cdot\underline{x}} u_t(\underline{x},t)\, d\underline{x} = \frac{\partial}{\partial t}\hat{u}(\underline{\omega},t)$$

and

$$\mathcal{F}\{\Delta u\} = \widehat{\Delta u}(\underline{\omega},t) = \sum_{j=1}^{n} (-i\omega_j)^2 \hat{u}(\underline{\omega},t) = -|\underline{\omega}|^2 \hat{u}(\underline{\omega},t).$$

The heat equation becomes

$$\frac{\partial}{\partial t}\hat{u}(\underline{\omega},t) = -|\underline{\omega}|^2 \hat{u}(\underline{\omega},t)$$

which is an ODE in t. The solution is

$$\hat{u}(\underline{\omega},t) = Ce^{-|\underline{\omega}|^2 t}.$$

The constant C can be determined from the Fourier transformed initial condition:

$$\hat{u}(\underline{\omega},0) = \hat{g}(\underline{\omega}).$$

Therefore

$$\hat{u}(\underline{\omega},t) = \hat{g}(\omega)e^{-|\underline{\omega}|^2 t}.$$

Applying the inverse transform yields

$$u(\underline{x},t) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-i\underline{x}\cdot\underline{\omega}}\,\hat{g}(\underline{\omega})\,e^{-|\underline{\omega}|^2 t}\,d\underline{\omega}$$

$$= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{2n}} e^{-i(\underline{x}-\underline{y})\cdot\underline{\omega} - |\underline{\omega}|^2 t}\,g(\underline{y})\,d\underline{y}\,d\underline{\omega}$$

$$= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} e^{-i(\underline{x}-\underline{y})\cdot\underline{\omega} - |\underline{\omega}|^2 t}\,d\underline{\omega}\right) g(\underline{y})\,d\underline{y}.$$

If we can evaluate

$$K(\underline{x},\underline{y},t) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-i(\underline{x}-\underline{y})\cdot\underline{\omega} - |\underline{\omega}|^2 t}\,d\underline{\omega}$$

then we have

$$u(\underline{x},t) = \int_{\mathbb{R}^n} K(\underline{x},\underline{y},t)\,g(\underline{y})\,d\underline{y}.$$

Consider now the special case $n=1$. We want to evaluate

$$\int_{-\infty}^{\infty} e^{i(x-y)\omega - \omega^2 t}\,d\omega.$$

We do this using integration in the complex plane. This is the same as integrating along

$$\omega = ai + z \quad \text{where } -\infty < z < \infty$$

where $a \in \mathbb{R}$ is fixed. Since x, y, t are fixed (in the integration), let

$$a = -\frac{(x-y)}{2t}$$

and introduce a change of variable

$$z = \frac{\eta}{\sqrt{t}} \Rightarrow dz = \frac{1}{\sqrt{t}}\,d\eta$$

and

$$-i(x-y)\omega - \omega^2 t = -i(x-y)\left(-\left(\frac{x-y}{2t}\right)i + \frac{\eta}{\sqrt{t}}\right) - \left(-\left(\frac{x-y}{2t}\right)i - \frac{\eta}{\sqrt{t}}\right)^2 t$$

$$= -\frac{(x-y)^2}{4t} - \eta^2.$$

Therefore

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(x-y)\omega-\omega^2 t}\,d\omega = \frac{e^{-\frac{(x-y)^2}{4t}}}{2\pi\sqrt{t}}\int_{-\infty}^{\infty} e^{-\eta^2}\,d\eta = \frac{e^{-\frac{(x-y)^2}{4t}}}{\sqrt{4\pi t}}.$$

Thus we obtain

$$K(\underline{x},\underline{y},t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|\underline{x}-\underline{y}|^2}{4t}}$$

since the integral over $\mathbb{R}^n$ can be expressed as the product of n 1-D integrals.

Theorem: If $g(\underline{x})$ is bounded and continuous on $\mathbb{R}^n$ then

$$(*) \qquad u(\underline{x},t) = \int_{\mathbb{R}^n} K(\underline{x},\underline{y},t)\, g(\underline{y})\, d\underline{y}$$

$$= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|\underline{x}-\underline{y}|^2}{4t}} g(\underline{y})\, d\underline{y}$$

is a C^∞ function satisfying

$$u_t = \Delta u \quad \text{for } \underline{x} \in \mathbb{R}^n$$

and extends continuously for $t \geq 0$ such that $u(\underline{x},0) = g(\underline{x})$.

This establishes existence for the pure IVP but uniqueness fails (there exist non-trivial solutions to the pure IVP for $g=0$).

Note: $u(\underline{x},t)$ given by $(*)$ for any $t>0$ shows that the "initial conditions" display infinite propagation speed in their effect on the solution (in contrast with finite propagation speed for the wave equation).

The heat kernel $K(\underline{x},\underline{y},t)$ can be ~~thought of~~ interpretted as the temperature at $(\underline{x},t)$ caused by a "burst" of temperature at $(\underline{y},0)$ (ie $g(\underline{z}) = \delta(\underline{z}-\underline{y})$). Then

$$u(\underline{x},t) = \int_{\mathbb{R}^n} K(\underline{x},\underline{z},t)\, \delta(\underline{z}-\underline{y})\, d\underline{z}$$

$$= K(\underline{x},\underline{y},t).$$

In particular, taking $\underline{y} = \underline{0}$, we get

$$K^*(\underline{x},t) = K(\underline{x},\underline{0},t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|\underline{x}|^2}{4t}}.$$

We expect the temperature distribution to tend to its initial value as $t \to 0$

ie $\lim_{t\to 0^+} K^*(\underline{x},t) = \delta(\underline{x})$.

ie $\forall v \in C_0^\infty(\mathbb{R}^n)$, we should have

$$\langle K^*(\underline{x},t), v\rangle = \int_{\mathbb{R}^n} K(\underline{x},0,t) v(\underline{x}) d\underline{x} \to v(\underline{0}).$$

This is an immediate consequence of the theorem with $g = v$.

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Inhomogeneous heat equation

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$$(*) \quad \begin{cases} u_t = \Delta u + f(\underline{x},t) & \underline{x} \in \mathbb{R}^n,\ t>0 \\ u(\underline{x},0) = 0 & \underline{x} \in \mathbb{R}^n \end{cases}$$

Def) The convolution of two functions f and g defined on $\mathbb{R}^n$ is

$$(f * g)(\underline{x}) = \int_{\mathbb{R}^n} f(\underline{x}-\underline{y}) g(\underline{y}) d\underline{y} \qquad (f,g \in L^1_{loc}(\mathbb{R}^n) \text{ or something})$$

A change of variables $\underline{z} = \underline{x}-\underline{y}$ shows that $f*g = g*f$.
The convolution property is

$$\mathcal{F}\{f*g\} = \mathcal{F}\{f\}\mathcal{F}\{g\}.$$

Taking the Fourier transform of $(*)$ yields

$$\begin{cases} \hat{u}_t = -|\underline{\omega}|^2 \hat{u} + \hat{f}(\underline{\omega},t), \\ \hat{u}(\underline{\omega},0) = 0. \end{cases}$$

Multiplying through by the integrating factor $e^{|\omega|^2 t}$ gives

$$\frac{\partial}{\partial t}\left(\hat{u} e^{|\omega|^2 t}\right) = e^{|\omega|^2 t} \hat{f}$$

$$\Rightarrow \hat{u} e^{|\omega|^2 t} = \int_0^t e^{|\omega|^2 s} \hat{f}(\underline{\omega},s) ds.$$

Therefore

$$\hat{u}(\underline{\omega},t) = \int_0^t e^{-|\omega|^2(t-s)} \hat{f}(\underline{\omega},s) ds.$$

Taking the inverse Fourier transform yields

$$u(x,t) = \int_0^t \mathcal{F}^{-1}\left\{e^{-|w|^2(t-s)}\right\} * f(x,s)\, ds.$$

Since

$$\mathcal{F}^{-1}\left\{e^{-|w|^2(t-s)}\right\} = \frac{1}{(4\pi(t-s))^{n/2}} e^{\frac{-|x|^2}{4(t-s)}},$$

we get

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \frac{1}{(4\pi(t-s))^{n/2}} e^{\frac{-|x-y|^2}{4(t-s)}} f(y,s)\, dy\, ds.$$

chap 4 p12-13 → Exercise: Solve (*) using du Hamel's principle.

Thus the solution of the IVP

$$\begin{cases} u_t = \Delta u + f(x,t) \\ u(x,0) = g(x) \end{cases}$$

is given (by linear superposition) by

$$u(x,t) = \int_{\mathbb{R}^n} K^*(x-y,t)\, g(y)\, dy \cancel{+ \int_0^t\int_{\mathbb{R}^n} \frac{1}{(4\pi(t-s))^{n/2}} e^{\frac{-|x-y|^2}{4(t-s)}}}$$

$$+ \int_0^t \int_{\mathbb{R}^n} K^*(x-y,t-s) f(y,s)\, dy\, ds.$$

Scaling Transformations

$$\bar{u} = \varepsilon^\alpha u, \quad \bar{x} = \varepsilon^\beta x, \quad \bar{t} = \varepsilon^\gamma t$$

i.e. $\bar{u}(x,\bar{t}) = \varepsilon^\alpha u(\varepsilon^{-\beta}x, \varepsilon^{-\gamma}t)$

Suppose we have α, β, γ such that for any $\varepsilon > 0$, $\bar{u}$ satisfies the PDE in $\bar{x}, \bar{t}$ if and only if u satisfies the PDE in x,t.

Taking $\varepsilon^\gamma = t^{-1} \Rightarrow \bar{t} = 1$. Introducing $w(\bar{x}) = \bar{u}(\bar{x},1)$ and $z = \bar{x}$, then

$$u(x,t) = t^{\alpha/\gamma} w(z), \quad z = t^{-\beta/\gamma} x \quad \text{——} \quad (*)$$

defines the similarity transformation to the similarity variables w, z.

In general, one of α, β, γ can be given an arbitrary value, so take $\gamma = 1$.
Then $(*)$ simplifies to

$$u(x,t) = t^{\alpha} w(z), \qquad z = t^{-\beta} x.$$

Substituting into PDE, we obtain an equation for w independent of t. For heat equation:

$$u_t = \alpha t^{\alpha-1} w - \beta t^{\beta-1} z \cdot \nabla z$$

$$\Delta_x u = t^{\alpha-2\beta} \Delta_z w$$

$$\Rightarrow \quad u_t - \Delta_x u = t^{\alpha-1}\left(\alpha w - \beta z \cdot \nabla_z w\right) - t^{\alpha-2\beta} \Delta_z w = 0.$$

Taking $\beta = 1/2$,

$$u(x,t) = t^{\alpha} w(z), \quad z = \frac{x}{\sqrt{t}}$$

$$\Rightarrow \cancel{t^{\alpha-1}}\left(\alpha w(z) - \frac{1}{2} z \cdot \nabla w(z) - \Delta_z w\right) = 0.$$

We have a DE involving n-independent variables $z = (z_1, \dots, z_n)$ (reduction of 1-independent variable).
If $n=1$, this is an ODE, but otherwise difficult to solve in general.
However, if we look for w which is a radial function of z, ie $w(z) = w(|z|)$, $z = x/\sqrt{t}$, substituting into PDE

$$\frac{d^2 w}{dr^2} + \left(\frac{n-1}{r}\right)\frac{dw}{dr} + \underbrace{\frac{1}{2} r \frac{dw}{dr}}_{\frac{1}{2} z \cdot \nabla w(z)} - \alpha w = 0.$$

Multiplying through by r^{n-1} and pick $\alpha = -n/2$

$$\Rightarrow \frac{d}{dr}\left(r^{n-1}\frac{dw}{dr}\right) + \frac{1}{2}\frac{d}{dr}\left(r^n w\right) = 0$$

$$\Rightarrow r^{n-1}\frac{dw}{dr} + \frac{1}{2} r^n w = a \qquad \text{arbitrary constant, so set } a = 0$$

Set $a = 0$

$$\Rightarrow \frac{dw}{dr} = \left(\frac{-r}{2}\right) w \Rightarrow w(r) = C e^{-r^2/4}.$$

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In terms of x, t, we have

$$u(\underline{x},t) = C\frac{1}{t^{n/2}}e^{\frac{-|\underline{x}|^2}{4t}}.$$

With an appropriate choice of C, this is the fundamental solution of the heat equation.

Laplace Transform

Given $f(x)$, $x>0$,

$$\mathcal{L}\{f\} = F(s) = \int_0^\infty e^{-sx}f(x)dx.$$

Note that if

$$\int_0^\infty e^{-cx}|f(x)|dx < \infty$$

for some c, then ~~[illegible]~~ $F(s)$ is defined for $s \geq c$.

Convolution Property

$$(f*g)(x) = \int_0^x f(x-y)g(y)dy$$

If we define $f(x) = g(x) = 0$ for $x \leq 0$, then

$$(f*g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy$$

see course notes

Note $f*g = g*f$, $\mathcal{L}\{f*g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}$.

ex Solve

$$\begin{cases} u_t = ku_{xx}, & x>0,\ t>0,\ (k>0.) \text{ constant} \\ u(x,0) = 0 \\ u_x(0,t) = g(t) \end{cases}$$

Take Laplace transform wrt t, to get

$$U(x,s) = \mathcal{L}\{u(x,t)\},$$

$$\Rightarrow \begin{cases} sU - \overbrace{u(x,0)}^{=0} = kU_{xx} \\ U_x(0,s) = G(s) \end{cases}$$

p 21 Watson's

For $s>0$,

$$U = c_1 e^{\sqrt{s/k}\,x} + c_2 e^{-\sqrt{s/k}\,x}$$

By Watson's lemma (p 21 of chap 4), the Laplace transform tends to 0 as $s \to \infty$. Hence $c_1 = 0$. The BC implies $c_2 = \sqrt{s/k}\,G(s)$.

$$\Rightarrow U(x,s) = -\sqrt{k}\,\frac{e^{-\sqrt{s/k}\,x}}{\sqrt{s}}\,G(s)$$