The solution to the original problem (*) is then

$$u(x,t)=\frac{1}{2c}\int_0^t\int_{x-c(t-s)}^{x+c(t-s)} F(\xi,s)\,d\xi\,ds$$

is a C^2 solution of (*). It's now straightforward to solve

$$\begin{cases} u_{tt}-c^2u_{xx}=F(x,t)\\ u(x,0)=g(x)\\ u_t(x,0)=h(x)\end{cases}$$

by considering

$$\begin{cases} u_{tt}-c^2u_{xx}=0\\ u(x,0)=g(x)\\ u_t(x,0)=h(x)\end{cases} \quad\text{and}\quad \begin{cases} u_{tt}-c^2u_{xx}=F(x,t)\\ u(x,0)=0\\ u_t(x,0)=0\end{cases}$$

and then using linear superposition.

Wave Equation in more than 1-D

Consider the pure initial value problem:
The spherical mean or the average over a sphere of radius r and centre $\underline{x}$ (for $u(\underline{x})$ defined on $\mathbb{R}^n$) is:

(*) —— $$I_u(\underline{x},r)=\frac{1}{\omega_n}\int_{|\xi|=1} u(\underline{x}+r\underline{\xi})\,dS_\xi .$$

ω_n = surface area of unit sphere

Since u is continuous in $\underline{x}$, $I_u(\underline{x},r)$ is continuous in $\underline{x}$ and $r\geq 0$. Letting $r\to 0^+$, we have $I_u(\underline{x},0)=u(\underline{x})$.

If $u\in C^k(\mathbb{R}^n)$ then $I_u\in C^k(\mathbb{R}^n\times[0,\infty))$. Using the chain rule on (*), 2013 11 13

(**) — $$\frac{\partial}{\partial r}I_u(\underline{x},r)=\frac{1}{\omega_n}\int_{|\xi|=1} u_{x_i}(\underline{x}+r\underline{\xi})\cdot\xi_i\,dS_\xi .$$

Let $\Omega=\{\xi\in\mathbb{R}^n;\ |\xi|<1\}$. Then $\partial\Omega=S^{n-1}$. The exterior unit normal is $\underline{n}=\xi$ $(|\xi|=1)$.
The RHS of (**) can be rewritten using the divergence theorem. Recall:

$$\int_\Omega \nabla\cdot V(\xi)\,d\xi=\int_{\partial\Omega} V\cdot n\,dS_\xi .$$

Take $V(\xi)=r^{-1}\nabla_\xi u(\underline{x}+r\xi)=\nabla_x u(\underline{x}+r\xi)$. Compute

$$\nabla \cdot \underline{v} = r\sum_{i=1}^{n} u_{x_i x_i}(\underline{x}+r\underline{\xi}) = r\,\Delta_{\underline{x}} u(\underline{x}+r\underline{\xi}).$$

Therefore (**) can be rewritten as

(***) — $$\frac{\partial}{\partial r} I_u(\underline{x}, r) = \frac{r}{\omega_n}\Delta_{\underline{x}} \int_{|\underline{\xi}|<1} u(\underline{x}+r\underline{\xi})\,d\underline{\xi}.$$

Making a change of variables, $\underline{\eta} = r\underline{\xi}$, $d\underline{\eta} = r^n d\underline{\xi}$, we have

$$\int_{|\underline{\xi}|<1} u(\underline{x}+r\underline{\xi})\,d\underline{\xi} = \frac{1}{r^n}\int_{|\underline{\eta}|<r} u(\underline{x}+\underline{\eta})\,d\underline{\eta}.$$

Using spherical coordinates:

$$\frac{1}{r^n}\int_{|\underline{\eta}|<r} u(\underline{x}+\underline{\eta})\,d\underline{\eta} = \frac{1}{r^n}\int_0^r \rho^{n-1}\int_{|\underline{\xi}|=1} u(\underline{x}+\rho\underline{\xi})\,dS_{\underline{\xi}}\,d\rho$$
$$= \frac{\omega_n}{r^n}\int_0^r \rho^{n-1} I_u(\underline{x},\rho)\,d\rho.$$

Substituting back into (***),

$$\frac{\partial}{\partial r} I_u(\underline{x}, r) = \frac{1}{r^{n-1}}\Delta_{\underline{x}}\int_0^r \rho^{n-1} I_u(\underline{x},\rho)\,d\rho.$$

Multiplying through by r^{n-1} and differentiating wrt r (FTC), and dividing again by r^{n-1}, we can write the resulting equation as

$$\left(\frac{\partial^2}{\partial r^2} + \left(\frac{n-1}{r}\right)\frac{\partial}{\partial r}\right) I_u(\underline{x}, r) = \Delta_{\underline{x}} I_u(\underline{x}, r).$$

This known as Darboux's equation.

Note: For a radial function, $u = u(r)$, $I_u(\underline{x}, r) = u$.

Consider

(*) — $$\begin{cases} u_{tt} - c^2\Delta u = 0 & \underline{x}\in\mathbb{R}^n,\ t>0 \\ u(\underline{x},0) = g(\underline{x}) & \\ u_t(\underline{x},0) = h(\underline{x}), & \underline{x}\in\mathbb{R}^n \end{cases}$$

Suppose $u(\underline{x},t)$ is a solution of (*). Thinking of t as a parameter, the spherical mean

$$I_u(\underline{x}, r, t) = \frac{1}{\omega_n}\int_{|\underline{\xi}|=1} u(\underline{x}+r\underline{\xi}, t)\,dS_{\underline{\xi}},$$

we have

(+)
$$\frac{\partial^2}{\partial t^2} I_u = \frac{1}{\omega_n}\int_{|\xi|=1} u_{tt}(x+r\xi,t)\,dS_\xi$$
$$= \frac{1}{\omega_n}\int_{|\xi|=1} c^2\Delta_x u(x+r\xi,t)\,dS_\xi \qquad \text{(by (*))}$$
$$= c^2\Delta_x I_u$$

Using Darboux's equation, we have

(1) —— $$\frac{\partial^2 I_u}{\partial t^2} = c^2\left(\frac{\partial^2}{\partial r^2} + \frac{(n-1)}{r}\frac{\partial}{\partial r}\right) I_u.$$

Taking spherical means of the ICs yields

(2) —— $$I_u(x,r,0) = \frac{1}{\omega_n}\int_{|\xi|=1} g(x+r\xi)\,dS_\xi$$

(3) —— $$\frac{\partial}{\partial t} I_u(x,r,0) = \frac{1}{\omega_n}\int_{|\xi|=1} h(x+r\xi)\,dS_\xi.$$

Solving the IVP (1), (2), (3), we obtain $u(x,t)$ by

$$u(x,t) = \lim_{r\to 0} I_u(x,r,t) = \lim_{r\to 0}\frac{1}{\omega_n}\int_{|\xi|=1} u(x+r\xi,t)\,dS_\xi$$

Now consider $n=3$ in more detail. Here (+) can be rewritten as

$$\frac{\partial^2}{\partial t^2}(r I_u) = c^2\frac{\partial^2}{\partial r^2}(r I_u).$$

Define

$$v^x(r,t) = r I_u(x,r,t).$$

Then v^x is the solution of the following 1-D wave equation:

$$\frac{\partial^2}{\partial t^2} v^x(r,t) = c^2\frac{\partial^2}{\partial r^2} v^x(r,t),$$
$$v^x(r,0) = \frac{r}{\omega_3}\int_{|\xi|=1} g(x+r\xi)\,dS_\xi =: \tilde{g}^x(r),$$
$$v^x_t(r,0) = \frac{r}{\omega_3}\int_{|\xi|=1} h(x+r\xi)\,dS_\xi =: \tilde{h}^x(r).$$

At $r=0$,

$$v^x(0,t) = \lim_{r\to 0} r I_u(x,r,t) = 0.$$

Since $\tilde{g}^{x}(0)=\hat{h}^{x}(0)=0$, we can extend $\tilde{g}^{x}$ and $\hat{h}^{x}$ to odd functions of r, and by d'Alembert's formula we get

$$v^{x}(r,t)=\frac{1}{2}\left(\tilde{g}^{x}(r+ct)+\tilde{g}^{x}(r-ct)\right)+\frac{1}{2c}\int_{r-ct}^{r+ct}\tilde{h}^{x}(\rho)\,d\rho.$$

Since $\tilde{g}^{x}$ and $\tilde{h}^{x}$ are odd functions, for $r<ct$ we have 2013 11 15

$$\tilde{g}^{x}(r-ct)=-\tilde{g}^{x}(ct-r),$$

$$\int_{r-ct}^{r+ct}\tilde{h}^{x}(\rho)\,d\rho=\int_{ct-r}^{ct+r}\tilde{h}^{x}(\rho)\,d\rho.$$

Thus

$$\begin{aligned}I_u(x,r,t)&=\frac{1}{r}v^{x}(r,t)\\&=\frac{\tilde{g}^{x}(ct+r)-\tilde{g}^{x}(ct-r)}{2r}+\frac{1}{2cr}\int_{ct-r}^{ct+r}\tilde{h}^{x}(\rho)\,d\rho\\&=\frac{(ct+r)I_g(x,ct+r)-(ct-r)I_g(x,ct-r)}{2r}\\&\qquad+\frac{1}{2cr}\int_{ct-r}^{ct+r}\rho I_h(x,\rho)\,d\rho.\end{aligned}$$

(see course notes pg 9-13 section 3.3)

As $r\to 0^+$ we have

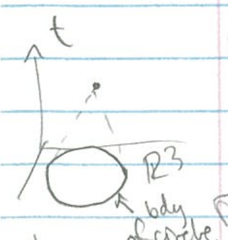
$$u(x,t)=\frac{\partial}{\partial\tau}\left(\tau I_g(x,\tau)\right)\Big|_{\tau=ct}+tI_h(x,ct).$$

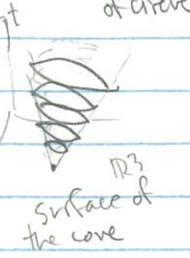
Theorem: If $g\in C^3(\mathbb{R}^3)$ and $h\in C^2(\mathbb{R}^3)$ then

$$u(x,t)=\frac{1}{4\pi}\frac{\partial}{\partial t}\left(t\int_{|\xi|=1}g(x+ct\xi)\,dS_\xi\right)+\frac{t}{4\pi}\int_{|\xi|=1}h(x+ct\xi)\,dS_\xi$$ Kirchoff's formula

defines a C^2 solution of

$$\begin{cases}u_{tt}-c^2\Delta u=0 & x\in\mathbb{R}^3,\ t>0,\\ u(x,0)=g(x) & x\in\mathbb{R}^3,\\ u_t(x,0)=h(x) & x\in\mathbb{R}^3.\end{cases}$$





The <u>domain of dependence</u> for a point (x,t) is the surface of the sphere $\{x+ct\xi;\ |\xi|=1\}\subseteq\mathbb{R}^3$, and similarly the <u>range of influence</u> of a point $x^*\in\mathbb{R}^3$ is the <u>forward light cone</u>

$$\{(x,t);\ |x-x^*|=ct\}$$

Physically experienced as a finite propagation speed of signals, and more specifically as the existence of sharp signals for 3D waves.
For example, an initial disturbance at $x = 0$ will be percieved at another point $x_1 \in \mathbb{R}^3$ only at time $t_1 = \|x_1\|/c_1$ and not thereafter (contrast with the 1-D case).

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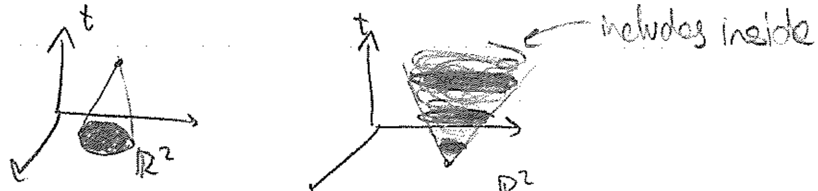
2D-wave Equation

Use Hadamard's method of descent and view 2-D problem as a special case of the 3-D problem with ICs independent of x_3.

K.F (*) — $$u(x,t) = \frac{1}{4\pi}\frac{\partial}{\partial t}\left(t\int_{|\xi|=1} g(x+ct\xi)\,dS_\xi\right) + \frac{t}{4\pi}\int_{|\xi|=1} h(x+ct\xi)\,dS_\xi$$

Upper hemisphere of unit disk is:
$$\psi(\xi) = \psi(\xi_1,\xi_2) = \sqrt{1-\xi_1^2-\xi_2^2}$$
over the unit disk $\xi_1^2+\xi_2^2 < 1$. So
$$dS_\xi = \sqrt{1+(\psi_{\xi_1})^2+(\psi_{\xi_2})^2}\,d\xi_1 d\xi_2 = \frac{1}{\sqrt{1-\xi_1^2-\xi_2^2}}\,d\xi_1 d\xi_2.$$
Using (*), integrating over upper hemisphere and multiplying by 2:
$$u(x_1,x_2,t) = \frac{1}{4\pi}\frac{\partial}{\partial t}\left(2t\int_{\xi_1^2+\xi_2^2<1} \frac{g(x_1+ct\xi_1,\, x_2+ct\xi_2)}{\sqrt{1-\xi_1^2-\xi_2^2}}\,d\xi_1 d\xi_2\right)$$
$$+\frac{t}{4\pi}\left(2\int_{\xi_1^2+\xi_2^2<1} \frac{h(x+ct\xi)}{\sqrt{1-\xi_1^2-\xi_2^2}}\,d\xi_1 d\xi_2\right)$$



Shows like 1-D case that the domain of dependence for a point (x,t) is the interior of a circle:
$$\{x+ct\xi,\ |\xi| \le 1\} \subseteq \mathbb{R}^2$$

Similarly, the range of influence ~~of~~ of a point $x^* \in \mathbb{R}^2$ is the interiorish of ~~the~~ a cone:

$$\{(\underline{x},t);\ |\underline{x}-x^*| \le ct\}.$$

Huygen's principle: "Sharp signals exist for n odd, $n>1$"

The pattern of using spherical means to solve the Cauchy problem for odd values of n, and the method of descent for even values, holds for all $n \ge 2$.

For $n = 2k+1$, the ~~solution~~ substitution ~~$v^x(r,t) = \frac{1}{r}\frac{\partial}{\partial r}$~~

$$v^x(r,t) = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left(r^{2k-1} I_u(x,r,t)\right)$$

don't worry bout this too much, should know 1,2,3-D though

satisfies the 1-D equation

$$\frac{\partial^2}{\partial t^2} v^x = c^2 \frac{\partial^2}{\partial r^2} v^x$$

$$v^x(r,0) = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left(r^{2k-1} I_g(x,r)\right) =: \tilde{g}^x(r)$$

$$v_t^x(r,0) = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left(r^{2k-1} I_h(\underline{x},r)\right) =: \hat{h}^x(r).$$

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So

$$v^x(r,t) = \frac{\tilde{g}(r+ct) + \tilde{g}(r-ct)}{2} + \frac{1}{2c}\int_{r-ct}^{r+ct} \hat{h}^x(\rho)\,d\rho$$

and then proceeding as before, (in 3D case) we obtain Kirchoff's formula:

$$u(x,t) = \frac{1}{C_n\omega_n}\frac{\partial}{\partial t}\left(\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}} t^{n-2}\int_{|\xi|=1} g(\underline{x}+ct\underline{\xi})\,dS_\xi\right)$$

$$+ \frac{1}{C_n\omega_n}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}} t^{n-2}\int_{|\xi|=1} h(\underline{x}+ct\underline{\xi})\,dS_\xi$$

where $C_n = 1\cdot 3\cdot 5\cdot \ldots \cdot (n-2)$.

Given a function $u(x,t)$, define its energy at time t to be

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^n} \left(u_t^2 + c^2|\nabla u|^2\right) dx.$$

Solutions of the n-dimensional wave equation satisfy conservation of energy.

Let $u \in C^2(\mathbb{R}^n \times (0,\infty))$ be a solution of

$$(*)\quad \begin{cases} u_{tt} = c^2 \Delta u, & x \in \mathbb{R}^n,\ t>0 \\ u(x,0) = g(x), & x \in \mathbb{R}^n \\ u_t(x,0) = h(x) & \text{"} \end{cases}$$

where g, h have compact support. ($B_R(0)$)

Due to finite propagation speed of signals, for any fixed t, $u(x,t)$ is zero outside the ball $|x| \le R + ct$.

Now,

$$\frac{dE}{dt} = \int_{\mathbb{R}^n} \left(u_t u_{tt} + c^2 \sum_{i=1}^{n} u_{x_i} u_{x_i t}\right) dx.$$

Integrating by parts yields

$$\frac{dE}{dt} = \int_{\mathbb{R}^n} u_t \underbrace{(u_{tt} - c^2 \Delta u)}_{=0}\, dx = 0.$$

That is to say, $E(t)$ is constant. In particular $E(t) = E(0)$.

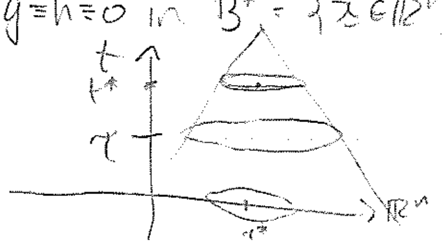
Corollary. The solution to $(*)$ is unique and depends continuously on the Cauchy data.

Domain of Dependence Inequality

Suppose u is a C^2 solution of

$$(*)\quad \begin{cases} u_{tt} = c^2 \Delta u \\ u(x,0) = g(x) \\ u_t(x,0) = h(x). \end{cases}$$

For any $x^* \in \mathbb{R}^n$, $t^* > 0$, if $g \equiv h \equiv 0$ in $\bar{B}^* = \{x \in \mathbb{R}^n;\ |x - x^*| \le ct^*\}$, then $u(x^*, t^*) \equiv 0$.



??

Proof: For any $\tau \in [0, t^*]$, let

$$\overline{B}_\tau = \{x \in \mathbb{R}^n;\ |x - x^*| \le c(\tau - t^*)\}.$$

Consider the local energy function

$$E^{(x^*, t^*)}(\tau) = \frac{1}{2}\int_{B_\tau} (u_t^2 + c^2|\nabla u|^2)\Big|_{t=\tau}\, dx.$$

We show that $E^{(x^*, t^*)}$ is a non-decreasing function of τ.

Define

$$\Omega_\tau = \{(x,t);\ |x - x^*| < c(t^* - t),\ 0 < t < \tau\},$$
$$C_\tau = \{(x,t);\ |x - x^*| = c(t^* - t),\ 0 < t < \tau\}.$$

Then $\partial\Omega_\tau = C_\tau \cup \overline{B}^* \cup (\overline{B}_\tau \times \{\tau\})$. The exterior unit normal ν on $\partial\Omega_\tau$ is given on $\overline{B}_\tau \times \{\tau\}$ by $\nu = (0, 0, \dots, 0, 1)$ and on $\overline{B}^*$ by $\nu = (0, \dots, 0, -1)$.

hm...

On C_τ, the normal ν satisfies

$$c^2(\nu_1^2 + \dots + \nu_n^2) = \nu_{n+1}^2.$$

Together with

$$\nu_1^2 + \dots + \nu_{n+1}^2 = 1,$$

we get

$$\left.\nu_1 + \dots + \nu_n = \frac{\nu_{n+1}^2}{c^2} = \frac{1}{1+c^2}.\right\} (\dagger)$$

Given a solution of the IVP $(*)$, we define the vector field

$$V = (2c^2 u_t u_{x_1}, \dots, 2c^2 u_t u_{x_n}, -(c^2|\nabla u|^2 + u_t^2)).$$

We show $E^{(x^*, t^*)}(\tau) \le E^{(x^*, t^*)}(0)$ for $0 \le \tau \le t^*$. 2013 11 20

$$\text{Then } \operatorname{div} V = 2c^2(\cancel{u_{tx_1}u_{x_1}} + u_t u_{x_1x_1} + \dots + \cancel{u_{tx_n}u_{x_n}} + u_t u_{x_nx_n}) - 2c^2(\cancel{u_{tx_1}u_{x_1}} + \dots + \cancel{u_{tx_n}u_{x_n}}) - 2u_t u_{tt}$$

$$= 0.$$

Therefore the divergence theorem implies

includes t ...

$$\int_{\partial\Omega_\tau} V \cdot \nu\, dS = \int_{\Omega_\tau} \operatorname{div} V\, dx = 0.$$

On C_τ, the following is true:

$$2u_t(u_{x_1}\nu_1 + \dots + u_{x_n}\nu_n) \le \frac{c}{\sqrt{1+c^2}}|\nabla u|^2 + \frac{1}{c\sqrt{1+c^2}}u_t^2.$$

(we don't show why, but look at $\|\nu_{n+1}\nabla u - u_t(\nu_1, \dots, \nu_n)\|^2 \ge 0$ and use $(\dagger)$)

Therefore $\nabla \cdot \underline{\nu} = 2c^2 u_t(u_{x_1}\nu_1 + \dots + u_{x_n}\nu_n) - (c^2|\nabla u|^2 + u_t^2)\nu_{n+1} \leq 0$.
So

$$\int_{C_\tau} \nabla \cdot \nu \, d\underline{S} \leq 0.$$

Therefore

$$\int_{B^*} (c^2|\nabla u|^2 + u_t^2)\Big|_{t=0} dx - \int_{B_\tau} (c^2|\nabla u|^2 + u_t^2)\Big|_{t=\tau} dx \geq 0$$

ie.:

$$E^{(x^*,t^*)}(\tau) \leq E^{(x^*,t^*)}(0).$$

We briefly consider the effect of lower order terms:

$$u_{tt} - u_{xx} + \alpha u_t + \beta u_x + \gamma u = f(x,t), \quad x \in \mathbb{R}, t > 0.$$

Consider first

$$(*) \quad u_{tt} - u_{xx} + \lambda u = 0, \quad \lambda \text{ constant}, \lambda \neq 0$$

Look for solutions

$$u(x,t) = U(\underbrace{k}_{\text{wave \#}}x - \underbrace{\omega}_{\text{frequency}} t)$$

These represent waves propagating with constant velocity $c = \omega/k$, and without change of shape.
Substituting $u = U(kx - \omega t)$ into $(*)$:

$$(\omega^2 - k^2)U'' + \lambda U = 0$$

For bounded solutions we require

$$\frac{\omega^2 - k^2}{\lambda} > 0,$$

in which case the solution can be written as a linear combination of sines and cosines.
However, simpler ~~to~~ and more general to write

$$U(kx - \omega t) = Ae^{i(kx-\omega t)} \quad (**)$$

and substitute into $(*)$ to get the <u>dispersion relation</u>

$$\omega^2 - k^2 = \lambda \iff \omega(k) = \pm\sqrt{k^2+\lambda},$$

which guarantees $(**)$ solves $(*)$.

The dispersal relation defines the frequency as a function of the wave number and similarly the velocity $c = c(k) = \omega/k$ becomes a function of the wave number.

If $dc/dk \neq 0$ then (**) gives dispersive wave solutions.

Significance:

$$u(x,t) = \sum_{j=1}^{N} A_j e^{i(k_j x - \omega_j t)} \quad \text{(discrete wave train)}$$

will satisfy

$$u_{tt} - u_{xx} + \lambda u = 0,$$

<u>provided</u> each $\omega_j = \omega_j(k_j)$ satisfies the dispersion relation

$$\omega_j(k_j) = \pm\sqrt{k_j^2 + \lambda}.$$

In the continuous limit (if A_j's decay sufficiently)

$$u(x,t) = \int_{-\infty}^{\infty} A(k) e^{i(kx - \omega(k)t)}\, dk.$$

Even in the dispersive case, solutions of

$$u_{tt} - u_{xx} + \lambda u = 0$$

satisfy <u>conservation of energy</u>, where

$$E(t) := \frac{1}{2}\int_{-\infty}^{\infty} (u_t^2 + u_x^2 + \lambda u^2)\, dx$$

<u>Dissipation</u>

constants

$$u_{tt} - u_{xx} + \alpha u_t + \beta u_x + \gamma u = 0$$

Consider the case where $\alpha > 0$. Changing to characteristic coordinates

$$\xi = x + t, \quad \eta = x - t$$

this becomes

$$u_{\xi\eta} - \left(\frac{\alpha+\beta}{4}\right)u_\xi + \left(\frac{\alpha-\beta}{4}\right)u_\eta + \frac{\gamma}{4}u = 0.$$

$$e^{\left(\frac{\alpha+\beta}{4}\right)\xi - \left(\frac{\alpha-\beta}{4}\right)\eta}$$

We make a change of dependent variable

$$w(\xi,\eta) = u(\xi,\eta)\, e^{\left(\frac{\alpha+\beta}{4}\right)\xi \;\cancel{}\; \left(\frac{\alpha-\beta}{4}\right)\eta}$$

(†) $\Longrightarrow \quad w_{\xi\eta} - \frac{\lambda}{4}w = 0 \quad$ where $\lambda = \frac{\alpha^2 - \beta^2 - 4\gamma}{4}$

Converting back to x,t coordinates,

$$u(x,t) = w(x,t)\exp\left(\frac{\beta}{2}x - \frac{\alpha}{2}t\right)$$

where w satisfies (†)

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For $\beta = 0$,

$$E(t) = \frac{1}{2}\int (u_t^2 + u_x^2 + \gamma u^2)\,dx$$

$$\Rightarrow \frac{dE}{dt} = \int (u_t u_{tt} + u_x u_{xt} + \gamma u u_t)\,dx$$

IBP, bdy term 0 as all are compactly supported implicit !!

$$= \int u_t (u_{tt} - u_{xx} + \gamma u)\,dx$$

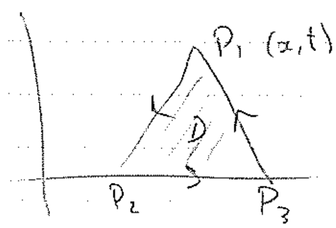
$$= -\alpha \int u_t^2\,dx < 0.$$

~1.5ish chap?

That is, the energy $E(t)$ of a solution decreases with (assuming initial data has compact support).

$$(*) \; — \; \begin{cases} u_{tt} - u_{xx} = f(x,t) \\ u(x,0) = g(x), \; u_t(x,0) = h(x) \end{cases}$$

The solution can be written as



$$(**) \; — \; u(P_1) = \frac{1}{2}\left(g(P_2) + g(P_3)\right) + \frac{1}{2}\int_{P_2}^{P_3} h(x)\,dx + \frac{1}{2}\iint_D f(\xi,\eta)\,d\xi\,d\eta$$

$P_2 < P_3$ are the points where the characteristics through P_1 intersect the x-axis.

Suppose we wish to study

$$(***) \; — \; \begin{cases} u_{tt} - u_{xx} + \alpha u_t + \beta u_x + \gamma u = f(x,t) \\ u(x,0) = g(x), \; u_t(x,0) = h(x). \end{cases}$$

$$\to u_{tt} - u_{xx} = F(x,t,u,u_x,u_y)$$

Using $(**)$, existence can be established (e.g. by contraction mapping principle).