$$= -\int_{\Omega} \left(\sum_n \frac{\varphi_n(\xi,\eta)\varphi_n(x,y)}{\lambda_n \|\varphi_n\|^2} \right) F(\xi,\eta)\, d\xi\, d\eta.$$

Therefore the Green's function representation is

$$u(\underline{x}) = \int_{\Omega} G(\underline{\xi};\underline{x}) F(\underline{\xi})\, d\underline{\xi}.$$

By comparison,

$$G(\underline{\xi};\underline{x}) = -\sum_n \frac{\varphi_n(\underline{\xi})\varphi_n(\underline{x})}{\lambda_n \|\varphi_n\|^2}.$$

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new topic

Hyperbolic Equations

$$(*) \quad \text{———} \quad u_{tt} - u_{xx} = 0$$

The characteristics are $x \pm t = \text{const.}$ Change of variables:

$$\xi = x+t, \quad \eta = x-t$$

leads to $(*)$ reducing to

$$u_{\xi\eta} = 0.$$

The general solution is then

$$u(\xi,\eta) = F(\xi) + G(\eta)$$

$$\Rightarrow u(x,t) = F(x+t) + G(x-t).$$

If $F \equiv 0$, then u is ~~a~~ constant along lines $x-t = \text{const.}$, and so can be described as a wave propagating in the positive x-direction, with speed $\frac{dx}{dt} = 1$. If $G \equiv 0$, then we have a wave propagating in the negative x-direction with speed $\frac{dx}{dt} = -1$.

Consider the Cauchy Problem

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(x,0) = g(x) \\ u_t(x,0) = h(x) \end{cases}$$

Well

$$u(x,t) = F(x+t) + G(x-t) \quad \text{———} \quad (**)$$

together with the above ICs yields

$$F(x)+G(x)=g(x), \quad \text{—— (1)}$$
$$F'(x)-G'(x)=h(x). \quad \text{—— (2)}$$

Integrating (2) wrt x yields

$$F(x)-G(x)=\int_0^x h(s)\,ds + C \quad \text{—— (3)}.$$

Now we solve for F and G using (1) and (3). We obtain

$$F(x)=\frac{1}{2}g(x)+\frac{1}{2}\int_0^x h(\xi)\,d\xi + C^*,$$

$$G(x)=\frac{1}{2}g(x)-\frac{1}{2}\int_0^x h(\xi)\,d\xi - C^*.$$

Since we have (☆☆), we combine these to get

don't use → (†) —— $$u(x,t)=\frac{1}{2}\big(g(x+t)+g(x-t)\big)+\frac{1}{2}\int_{x-t}^{x+t} h(\xi)\,d\xi,$$

which is known as d'Alembert's ~~soln~~ formula.

Theorem: If $g\in C^2$, $h\in C^1$, then

(*) —— $$u(x,t)=\frac{1}{2}\big(g(x+ct)+g(x-ct)\big)+\frac{1}{2c}\int_{x-ct}^{x+ct} h(\xi)\,d\xi$$

defines a C^2 solution of

(**) —— $$\begin{cases} u_{tt}-c^2u_{xx}=0, \\ u(x,0)=g(x), \quad u_t(x,0)=h(x). \end{cases}$$

Note: (1) (*) ⇒ a solution to (**) is unique — MAJOR (SKETCH)

(2) Small perturbations in Cauchy data g,h will affect the solution by "small" amounts (ie cont. dependence on ICs)

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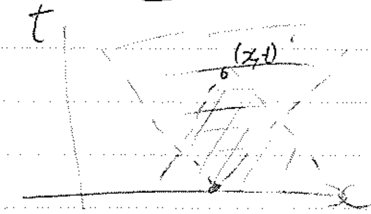
Note (cont.): (3) At any (x,t), u is determined by values of g and h in the interval $[x-ct, x+ct]$, known as the <u>domain of dependence</u>.



Conversely, any pt ξ on the x-axis lies only in the domain of dependence of pts (x,t) in an inverted wedge shape known as the <u>range of influence</u>

Physically, these ideas express the "finite propagation speed" of disturbances.

An initial disturbance near $x=\xi$ $(t=0)$ will not be felt at x^* until $t^* = |x^* - \xi|$. (c is gone)

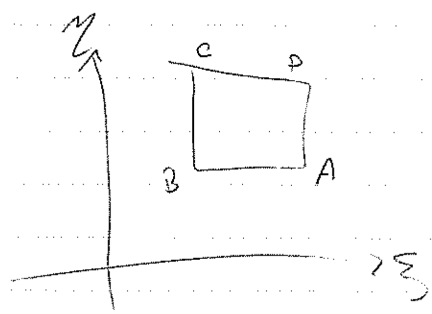
drop c dependence from now on

We might expect

$$u(x,t) = F(x+t) + G(x-t)$$

to represent a weak solution of the Cauchy problem when F and G no longer are required to be C^2.

Let $\xi = x+t$, $\eta = x-t$.



Consider ABCD. Since F is constant along vertical lines and G is " " horizontal lines,

$u_{tt} - u_{xx} = 0$

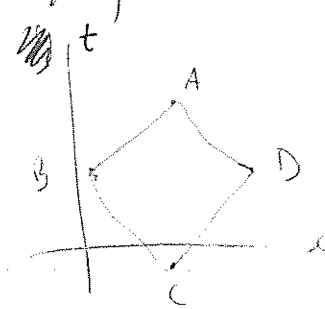
$$F(A) = F(D), \quad F(C) = F(B)$$
$$G(C) = G(D), \quad G(B) = G(A)$$

but $u(\xi,\eta) = F(\xi) + G(\eta)$, so

$$u(A) + u(C) = u(B) + u(D).$$

(ie sums of the value of u at opposite corners are equal.)

This is known as the <u>parallelogram rule</u>.



Consider oscillations of a string of length L:

$$\begin{cases} u_{tt} - u_{xx} = 0 & 0<x<L,\ t>0, \\ u(x,0)=g(x),\quad u_t(x,0)=h(x), & 0<x<L, \\ u(0,t)=\alpha(t),\quad u(L,t)=\beta(t), & t>0. \end{cases}$$

We can use separation of variables to solve this:

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin\left(\frac{n\pi x}{L}\right).$$

Formally, u satisfies $u(0,t)=0=u(L,t)$. Upon substituting into our DE, we ~~get~~ recover an ODE for $a_n(t)$:

$$a_n'' + \left(\frac{n\pi}{L}\right)^2 a_n = 0.$$

General solution is

$$a_n(t) = c_n \sin\left(\frac{n\pi t}{L}\right) + d_n \cos\left(\frac{n\pi t}{L}\right).$$

Since $u(x,0)=g(x)$,

$$g(x) = \sum_{n=1}^{\infty} d_n \sin\left(\frac{n\pi x}{L}\right)$$

Since $u_t(x,0)=h(x)$,

$$h(x) = \sum_{n=1}^{\infty} c_n \frac{n\pi}{L} \sin\left(\frac{n\pi x}{L}\right)$$

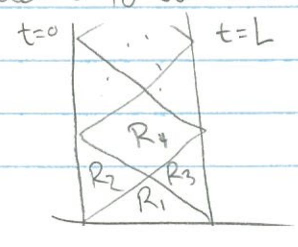
$$d_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

$$c_n = \frac{2}{n\pi}\int_0^L h(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$(*) \quad u(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin\left(\frac{n\pi x}{L}\right).$$

Disadvantage: infinite series.

Another approach is to use the d'Alembert solution and the parallelogram rule



$$R = \{x\in[0,L],\ t>0\}$$
$$= \cup_{j=1}^{\infty} R_j$$

where each R_j contains the characteristics ... upper edges but not lower edges

see pg 1-8 of chap 3

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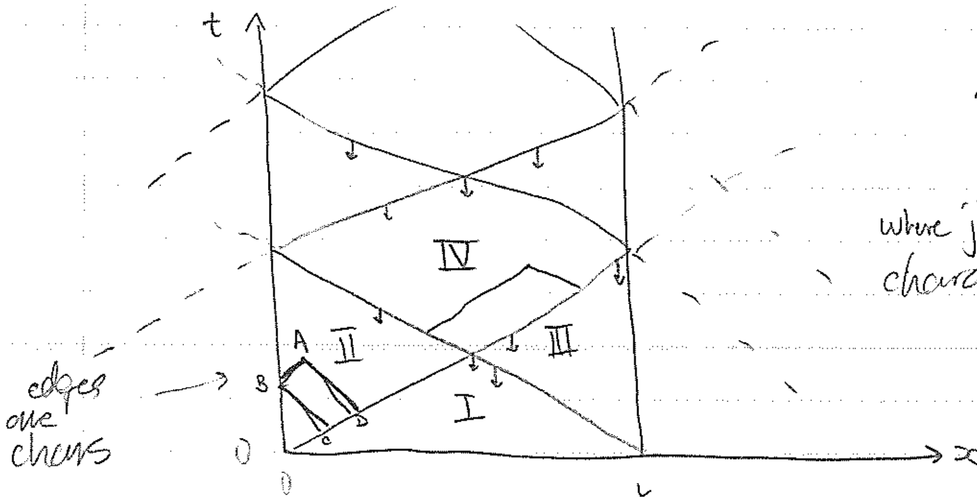
$$\begin{cases} u_{tt} - u_{xx} = 0 & 0<x<L, \ t>0 \\ u(x,0) = g(x), \ u_t(x,0) = h(x), & 0<x<L \\ u(0,t) = \alpha(t), \ u(L,t) = \beta(t), & t \geq 0 \end{cases}$$



$R = \mathrm{I} \cup \mathrm{II} \cup \mathrm{III} \cup \cdots$

where j^{th} region contains characteristics directly above it

In I: Solution is given by d'Alembert formula

$$u(x,t) = \frac{1}{2}\left(g(x+t) + g(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} h(\xi)\,d\xi$$

In II: $u(A) = \underbrace{-u(C)}_{\text{known from I}} + \underbrace{u(B)}_{\text{known from BC, } \alpha(t)} + \underbrace{u(D)}_{\text{known from I}}$

Similarly we build up u in all regions based on the ones below.

As $A \to D$, $B, C \to 0$ and so we get

$$\cancel{u(A)} = -g(0) + \alpha(0) + u(A)$$

To have continuity in $\mathrm{I} \cup \mathrm{II}$, we need $\alpha(0) = g(0)$

Similarly we require $\alpha'(0) = h(0)$ in order for $u \in C'$

Inhomogeneous Wave Equation

$$(*) \quad \begin{cases} u_{tt} - c^2 u_{xx} = F(x,t) \\ u(x,0) = 0 = u_t(x,0) \end{cases}$$

Duhamel's principle reduces this problem to the following:

$$(**) \quad \begin{cases} v_{tt} - c^2 v_{xx} = 0 \\ v(x,0,s) = 0, \quad x \in \mathbb{R},\ s \geq 0 \\ v_t(x,0,s) = F(x,s), \quad x \in \mathbb{R},\ s \geq 0 \end{cases}$$

which has to be solved for each parameter value s.

Duhamel's principle: If $v(x,t,s)$ is C^2 in x,t and C^0 in s, and is a solution of $(**)$, then

$$(+) \quad u(x,t) = \int_0^t v(x,t-s,s)\,ds$$

solves $(*)$.

Proof: Define u by $(+)$. Then

$$u_t(x,t) = v(x,0,t) + \int_0^t v_t(x,t-s,s)\,ds = \int_0^t v_t(x,t-s,s)\,ds,$$

$$u_{tt}(x,t) = v_t(x,0,t) + \int_0^t v_{tt}(x,t-s,s)\,ds = F(x,t) + \int_0^t v_{tt}(x,t-s,s)\,ds,$$

$$u_{xx}(x,t) = \int_0^t v_{xx}(x,t-s,s)\,ds.$$

Therefore

$$u_{tt} - c^2 u_{xx} = F(x,t) + \int_0^t \Big(\overbrace{v_{tt}(x,t-s,s) - c^2 v_{xx}(x,t-s,s)}^{=0 \text{ by } (**)}\Big)\,ds = F(x,t).$$

Clearly

$$u(x,0) = \int_0^0 \cdots = 0, \quad u_t(x,0) = \int_0^0 \cdots = 0$$

and thus u is a solution to $(*)$. ∎

However, we would still need to solve $(**)$. In the 1-D case, we can use D'Alembert's formula to solve $(**)$:

$$v(x,t,s) = \frac{1}{2c}\int_{x-ct}^{x+ct} F(\xi,s)\,d\xi \quad \text{provided } F \text{ is } C^1 \text{ in } x \text{ and } C^0 \text{ in } t.$$

The solution to the original problem (*) is then

$$u(x,t) = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} F(\xi,s)\,d\xi\,ds$$

is a C^2 solution of (*). It's now straightforward to solve

$$\begin{cases} u_{tt} - c^2 u_{xx} = F(x,t) \\ u(x,0) = g(x) \\ u_t(x,0) = h(x) \end{cases}$$

by considering

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x,0) = g(x) \\ u_t(x,0) = h(x) \end{cases} \quad \text{and} \quad \begin{cases} u_{tt} - c^2 u_{xx} = F(x,t) \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases}$$

and then using linear superposition.

Wave Equation in more than 1-D

Consider the pure initial value problem:

The spherical mean or the average over a sphere of radius r and centre x (for $u(x)$ defined on $\mathbb{R}^n$) is:

(*) —— $$I_u(x,r) = \frac{1}{\omega_n}\int_{|\xi|=1} u(x + r\xi)\,dS_\xi .$$

ω_n = surface area of unit sphere

Since u is continuous in x, $I_u(x,r)$ is continuous in x and $r \geq 0$. Letting $r \to 0^+$, we have $I_u(x,0) = u(x)$.

If $u \in C^k(\mathbb{R}^n)$ then $I_u \in C^k(\mathbb{R}^n \times [0,\infty))$. Using the chain rule on (*), 2013 11 13

(**) — $$\frac{\partial}{\partial r} I_u(x,r) = \frac{1}{\omega_n}\int_{|\xi|=1} u_{x_i}(x + r\xi)\cdot \xi_i \, dS_\xi .$$

Let $\Omega = \{\xi \in \mathbb{R}^n;\ |\xi| < 1\}$. Then $\partial\Omega = S^{n-1}$. The exterior unit normal is $\underline{n} = \xi \quad (|\xi| = 1)$.

The RHS of (**) can be rewritten using the divergence theorem. Recall:

$$\int_\Omega \nabla\cdot V(\xi)\,d\xi = \int_{\partial\Omega} V\cdot n\, dS_\xi .$$

Take $V(\xi) = r^{-1}\nabla_\xi u(x + r\xi) = \nabla_x u(x + r\xi)$. Compute