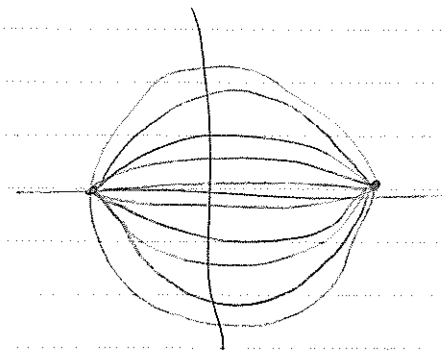
The level curves are given by

$$\frac{-2y}{1-x^2-y^2} = \frac{1}{c} \qquad \text{(c arbitrary constant)}$$

$$\Rightarrow 1-x^2-y^2 = -2cy$$

$$\Rightarrow x^2+(y-c)^2 = 1+c^2$$

All these circles pass through $(\pm 1, 0)$:



Alternatively, see chap 2 pg 30-31 and use the Poisson integral formula

new topic

Fourier Series and Eigenvalues of the Laplacian

$$(1) — u_{tt} = c^2 \Delta u \qquad (2) — u_t = k\Delta u$$

in a bounded domain $\Omega \subset \mathbb{R}^3$.

Let $u(x,y,z,t) = T(t)v(x,y,z)$. Then

$$(1) \to -\lambda = \frac{T''}{c^2 T} = \frac{\Delta v}{v}, \qquad -\lambda = \frac{T'}{kT} = \frac{\Delta v}{v} \leftarrow (2)$$

Both problems lead to the eigenvalue problem

$$\Delta v + \lambda v = 0 \quad \text{in } \Omega$$

with some Dirichlet, Neumann, or Robin condition on $\partial\Omega$.

If the problem has eigenvalues λ_n (all positive) and associated eigenfunctions $v_n(x)$, then solutions of (1) 2013 11 04

$$u(x,t) = \sum_{m,n} \left(a_n \cos(\sqrt{\lambda_n}ct) + b_n \sin(\sqrt{\lambda_n}ct)\right) v_m(x)$$

and for (2)

$$u(x,t) = \sum_n a_n e^{-\lambda_n k t} v_n(x).$$

$$(*) \quad \begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Note $(*)$ always admits the trivial solution $u=0$, but values of λ which admit nontrivial solutions.

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ex $\Omega : (0,a) \times (0,b)$

in this case $(*)$ becomes

$$\begin{cases} u_{xx} + u_{yy} + \lambda u = 0 \\ u(0,y) = u(a,y) = 0 & y \in (0,b) \\ u(x,0) = u(x,b) = 0 & x \in (0,a) \end{cases}$$

Let $u(x,y) = X(x)Y(y)$, do separating of variables to get

$$\lambda_{m,n} = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$$

$$u_{m,n} = \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \quad , \quad m,n = 1, \ldots$$

We see $\{\lambda_{m,n}\}_{m,n=1}^{\infty}$ forms a countable set of positive reals, with no accumulation points.

The eigenvalue $\lambda_{1,1}$ has only one eigenfunction (up to scaling).

If $a=b$, then $\lambda_{m,n} = \lambda_{n,m}$, so $\lambda_{n,m}$ has two eigenfunctions. $(n \neq m)$

Further,

$$\int_0^a \int_0^b u_{m,n} u_{m^*,n^*} \, dx\, dy = \begin{cases} 0 & \text{if } m \neq m^* \text{ or } n \neq n^* \\ ab/4 & \text{if } (m,n) = (m^*,n^*). \end{cases}$$

Recall, we can associate any $f(x,y)$ on $[0,a] \times [0,b]$ to a Fourier double sine series

$$f(x,y) \sim \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{m,n} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

where

$$A_{m,n} = \frac{4}{ab} \int_0^a \int_0^b f(x,y) \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) dx\,dy.$$

The double Fourier series can be considered to be an expansion in terms of the eigenfunctions of $(*)$.

We can study $(*)$ by various methods – one approach is to use the Green's function to rewrite $(*)$ as an integral equation

$$u(x) + \lambda \int_\Omega G(x,y)u(y)\,dy = 0.$$

Another approach is to look for solutions of $(*)$ using variational techniques, to deduce the following properties:

(i) eigenvalues of $(*)$ form a countable set $\{\lambda_n\}_{n=1}^\infty$ of positive real numbers with $\lambda_n \to \infty$ as $n \to \infty$;
(ii) for each eigenvalue λ_n, there exists a finite number of linearly independent eigenfunctions (known as the multiplicity of λ_n);
(iii) the first/principle eigenvalue λ_1 has multiplicity 1, and the corresponding eigenfunction u_1 does not change sign in Ω;
(iv) eigenfunctions corresponding to distinct eigenvalues are orthogonal;
(v) the eigenfunctions can be used to expand certain functions on Ω.

Note: $\{\lambda_n\}$ is the spectrum of the Laplacian wrt Dirichlet BCs

Change notation so that λ_n corresponds to the eigenfunction ϕ_n and

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \lambda_4 \leq \cdots \longrightarrow \infty$$

(repeating according to multiplicity).
Choosing an orthonormal basis of eigenfunctions in each eigenspace, we can arrange for $\{\phi_n\}_{n=1}^\infty$ to be orthonormal. Then

$$f(x) \sim \sum_{n=1}^\infty a_n \phi_n(x)$$

where

$$a_n = \int_\Omega f(x)\phi_n(x)\,dx.$$

Important consequences of (iv) is <u>uniqueness</u> of the eigenfunction expansion.
The sense of which $f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x)$ is convergence in the mean (ie L^2 norm):

$$\int_{\Omega} |f(\underline{x}) - \sum_{n=1}^{N} a_n \phi_n(\underline{x})| \, d\underline{x} \longrightarrow 0 \quad \text{as } N \to \infty.$$

Remark: Pointwise convergence of the eigenfunction expansion to f requires additional regularity (such as $f \in C^2(\Omega)$); then the eigenfunction expansion converges absolutely and uniformly.

ex
$$(1) - \ -\Delta u = f \quad \text{in } \Omega$$
$$(2) - \quad u = 0 \quad \text{on } \partial\Omega$$

If
$$f(\underline{x}) \sim \sum_{n=1}^{\infty} a_n \phi_n(\underline{x})$$
then substituting
$$u(\underline{x}) = \sum_{n=1}^{\infty} c_n \phi_n(\underline{x})$$
into (1) and comparing coefficients gives
$$c_n = -\frac{1}{\lambda_n} a_n$$
and so our solution is
$$(3) - \quad u(x) \sim \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n} \phi_n(\underline{x})$$
(assuming sufficient regularity of f for the expanssion to converge absolutely and uniformly on $\overline{\Omega}$ since the factor $1/\lambda_n$ only improves our convergence assumption of $f(\underline{x}) \sim \sum_{n=1}^{\infty} a_n \phi_n(\underline{x})$).

ex. Expansion of Green's function using eigenfunctions)

Fix $\underline{x}$ and try to expand Green's function $G(\underline{x};\underline{y})$ in the orthonormal eigenfunctions $\phi_n(\underline{y})$

$$\begin{cases} \Delta u = F & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$$u \sim \sum_n a_n \phi_n, \quad F \sim \sum_n c_n \phi_n$$

$$\Longrightarrow \sum a_n \Delta \phi_n = \sum c_n \phi_n$$

$$\Longrightarrow -\sum a_n \lambda_n \phi_n = \sum c_n \phi_n$$

$$\vdots$$

$$a_n = -\frac{c_n}{\lambda_n}$$

$$\therefore \quad u(\underline{x}) \sim \sum \frac{c_n}{\lambda_n} \phi_n = \sum \frac{\langle F, \phi_n \rangle}{\lambda_n \langle \phi_n, \phi_n \rangle} \phi_n$$

see course notes chap 2 38-41

$$= -\int_\Omega \left(\sum_{n=1}^{\infty} \frac{\phi_n(\underline{\xi}) \phi_n(\underline{x})}{\lambda_n \|\phi_n\|^2} \right) F(\underline{\xi}) \, d\underline{\xi}$$

$$= +\int_\Omega G(\underline{\xi}; \underline{x}) F(\underline{\xi}) \, d\underline{\xi}.$$

~~Note~~ $\cancel{\langle f, g \rangle = \int_\Omega f(\underline{x}) \overline{g(\underline{x})} \, d\underline{x}}$

So

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$$G(\underline{\xi}; \underline{x}) = -\sum_{n=1}^{\infty} \frac{\phi_n(\underline{\xi}) \phi_n(\underline{x})}{\lambda_n \|\phi_n\|^2}$$

Recall

$$\langle f, g \rangle := \int_\Omega f(\underline{x}) \overline{g(\underline{x})} \, d\underline{x}.$$

If u, v satisfy homogeneous (Dirichlet or Neumann or) Robin conditions:

$$u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} = u(\alpha v) - v(\alpha u) = 0$$

so by Green's second identity, (RHS vanishes)

$$\langle u, \Delta v \rangle = \langle \Delta u, v \rangle. \quad \text{—} \quad (*)$$

Now suppose u, v are real eigenfunctions of

$$\Delta u + \lambda_1 u = 0, \quad \Delta v + \lambda_2 v = 0$$

where u, v satisfy one of the 3 classical boundary conditions. Then

$$0 = \langle u, \Delta v\rangle - \langle \Delta u, v\rangle = (\lambda_1 - \lambda_2)\langle u, v\rangle$$

by (*). Therefore u and v are orthogonal if $\lambda_1 \neq \lambda_2$.
Note: eigenvalues are positive since

$$\int_\Omega (u\Delta u + |\nabla u|^2)\,d\underline{x} = \int_{\partial\Omega} u \frac{\partial u}{\partial n}\,dS,$$

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$$\Longrightarrow \lambda \int_\Omega u^2\,d\underline{x} = \int_\Omega |\nabla u|^2\,d\underline{x}$$

$$\Longrightarrow \lambda = \frac{\int_\Omega |\nabla u|^2\,d\underline{x}}{\int_\Omega u^2\,d\underline{x}} > 0$$

since $u \neq 0$ (as it is an eigenfunction).

Consider

$$\begin{cases} \Delta u = F \text{ in } \Omega & \text{— (1)} \\ u = 0 \text{ on } \partial\Omega. & \text{— (2)} \end{cases}$$

Since eigenfunctions (φ_n) are complete, let

$$u = \sum_n a_n \varphi_n, \qquad F = \sum_n c_n \varphi_n$$

where $c_n = \dfrac{\langle F, \varphi_n\rangle}{\|\varphi_n\|^2}$.

What about the a_n's? Substitute into (1) to see

$$\sum_n a_n \Delta\varphi_n = \sum_n c_n \varphi_n$$

$$\Longrightarrow -\sum_n a_n \lambda_n \varphi_n = \sum_n c_n \varphi_n$$

$$\Longrightarrow a_n = \frac{-c_n}{\lambda_n}.$$

Therefore

$$u = -\sum_n \frac{c_n}{\lambda_n}\varphi_n = -\sum_n \frac{\langle F, \varphi_n\rangle}{\lambda_n \|\varphi_n\|^2}\varphi_n.$$

$$= -\sum_n \int_\Omega \frac{F(\xi,\eta)\varphi_n(\xi,\eta)}{\lambda_n\|\varphi_n\|^2}\varphi_n(x,y)\,d\xi\,d\eta$$

$$= -\int_{\Omega}\left(\sum_n \frac{\varphi_n(\xi,\eta)\varphi_n(x,y)}{\lambda_n\|\varphi_n\|^2}\right)F(\xi,\eta)\,d\xi\,d\eta.$$

Therefore the Green's function representation is

$$u(\underline{x}) = \int_{\Omega} G(\underline{\xi};\underline{x})F(\underline{\xi})\,d\underline{\xi}.$$

By comparison,

$$G(\underline{\xi};\underline{x}) = -\sum_n \frac{\varphi_n(\underline{\xi})\varphi_n(\underline{x})}{\lambda_n\|\varphi_n\|^2}.$$

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new topic

Hyperbolic Equations

$$(*) \quad —— \quad u_{tt} - u_{xx} = 0$$

The characteristics are $x \pm t = \text{const.}$ Change of variables:

$$\xi = x+t, \quad \eta = x-t$$

leads to $(*)$ reducing to

$$u_{\xi\eta} = 0.$$

The general solution is then

$$u(\xi,\eta) = F(\xi) + G(\eta)$$

$$\Rightarrow u(x,t) = F(x+t) + G(x-t).$$

If $F \equiv 0$, then u is constant along lines $x - t = \text{const.}$, and so can be described as a wave propagating in the positive x-direction, with speed $\frac{dx}{dt} = 1$. If $G \equiv 0$, then we have a wave propagating in the negative x-direction with speed $\frac{dx}{dt} = -1$.

Consider the Cauchy Problem

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(x,0) = g(x) \\ u_t(x,0) = h(x) \end{cases}$$

Well

$$u(x,t) = F(x+t) + G(x-t) \quad —— \quad (**)$$

together with the above ICs yields