AMATH 453 - Partial Differential Equations II

2013 09 09

Cauchy Problem

Recall from ODEs that

$$u_t = F(t,u) \quad (1)$$

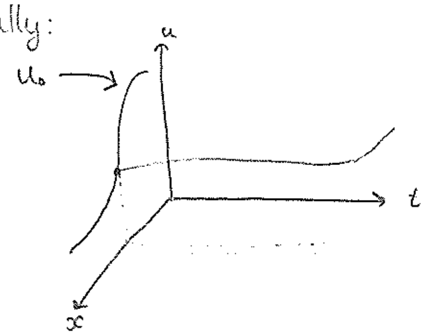
can be uniquely solved for each IC,

$$u(0) = u_0,$$

if F is continuous in t and Lipschitz continuous in u.
Here the solution may exist globally or blow-up in finite time.
Now let u, F, u_0 depend on x. Then $u = u(x,t)$ is the solution to the IVP:

$$\begin{cases} u_t = F(x,t,u) \\ u(x,0) = u_0(x) \end{cases} \quad (2)$$

If F and u_0 are continuous in x, then the solution will be continuous in x (and t). Geometrically:



The graph of u is a surface containing the initial curve, $(x, 0, u_0(x))$.
Note the surface may be defined for all $t>0$, or develop a fold or singularity at some finite time, which may depend on x.
These ideas underlying the "method of characteristics" which applies to general first order quasi-linear PDEs.

ex. (P1) - $\begin{cases} u_t + a u_x = 0 \\ u(x,0) = h(x) \end{cases}$

is called the transport equation. Find $x(t)$ such that

(*) $$\frac{d}{dt} u(x(t), t) = u_t + a u_x = 0.$$

Using chain rule, this is

$$u_t + \frac{dx}{dt} u_x = u_t + a u_x = 0.$$

That is, we want $\frac{dx}{dt} = a$, or $x = at + x_0$.

Along this curve, $u_t = 0$ (from (*)), so u = constant since $u(x_0, 0) = h(x_0) = h(x - at)$ along the curve.
Solution is initial data $h(x_0)$ "transported" along at $\frac{dx}{dt} = a$ without change.
The curves $x = at + x_0$ are the characteristic curves for (P1) (actually the base characteristics in the xt-plane).

Now consider PDEs of the form

$$\text{(P2)} \qquad a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} = c(x,y,u)$$

where x, y are independent variables and u is a dependent variables (and a, b, c are given smooth functions). We want to solve for $u(x,y)$.
This is a <u>first order quasi-linear PDE</u>.
If a, b are independent of u then (P2) is a <u>semi-linear PDE</u>.
If, in addition, $c(x,y,u) = \alpha(x,y)u + \beta(x,y)$ is a ~~linear~~ affine function of u, then (P2) is a <u>linear PDE</u>.
Consider the graph of a solution of (P2):

$$z = u(x,y)$$

This surface has normal $\underline{n}_0 = \left(-\frac{\partial u}{\partial x}\Big|_{(x_0,y_0)}, -\frac{\partial u}{\partial y}\Big|_{(x_0,y_0)}, 1\right)$ at $(x_0, y_0, u(x_0,y_0))$.
Let $z_0 = u(x_0, y_0)$.
From (P2), it is clear that $V(x_0,y_0,z_0) = (a(x_0,y_0,z_0), b(x_0,y_0,z_0), c(x_0,y_0,z_0))$ is perpendicular to the normal $\underline{n}_0$. Thus $V(x,y,z)$ defines a vector field in $\mathbb{R}^3$ to which graphs of solutions of (P2) must be tangent at each point.

<u>Def</u>) Surfaces that are tangential at each point to a vector field V in $\mathbb{R}^3$ are known as <u>integral surfaces</u>.
Curves that are tangential at each point to a vector field V in $\mathbb{R}^3$ are known as <u>integral curves</u>.

To solve (P2), we must find the integral surface $z = u(x,y)$. 2013 09 11
In general, there are many integral surfaces, but we can be more specific and find the integral surface containing a given curve $\Gamma \subset \mathbb{R}^3$. This leads to a formulation of the <u>Cauchy Problem</u>:

Given a curve $\Gamma \subset \mathbb{R}^3$, find a solution $z = u(x,y)$ of (P2) whose graph contains Γ.

Note that in the special case $\Gamma = (x, 0, h(x))$, the Cauchy Problem is just an initial value problem. (Hence we have generalized from the motivating ODE example we started with)

Construction of integral surface

definition of characteristic

We can use the characteristic curves, which are the integral curves of the vector field (a,b,c). That is, $(x(t), y(t), z(t))$ is a characteristic if it satisfies

$$\frac{dx}{dt} = a(x,y,u), \quad \frac{dy}{dt} = b(x,y,u), \quad \frac{dz}{dt} = c(x,y,u),$$

for a vector field the curve B in $\mathbb{R}^3$

as well as

$$x(t_0) = x_0, \quad y(t_0) = y_0, \quad u(t_0) = u_0. \qquad \text{(what are } x_0, y_0, u_0\text{?)}$$

We assume that a, b, c are C^1 functions of (x,y,u).
These can be solved locally for small $|t - t_0|$.
If the graph of $z = u(x,y)$ is a smooth surface S which is the union of such characteristics, then at each (x_0, y_0, z_0) the tangent plane contains $V(x_0, y_0, z_0)$. Hence S is an integral surface. (That is, a smooth union of characteristics is an integral surface)

$$\Gamma: (f(s), g(s), h(s))$$

with ICs: $x_0 = f(s), \quad y_0 = g(s), \quad z_0 = h(s)$

From this we obtain an integral surface parameterized by s and t.
To obtain the solution $u(x,y)$ of (P2), we replace s and t by expressions involving x and y.

Note that it is always true that an integral surface $z = u(x,y)$ of the vector field $V = (a,b,c)$ is always a union of characteristic curves.

Theorem: If Γ is non-characteristic for

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u)$$

then there exists a unique solution of the Cauchy Problem near Γ.

ex. Solve

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 1$$

for $u(x,t)$ in $t>0$, subject to the IC $u=x$ on $t=0$.

Solution: char. eqns: $\frac{dt}{d\tau} = 1$, $\frac{dx}{d\tau} = u$, $\frac{du}{d\tau} = 1$

$$\Gamma: (0,s,s)$$

So we get

$$t=0, \quad x = x_0 = s, \quad u = u_0 = s.$$

Solving for t first yields $t = \tau$. The third equation gives $u = \tau + s = t + s$ ⊛ when combined with the IC. The second equation becomes

$$\frac{dx}{d\tau} = \tau + s, \quad x(0) = s$$

which has solution $x = s + s\tau + \frac{1}{2}\tau^2 = \frac{1}{2}t^2 + st + s$.

Now we want to substitute to get the solution in terms of u, x, y.

The above says

$$s = \frac{x - \frac{1}{2}t^2}{1+t}$$

so we find by ⊛ that

$$u(x,t) = \frac{x + t + \frac{1}{2}t^2}{1+t}.$$

Alternatively, the characteristic equations can be written as

$$\frac{dx}{a(x,y,u)} = \frac{dy}{b(x,y,u)} = \frac{du}{c(x,y,u)}.$$

If we can "spot" two first* integrals of these ODEs, ie

$$f(x,y,u) = C_1 \quad \text{(constant)},$$
$$g(x,y,u) = C_2 \quad (\text{"}),$$

then the general solution of the PDE can be written implicitly as

$$f(x,y,u) = F(g(x,y,u))$$

where F is arbitrary.

ex. Returning to the previous example, we would have

$$\frac{dt}{1} = \frac{dx}{u} = \frac{du}{1}.$$

These can be arranged as

$$\frac{du}{dt} = 1, \quad \frac{dx}{dt} = u.$$

Solving gives

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$$u = t + c_1 \Rightarrow c_1 = \underbrace{u - t}_{f}, \qquad dx/dt = t + c_1 \Rightarrow x = \tfrac{1}{2}t^2 + c_1 t + c_2$$

$$\Rightarrow x = \tfrac{1}{2}t^2 + (u-t)t + c_2$$

$$\Rightarrow c_2 = \underbrace{x - \tfrac{1}{2}t^2 - (u-t)t}_{g}$$

The general solution is found by ~~~~ setting $f = F(g)$.
Hence

$$u - t = F(\tfrac{1}{2}t^2 - ut + x)$$

$$\Rightarrow u = t + F(x + \tfrac{1}{2}t^2 - ut).$$

Using IC gives $u = F(x)$ and we know $u = x$ here so $F(x) = x$.
Thus

$$\Rightarrow u = t + x + \tfrac{1}{2}t^2 - ut$$

$$\Rightarrow u(x,t) = \frac{t + x + \frac{1}{2}t^2}{1+t}$$

as we had before.

Why does this work?
Well $f(x,y,u) = c_1$ defines a one parameter family of solutions surfaces. Similarly $g(x,y,u) = c_2$ defines another one parameter family of solution surfaces.
Any surface defined by an equation of the form $f = F(g)$ has the property that f is constant whenever g is constant. Hence the surface is composed of the union of a family of characteristics, and is therefore a solution of the PDE.

ex Find the general solution for:

$$yu\frac{\partial u}{\partial x} + -xu\frac{\partial u}{\partial y} = x - y$$

Solution: The characteristic equations are

$$\frac{dx}{dt} = yu, \quad \frac{dy}{dt} = -xu, \quad \frac{dz}{dt} = x - y$$

Note $\frac{d}{dt}(x^2+y^2) = 0$ and $\frac{d}{dt}(\tfrac{1}{2}u^2 + x + y)$

So $\underbrace{x^2+y^2}_{g} = c_1$ and $\underbrace{\tfrac{1}{2}u^2 + x+y}_{f} = c_2$.

So $\tfrac{1}{2}u^2+x+y = F(x^2+y^2) \Rightarrow u^2 = -2x-2y+F^*(x^2+y^2)$

Generalization to n-independent variables

$$\sum_{i=1}^{n} a_i(x_1, \dots, x_n, u)\frac{\partial u}{\partial x_i} = c(x_1, \dots, x_n, u)$$

The characteristic curves are (the integral curves) given by the system of $n+1$ ODEs in $n+1$ unknowns:

$$\frac{dx_i}{dt} = a_i(x_1, \dots, x_n, u) \qquad i=1, \dots, n$$

$$\frac{du}{dt} = c(x_1, \dots, x_n, u)$$

This can be solved if we are given initial conditions on an $(n-1)$-dimensional manifold:

$$x_i = f_i(s_1, \dots, s_{n-1}) \qquad i=1, \dots, n$$
$$u = h(s_1, \dots, s_{n-1})$$

This will generate an n-dimensional integral manifold M parameterized by $(s_1, \dots, s_{n-1}, t)$. The solution $u(x_1, \dots, x_n)$ is obtained by solving for $(s_1, \dots, s_{n-1}, t)$ in terms of $(x_1, \dots, x_n)$

Semi-linear DEs

$$\left(\sum_{|\alpha|=k} a_\alpha(\underline{x}) D^\alpha \{u\}\right) + a_0(x, u, Du, \dots, D^{k-1}u) = 0$$

$\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_i \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$

$|\alpha| = \alpha_1 + \dots + \alpha_n$

$\underline{x} = (x_1, \dots, x_n)$

$\underline{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$

$\alpha! = (\alpha_1!)(\cdot)\cdots(\ )(\alpha_n!)$

$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\left(\quad\right) \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$ (composition)