Finite Element Method

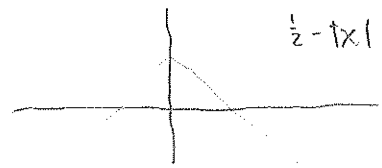
2015 11 09

start with 1D:

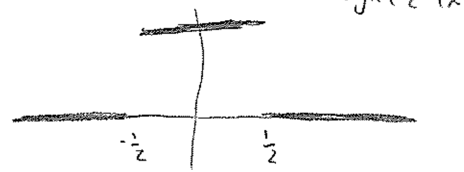
$$\begin{cases} -u'' = f, & x \in (-1,1) \\ u(-1) = u(1) = 0 \end{cases}$$

$u \in C^2(\Omega) \cap C^1(\bar{\Omega})$

u is a classical solution



$$\begin{cases} -u'' = \operatorname{sgn}\left(\frac{1}{2} - |x|\right), & x \in (-1,1) \\ u(-1) = u(1) = 0 \end{cases}$$



- FD $\rightarrow$ approximate the partial derivatives

$$\left.\frac{\partial u}{\partial t}\right|_{x_j, t_n} \approx \frac{U_j^{n+1} - U_j^n}{\Delta t}$$

- FV $\rightarrow$ discretized integral form of the PDE
- FEM $\rightarrow$ discretize the weak form of the PDE

Model Problem

$$\begin{cases} \frac{d}{dx}\left(a(x)\frac{du}{dx}\right) + \frac{d}{dx}\left(b(x)u\right) + c(x)u = f(x), & x \in \Omega \\ u(0) = \alpha \\ a(1)\frac{du}{dx}(1) + b(1)u(1) = \beta \end{cases}$$

$\Omega = (0,1)$, $a, b \in C^1(\bar{\Omega})$, $c, f \in C^0(\bar{\Omega})$

don't worry too much, can be relaxed

- v is a sufficiently smooth function
- first assume that u is a classical solution
- $v(x)\left(\frac{d}{dx}\left(a(x)\frac{du}{dx}\right) + \frac{d}{dx}(b(x)u) + c(x)u(x)\right) = v(x)f(x)$
- Integrate over Ω

$$\int_\Omega v(x)\left(\frac{d}{dx}\left(a(x)\frac{du}{dx}\right) + \frac{d}{dx}(b(x)u) + c(x)u(x)\right) = \int_\Omega v(x)f(x)\,dx$$

- Integrate by parts

$$\int_0^1 \left(-a(x)\frac{du}{dx}\frac{dv}{dx} - b(x)u(x)\frac{dv}{dx} + v(x)u(x)c(x)\right)dx + v(x)\left(a(x)\frac{du}{dx} + b(x)u(x)\right)\Big|_0^1 = \int_0^1 v(x)f(x)\,dx$$

$$\int_0^1 \left(-a(x)\frac{du}{dx}\frac{dv}{dx} - b(x)u(x)\frac{dv}{dx} + v(x)u(x)c(x)\right)dx$$

$$+ v(1)\beta - v(0)\left(\alpha(0)\frac{du}{dx}(0) + b(0)u(0)\right) = \int_0^1 fv\,dx$$

We are looking for

$$u \in V_\alpha = \{v \in C^1(\bar{\Omega})\ ;\ v(0) = \alpha\}$$

for all $v \in V_0$.

Define the bilinear form

$$a(u,v) : V_\alpha \times V_0 \to \mathbb{R}$$

and linear form

$$l(v) : V_0 \to \mathbb{R}$$

as

$$a(u,v) = \int_0^1 \left(-a(x)\frac{du}{dx}\frac{dv}{dx} - b(x)u(x)\frac{dv}{dx} + u(x)v(x)c(x)\right)dx$$

$$l(v) = \int_0^1 f(x)v(x)\,dx - v(1)\beta$$

Weak formulation

Find $u \in V_\alpha$ such that

$$a(u,v) = l(v) \quad \forall v \in V_0.$$

Terminology

- The arbitrary function $v \in V_0$ is also known as test function
- The function $u \in V_\alpha$ is known as trial function
- b.c. $u(0) = \alpha$ is called essential or Dirichlet b.c.
  (appears explicitly in the weak formulation through definition of V_α)
- b.c. $a(1)\frac{du}{dx}(1) + b(1)u(1) = \beta$ is called a natural or Neumann b.c.
  (the b.c. is incorporated implicitly into the weak formulation)
- if u satisfies the weak formulation it is called a weak solution of the PDE
- if u is a classical solution, it also satisfies the weak formulation
- the reverse is not true: a solution that satisfies the weak form is not necessarily a classical solution

To implement the weak formulation we introduce finite-dimensional subspaces of V_α and V_0.

$$V_{h,\alpha} \subset V_\alpha \qquad V_{h,0} \subset V_0$$

finite dimensional

Let these finite dimensional subspaces be N-dimensional vector spaces with basis

$$\{\varphi_1, \dots, \varphi_N\}$$

$$V_{h,\alpha} = \text{span}\{\varphi_1, \dots, \varphi_N\}$$

$u_h \in V_{h,\alpha}$ can be written as

$$u_h = \alpha \varphi_0(x) + \sum_{j=1}^{N} U_j \varphi_j(x)$$

$v_h \in V_{h,0}$

$$v_h = \sum_{j=1}^{N} V_j \varphi_j(x)$$

The FEM:

Find $u_h = V_{h,\alpha}$ such that

$$a(u_h, v_h) = \ell(v_h) \quad \forall v_h \in V_{h,0}$$

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How do we get a matrix?

$$a(u_h, v_h) = \int_0^1 \left(-a(x) \frac{du_h}{dx} \frac{dv_h}{dx} - b(x) u_h(x) v_h(x) + u_h(x) c(x) u_h(x)\right) dx$$

$$\ell(v_h) = \int_0^1 f(x) v_h(x)\, dx - v_h(1) \beta$$

Substitute definitions of u_h and v_h:

$$a(u_h, v_h) = \int_0^1 \left(-a(x) \frac{d}{dx}\left(\sum_{j=1}^{N} U_j \varphi_j\right) \frac{d}{dx}\left(\sum_{i=1}^{N} V_i \varphi_i\right) - b(x)\left(\sum_{j=1}^{N} U_j \varphi_j\right)\left(\sum_{i=1}^{N} V_i \varphi_i\right)\right.$$

$$\left. + c(x)\left(\sum_{j=1}^{N} U_j \varphi_j\right)\left(\sum_{i=1}^{N} V_i \varphi_i\right)\right) dx$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} V_i U_j \int_0^1 \left(-a(x) \frac{d\varphi_j}{dx} \frac{d\varphi_i}{dx} - b(x) \varphi_j \frac{d\varphi_i}{dx} + c(x) \varphi_j(x) \varphi_i(x)\right) dx$$

$$\ell(V_h) = \int_0^1 f(x)\left(\sum_{i=1}^N V_i \varphi_i\right)dx - \left(\sum_{i=1}^N \varphi_i(1) V_i\right)\beta$$
$$= \sum_{i=1}^N V_i\left(\int_0^1 f(x)\varphi_i\,dx - \varphi_i(1)\beta\right)$$

$$\sum_{i=1}^N \sum_{j=1}^N V_i U_j \int_0^1 \left(-a(x)\frac{d\varphi_j}{dx}\frac{d\varphi_i}{dx} - b(x)\varphi_j \frac{d\varphi_i}{dx} + c(x)\varphi_j(x)\varphi_i(x)\right)dx$$
$$= \sum_{i=1}^N V_i\left(\int_0^1 f(x)\varphi_i(x)\,dx - \varphi_i(1)\beta\right)$$

holds for all test functions

Find U_j, $j=1,\dots,N$ such that

$$\sum_{j=1}^N U_j \int_0^1\left(-a(x)\frac{d\varphi_j}{dx}\frac{d\varphi_i}{dx} - b(x)\varphi_j(x)\frac{d\varphi_i}{dx} + c(x)\varphi_j\varphi_i\right)dx$$
$$= \int_0^1 f(x)\varphi_i(x)\,dx - \varphi_i(1)\beta.$$

for all $i=1,\dots,N$.

Define

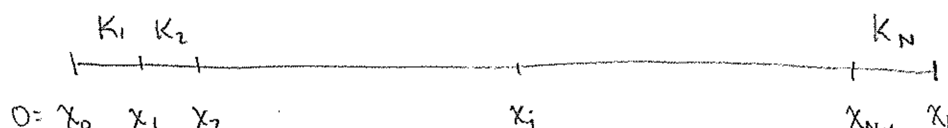
$$a_{ij} = \int_0^1\left(-a(x)\frac{d\varphi_j}{dx}\frac{d\varphi_i}{dx} - b(x)\varphi_j\frac{d\varphi_i}{dx} + c(x)\varphi_j\varphi_i\right)dx$$
$$F_i = \int_0^1 f(x)\varphi_i(x) - \varphi_i(1)\beta$$

then

$$\sum_{j=1}^N a_{ij} U_j = F_i$$

for $i=1,\dots,N$ if and only if $AU=F$, where $A=[a_{ij}]\in\mathbb{R}^{N\times N}$ and $F=[F_i]\in\mathbb{R}^N$ and $U=[U_i]\in\mathbb{R}^N$

Grid



Define an element $K_j=[x_{j-1},x_j]$, define length of element $h_j=x_j-x_{j-1}$.

We need a basis for $V_{h,a}$. We use continuous <u>piecewise</u> polynomial approximations

$$V_{h,\alpha}^k = \{ u \in C^0;\ u \in P^k(K_j)\ \forall K_j \in \Omega,\ u(0) = \alpha \}$$

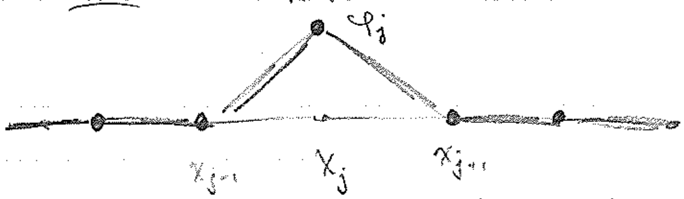
continuous — is a polynomial of degree k in element K_j

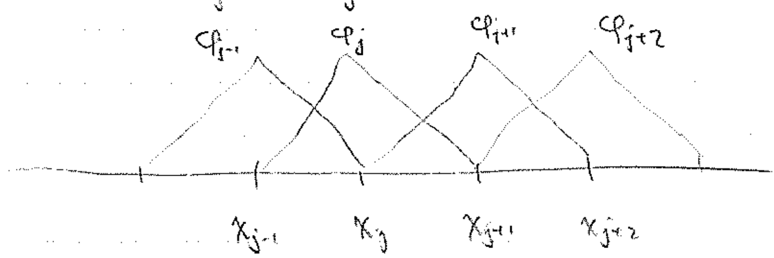
The basis functions are defined as $\varphi_j \in P^k(K_j)$ if $x \in K_i$ with the condition that $\varphi_i(x_j) = \delta_{ij}$

$\Rightarrow \varphi_j$ is a polynomial in K_i and continuous at element boundaries

$$\varphi_j(x) = \begin{cases} 1 & x = x_j \\ 0 & x = x_k,\ k \neq j \end{cases}$$

Example <u>linear</u> basis functions





$$\varphi_j = \begin{cases} \dfrac{x - x_{j-1}}{x_j - x_{j-1}} & x \in K_j \\[2ex] \dfrac{x_{j+1} - x}{x_{j+1} - x_j} & x \in K_{j+1} \\[2ex] 0 & \text{otherwise} \end{cases}$$

These basis functions are defined on the <u>whole</u> domain $\rightarrow$ they are called <u>global</u> basis functions.

For implementation reasons, we introduce <u>local</u> basis functions, defined only on 1 element (K_j)

$$\psi_{j,1} = \frac{x_j - x}{x_j - x_{j-1}} \qquad \psi_{j,2} = \frac{x - x_{j-1}}{x_j - x_{j-1}}$$

then on element K_j,

local $\longrightarrow$ $\psi_{j,1}(x) = \varphi_{j-1}(x)$ $\longleftarrow$ global

$\psi_{j,2}(x) = \varphi_j(x)$

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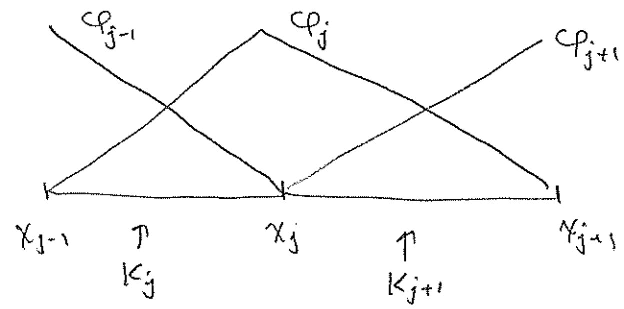
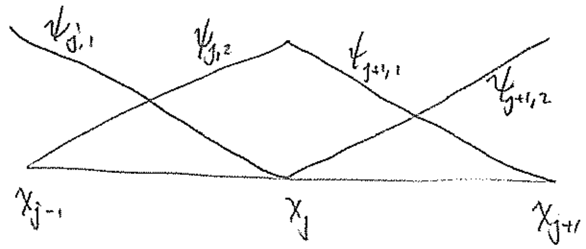


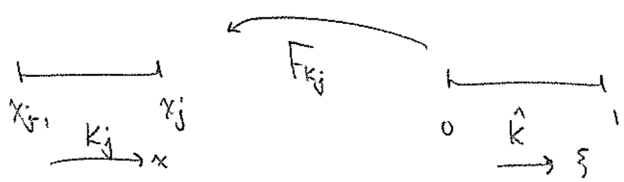
(global)



(local)

Introduce a reference element $\hat{K} = [0,1]$, with local coordinates ξ.
To map between physical element K_j and reference element $\hat{K}$ we need a mapping F_{K_j}

$$F_{K_j} : (0,1) \longrightarrow K_j : \xi \longmapsto x = h_j \xi + x_{j-1}$$



$F_{K_j}(0) = x_{j-1}$ $\quad$ $F_{K_j}(1) = h_j + x_{j-1} = (x_j - x_{j-1}) + x_{j-1} = x_j$ $\qquad$ 2015 11 13

For $\xi \in (0,1) \longrightarrow F_{K_j}(\xi) \in (x_{j-1}, x_j)$

$$\psi_{j,2}(x) = \frac{x - x_{j-1}}{x_j - x_{j-1}} = \frac{x - x_{j-1}}{h_j} = \frac{h_j \xi + x_{j-1} - x_{j-1}}{h_j} = \xi$$

$$\psi_{j,1}(x) = \cdots = 1 - \xi$$

On the reference element the local basis functions are

$$\hat{\psi}_1(\xi) = 1 - \xi \quad , \quad \hat{\psi}_2(\xi) = \xi$$

These are related to $\psi_{j,m}(x)$ $(m = 1, 2)$ by

$$\hat{\psi}_m(\xi) = \hat{\psi}_m(F_{K_j}^{-1}(x)) = \psi_{j,m}(x)$$

Consider the element $K_k = [x_{k-1}, x_k]$

$$u_h = \sum_{j=1}^{N} \varphi_j(x) U_j + \alpha \varphi_0(x)$$

but $\varphi_j(x) = 0$ almost everywhere except on those two elements connected to x_j.

Restrict u_h to K_k we write u_h in terms of terms of local basis functions

$$u_h|_{K_k} = \begin{cases} \sum_{m=1}^{2} U_m^{(k)} \psi_{k,m}(x), & k=2,\dots,N \\ \alpha \psi_{k,1}(x) + U_2^{(k)} \psi_{k,2}(x) & k=1 \end{cases}$$

Remember

$\psi_{k,1}(x)$ corresponds to $\varphi_{k-1}(x)$
$\psi_{k,2}(x)$ corresponds to $\varphi_k(x)$

Likewise

$U_1^{(k)}$ corresponds to U_{k-1}
$U_2^{(k)}$ corresponds to U_k

$\uparrow$ local $\qquad \uparrow$ global

Local element matrices A_k and vectors F_k

$A_k = [a_{i,j}^{(k)}] \in \mathbb{R}^{2\times 2}$

$$a_{ij}^{(k)} = \int_{x_{k-1}}^{x_k} \left(-a(x)\frac{d\psi_{k,j}}{dx}\frac{d\psi_{k,i}}{dx} - b(x)\psi_{k,j}\frac{d\psi_{k,i}}{dx} + c(x)\psi_{k,j}(x)\psi_{k,i}(x)\right)dx$$

$F_k = [F_i^{(k)}] \in \mathbb{R}^2$

$$F_i^{(k)} = \int_{x_{k-1}}^{x_k} f(x)\psi_{k,i}(x) - \delta_{kN}\beta\psi_{k,i}(x=1)$$

Transform to reference element $\hat{K}$

$$a_{ij}^{(k)} = \int_{x_{k-1}}^{x_k} \left(-a(x)\frac{d\psi_{k,j}}{dx}\frac{d\psi_{k,i}}{dx} - b(x)\psi_{k,j}\frac{d\psi_{k,i}}{dx} + c(x)\psi_{k,j}(x)\psi_{k,i}(x)\right)dx$$

$x = h_j\xi + x_{j-1} \longrightarrow dx = h_j\, d\xi$

$x = x_{j-1} \longleftrightarrow \xi = 0$

$x = x_j \longleftrightarrow \xi = 1$

$\psi_{k,j}(x) = \hat{\psi}_j(F_{k,j}^{-1}(x)) = \hat{\psi}_j(\xi)$

$$\frac{d\psi_{k,j}(x)}{dx} = \frac{1}{h_j}\frac{d\hat{\psi}_j(\xi)}{d\xi}$$

$$a_{ij}^{(k)} = \int_0^1 \left(-a(x(\xi))\frac{1}{h_k^2}\frac{d\hat{\psi}_j}{d\xi}\frac{d\hat{\psi}_i}{d\xi} - b(x(\xi))\frac{1}{h_k}\hat{\psi}_j(\xi)\frac{d\hat{\psi}_i}{d\xi} + c(x(\xi))\hat{\psi}_j(\xi)\hat{\psi}_i(\xi)\right)h_k\, d\xi$$

$$F_{ij}^{(k)} = \begin{cases} \int_0^1 f(x(\xi))\hat{\psi}_i(\xi)h_k\, d\xi & k=1,2,\dots,N-1 \\ \int_0^1 f(x(\xi))\hat{\psi}_i(\xi)h_k\, d\xi - \beta\hat{\psi}_i(\xi=1), & k=N \end{cases}$$

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$$A^{(k)} U^{(k)} = F^{(k)}$$

$$\begin{bmatrix} a_{11}^{(k)} & a_{12}^{(k)} \\ a_{21}^{(k)} & a_{22}^{(k)} \end{bmatrix} \underbrace{\begin{bmatrix} U_1^{(k)} \\ U_2^{(k)} \end{bmatrix}} = \begin{bmatrix} F_1^{(k)} \\ F_2^{(k)} \end{bmatrix}$$

in terms of global expansion coeffs $\longrightarrow \begin{bmatrix} U_{k-1} \\ U_k \end{bmatrix}$ $\quad k = 2, 3, 4, \dots, N$

$$\begin{bmatrix} \cancel{a_{11}^{(1)}} \to 0 & \cancel{a_{12}^{(1)}} \to 0 \\ a_{21}^{(1)} & a_{22}^{(1)} \end{bmatrix} \begin{bmatrix} \alpha \\ U_1 \end{bmatrix} = \begin{bmatrix} F_1^{(1)} \\ F_2^{(1)} \end{bmatrix}$$

$a_{11}^{(1)} = a_{12}^{(1)} = 0$ because $\varphi_0 = 0$ (test functions)

$$a_{22}^{(1)} U_1 = F_2^{(1)} - a_{21}^{(1)} \alpha$$

Global System $AU = F$, $U = [U_1\ U_2\ \cdot\ U_N]^T$

$$\begin{bmatrix} a_{11}^{(2)} & a_{12}^{(2)} & & & \\ a_{21}^{(2)} & a_{22}^{(2)} + a_{11}^{(3)} & a_{12}^{(3)} & & \\ & a_{21}^{(3)} & a_{22}^{(3)} + a_{11}^{(4)} & a_{12}^{(4)} & \\ & & a_{21}^{(4)} & a_{22}^{(4)} + a_{11}^{(5)} & \ddots \\ & & & \ddots & \ddots \end{bmatrix} \begin{bmatrix} U_1 \\ \cdot \\ \cdot \\ \cdot \\ U_N \end{bmatrix}$$

$$A_{j-1,j} = a_{21}^{(j)}$$
$$A_{jj} = a_{22}^{(j)} + a_{11}^{(j+1)}$$
$$A_{j,j+1} = a_{12}^{(j+1)}$$

$j = 1$: $A_{j,j} = a_{22}^{(j)} + a_{11}^{(j+1)}$, $\quad A_{j,j+1} = a_{12}^{(j+1)}$

$j = N$: $A_{j-1,j} = a_{21}^{(j)}$, $\quad A_{j,j} = a_{22}^{(j)}$

$$F = \begin{bmatrix} F_2^{(1)} + F_1^{(2)} - a_{21}^{(1)} \alpha \\ F_2^{(2)} + F_1^{(3)} \\ \vdots \\ F_2^{(k)} + F_1^{(k+1)} \\ \\ F_2^{(N-1)} + F_1^{(N)} \\ F_2^{(N)} - \beta \end{bmatrix}$$

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$$\begin{cases} -u'' = c & x \in (0,1), \quad c = \text{constant} \\ u(0) = \alpha & \text{(Dirichlet bc)} \\ -u'(1) = \beta & \text{(Neumann bc)} \end{cases}$$

FEM

Find $u_h \in V'_{h,\alpha}$ st $a(u_h, v_h) = f(v_h) \quad \forall v_h \in V'_{h,0}$

$$a(u_h, v_h) = \int_0^1 u_h'(x) v_h'(x)\, dx = \sum_{k=1}^{N} \int_{x_{k-1}}^{x_k} u_h'(x) v_h'(x)\, dx$$

$$f(v_h) = \int_0^1 c\, v_h(x)\, dx - v(1)\beta = \sum_{k=1}^{N} \int_{x_{k-1}}^{x_k} c\, v_h(x)\, dx - v(1)\beta$$

On the element K_k we have a local system $A_k U^{(k)} = F_k$

$$A_k = \left[a_{ij}^{(k)}\right] \quad \text{where} \quad a_{ij}^{(k)} = \int_{x_{k-1}}^{x_k} \frac{d\psi_{k,i}}{dx} \frac{d\psi_{k,j}}{dx}\, dx$$

$$F_k = \left[F_i^{(k)}\right] \quad \text{where} \quad F_i^{(k)} = \int_{x_{k-1}}^{x_k} c\, \psi_{k,i}\, dx - \delta_{kN} \beta\, \psi_{k,j}(1)$$

On the reference element $\hat{K} = (0,1)$

$$a_{ij}^{(k)} = \int_0^1 \frac{1}{h_k} \frac{d\hat{\psi}_j}{d\xi} \frac{d\hat{\psi}_i}{d\xi}\, d\xi$$

$$F_i^{(k)} = \int_0^1 c\, \hat{\psi}_i(\xi)\, h_k\, d\xi - \beta\, \hat{\psi}_i(\xi = 1)\, \delta_{kN}$$

Assume uniform grid: $h_k = h$

On $\hat{K}$: $\hat{\psi}_1 = 1 - \xi, \quad \hat{\psi}_2 = \xi$

$$\longrightarrow \frac{d\hat{\psi}_1}{d\xi} = -1, \quad \frac{d\hat{\psi}_2}{d\xi} = +1$$

$$\left.\begin{aligned} a_{11}^{(k)} &= \int_0^1 \frac{1}{h}(-1)(-1)\, d\xi = \frac{1}{h} & F_1^{(k)} &= \int_0^1 ch(1-\xi)\, d\xi = \frac{1}{2}ch \\ a_{12}^{(k)} &= \int_0^1 \frac{1}{h}(+1)(-1)\, d\xi = -\frac{1}{h} & F_2^{(k)} &= \int_0^1 ch\, \xi\, d\xi = \frac{1}{2}ch \end{aligned}\right\} k = 1, \ldots, N-1$$

$$a_{21}^{(k)} = \int_0^1 \frac{1}{h}(-1)(+1)\, d\xi = -\frac{1}{h} \qquad F_1^{(N)} = \int_0^1 ch(1-\xi)\, d\xi - \beta(0) = \frac{1}{2}ch$$

$$a_{22} = \int_0^1 \frac{1}{h}(+1)(+1)\, d\xi = \frac{1}{h} \qquad F_2^{(N)} = \int_0^1 ch\, \xi\, d\xi - \beta(1) = \frac{1}{2}ch - \beta$$

Global Matrix

$$\left.\begin{array}{lll} A_{j-1,j} = a_{21}^{(j)} = -\frac{1}{h}, & A_{j,j} = a_{22}^{(j)} + a_{11}^{(j+1)} = \frac{2}{h}, & A_{j,j+1} = a_{12}^{(j+1)} = -\frac{1}{h} \end{array}\right\} j = 2, \dots, N-1$$

$\underline{j=1}$: $A_{jj} = a_{22}^{(j)} + a_{11}^{(j+1)} = 2/h$

$A_{j,j+1} = a_{12}^{(j)} = -1/h$

$\underline{j=N}$: $A_{j-1,j} = a_{21}^{(j)} = -1/h$

$A_{jj} = a_{22}^{(j)} = 1/h$

$\underline{j=1}$: $F_j = F_2^{(1)} + F_1^{(2)} - a_{21}^{(1)}\alpha = ch + \alpha/h$

$\underline{j=2,\dots,N-1}$: $F_j = F_2^{(j)} + F_1^{(j+1)} = ch$

$\underline{j=N}$: $F_j = F_2^{(j)} - \beta = \frac{1}{2}ch - \beta$

$$\frac{1}{h}\begin{bmatrix} 2 & -1 & 0 & \cdots & & & & 0 \\ -1 & 2 & -1 & 0 & \cdots & & & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & & & \vdots \\ 0 & \cdots & 0 & -1 & 2 & -1 & 0 & 0 \\ & & & \ddots & \ddots & \ddots & & 0 \\ \vdots & & & & 0 & -1 & 2 & -1 \\ 0 & & \cdots & & & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_k \\ \vdots \\ U_{N-1} \\ U_N \end{bmatrix} = \begin{bmatrix} ch + \alpha/h \\ ch \\ \vdots \\ ch \\ \vdots \\ ch \\ \frac{1}{2}ch - \beta \end{bmatrix}$$

2-D

$$\begin{cases} -\Delta u = f, & \vec{x} \in \Omega \subset \mathbb{R}^2 \\ u = g_0, & \vec{x} \in \partial\Omega_0 \\ \nabla u \cdot \vec{n} = g_N, & \vec{x} \in \partial\Omega_N \end{cases}$$

Notation:
- $\Delta u = \nabla^2 u = \nabla \cdot \nabla u = \partial_{xx} u + \partial_{yy} u$
- $\vec{x} = (x, y)$
- Domain Ω, boundary $\partial\Omega$
- the boundary is decomposed as

$$\underset{\text{Dirichlet}}{\partial\Omega = \partial\Omega_0} \cup \underset{\text{Neumann}}{\partial\Omega_N}, \quad \partial\Omega_0 \cap \partial\Omega_N = \emptyset$$

assumption: $\partial\Omega_N = \emptyset$

Model Problem

$$\begin{cases} -\Delta u = f, & \vec{x} \in \Omega \\ u = g_D, & \vec{x} \in \partial\Omega \end{cases}$$

u is a classic solution if $u \in C^2(\Omega)$

We need a weak formulation.

Let v be a "smooth enough" <u>test</u> function.

If u is a classical solution to the model problem, then u satisfies

$$\int_\Omega v(\Delta u + f)\,dx = 0$$

With v being smooth enough, the smoothness requirement of u can be reduced

$$\int_\Omega v f\,dx = -\int_\Omega v\,\Delta u\,dx$$

(Note: $\nabla\cdot(v\nabla u) = v\nabla\cdot\nabla u + \nabla v\cdot\nabla u = v\Delta u + \nabla v\cdot\nabla u$

$-v\Delta u = \nabla v\cdot\nabla u - \nabla\cdot(v\nabla u)$)

$$\int_\Omega v f\,dx = \int_\Omega \nabla v\cdot\nabla u\,dx - \int_\Omega \nabla\cdot(v\nabla u)\,dx$$

$\downarrow$ divergence thm

$$= \int_\Omega \nabla v\cdot\nabla u\,dx - \int_{\partial\Omega} v\nabla u\cdot\vec{n}\,ds$$

We are looking for a smooth enough function u that satisfies

$$\int_\Omega \nabla u\cdot\nabla v\,dx = \int_\Omega v f\,dx - \int_{\partial\Omega} v\nabla u\cdot\vec{n}\,ds$$

for all test functions v that are "smooth enough"

Question: What is "smooth enough"?

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Find $u \in U$ such that

$$\int_\Omega \nabla u\cdot\nabla v\,dx = \int_\Omega v f\,dx - \int_{\partial\Omega} v\nabla u\cdot\vec{n}\,ds$$

for all $v \in V$. We still need definition of U, V.

Definitions:

- The Lebesgue space of square-integrable functions is defined as

$$L^2(\Omega) = \{u : \Omega \to \mathbb{R};\ \int_\Omega u^2\,dx < \infty\}$$

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- The norm is defined as

$$\|u\| = \left(\int_\Omega u^2 dx\right)^{1/2}$$

- Cauchy-Schwarz: let $f, g \in L^2(\Omega)$. then

$$\int_\Omega |f(x)g(x)| dx \le \|f\| \cdot \|g\|$$

If $\partial_x u, \partial_y u, \partial_x v, \partial_y v \in L^2(\Omega)$ then

$$\int \nabla u \cdot \nabla v \, dx = \int_\Omega \partial_x u \, \partial_x v \, dx + \int_\Omega \partial_y u \, \partial_y v \, dx$$

$$\overset{C.S.}{\le} \|\partial_x u\| \, \|\partial_x v\| + \|\partial_y u\| \, \|\partial_y v\| < \infty$$

So, if $\partial_x u, \partial_x v, \partial_y u, \partial_y v \in L^2(\Omega)$ then

$$\int_\Omega \nabla u \cdot \nabla v \, dx < \infty$$

Definitions:

- Sobolev space

$$H^1(\Omega) = \{u : \Omega \to \mathbb{R} ; \; u, \partial_x u, \partial_y u \in L^2(\Omega)\}$$

- To add the Dirichlet b.c.

$$H^1_g(\Omega) = \{u \in H^1(\Omega) ; \; u = g_0 \text{ on } \partial\Omega\}$$

Test space:

$$H^1_0(\Omega) = \{u \in H^1(\Omega) ; \; u = 0 \text{ on } \partial\Omega\}$$

Weak formulation

Find $u \in H^1_g(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \qquad \forall v \in H^1_0(\Omega)$$

Note:

$$\int_{\partial\Omega} v \, \nabla u \cdot \vec{n} \, ds = 0 \qquad \text{as } v = 0 \text{ on } \partial\Omega$$

Bilinear form:

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx$$

Linear form:

$$l(v) = \int_\Omega f v \, dx$$

Finite dimensional form of the weak formulation

- Let V_g^h be a finite dim subspace of $H_g^1(\Omega)$, $V_g^h \subset H_g^1(\Omega)$.
- Let V_g^h be an N-dim vector space with basis

$$\{\varphi_1, \varphi_2, \dots, \varphi_N\}$$

- We need $u = g_D$ on $\partial\Omega$

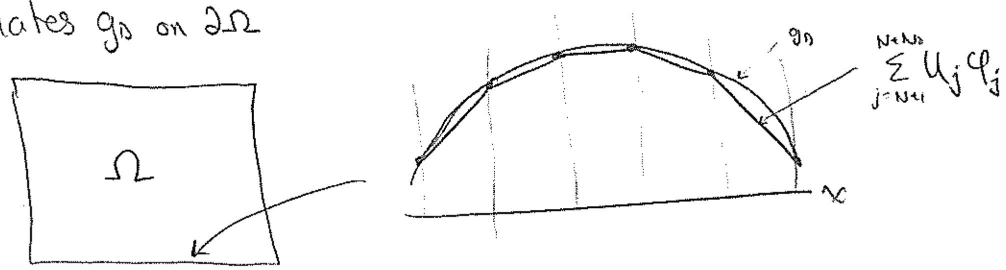
$\Rightarrow$ Define additional basis functions

$$\varphi_{N+1}, \dots, \varphi_{N+N_\partial}$$

and select U_j ($j = N+1, \dots, N+N_\partial$) such that

$$\sum_{j=N+1}^{N+N_\partial} U_j \varphi_j$$

interpolates g_D on $\partial\Omega$



$$U_h = \underbrace{\sum_{j=1}^{N} U_j \varphi_j}_{\text{unknowns}} + \underbrace{\sum_{j=N+1}^{N+N_\partial} U_j \varphi_j}_{\text{known}}$$

FEM weak formulation

Find $u_h \in V_g^h$ such that

$$\int_\Omega \nabla u_h \cdot \nabla v_h \, dx = \int_\Omega f v_h \, dx \qquad \forall v_h \in V_0^h$$

Use

$$u_h = \sum_{j=1}^{N} U_j \varphi_j + \sum_{j=N+1}^{N+N_\partial} U_j \varphi_j \qquad v_h = \sum_{j=1}^{N} V_j \varphi_j.$$

The weak formulation is equivalent to

Find U_j $(j=1,\dots,N)$ such that

$$\sum_{j=1}^{N} U_j \int_\Omega \nabla\varphi_j \cdot \nabla\varphi_i \, dx + \sum_{j=N+1}^{N+N_\partial} U_j \int_\Omega \nabla\varphi_j \cdot \nabla\varphi_i \, dx = \int_\Omega \varphi_i f \, dx$$

for $i=1,\dots,N$

Take known data to RHS

$$\sum_{j=1}^{N} U_j \int_\Omega \nabla\varphi_j \cdot \nabla\varphi_i \, dx = \int_\Omega \varphi_i f \, dx - \sum_{j=N+1}^{N+N_\partial} U_j \int_\Omega \nabla\varphi_j \cdot \nabla\varphi_i \, dx$$

$$a_{ij} = \int_\Omega \nabla\varphi_j \cdot \nabla\varphi_i \, dx$$

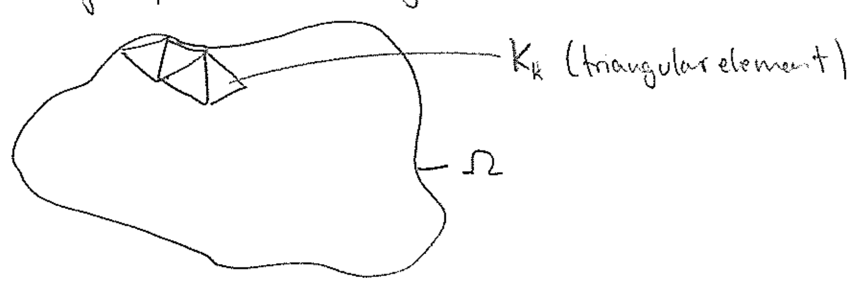
$$F_i = \int_\Omega \varphi_i f \, dx - \sum_{j=N+1}^{N+N_\partial} U_j \int_\Omega \nabla\varphi_j \cdot \nabla\varphi_i \, dx$$

We can write the weak form as

$$\sum_{j=1}^{N} a_{ij} U_j = F_i \qquad A = [a_{ij}] \in \mathbb{R}^{N\times N} \qquad \text{for } i=1,\dots,N$$

$$AU = F \qquad F = [F_i] \in \mathbb{R}^N$$

To compute the integrals, we need a grid.



The triangulation of Ω is given by

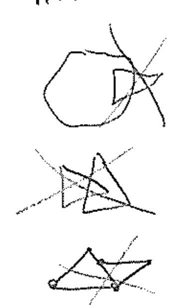
$$T_h = \{K_k\}$$

triangles

Requirements on triangles

- $\bigcup_k \overline{K}_k = \overline{\Omega}$
- $K_k \cap K_m = \emptyset$ (no overlap of triangles)
- vertices of neighbouring triangle coincide

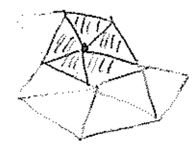
not allowed:



Terminology:

- point where triangle vertices meet are <u>nodes</u>

- surrounding a node is a <u>patch</u> of triangles



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$$\int_\Omega (\ )\,dx = \sum_{K_k \in T_h} \int_{K_k} (\ \ )\,dx$$

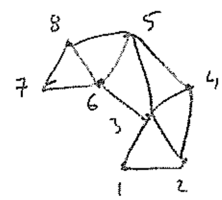
So

$$a_{ij} = \sum_{K_k \in T_h} \nabla\varphi_i \cdot \nabla\varphi_j \,dx$$

$$F_i = \sum_{K_k \in T_h} \int_{K_k} \varphi_i f \,dx - \sum_{j=N+1}^{N+N_\partial} U_j \left(\sum_{K_k \in T_h} \int_{K_k} \nabla\varphi_i \cdot \nabla\varphi_j \,dx \right)$$

<u>Basic Functions</u>

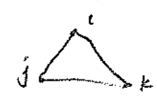
- Label each node in grid by $j = 1, \ldots, N$
- For each node j define a basis function φ_j that is nonzero on a <u>patch</u> of node j:



Let

$$\varphi_j = \begin{cases} 1 & \text{at node } j \\ 0 & \text{elsewhere} \end{cases}$$

- For every element K with vertices i, j, k, there are only three basis functions $\varphi_i, \varphi_j, \varphi_k$ that are not 0

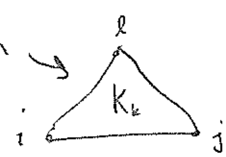


- $\varphi_i, \varphi_j, \varphi_k$ are global basis functions

Better to consider integrals in terms of <u>element</u> basis functions

Element K_k

- This element has 3 local degrees of freedom
- To each global basis function we associate a local (element) basis function



$$\psi_{k,1}, \psi_{k,2}, \psi_{k,3}$$

- Likewise, to each global expansion coefficient U_j we associate local expansion coefficients

$$U_1^{(k)}, U_2^{(k)}, U_3^{(k)}$$

The solution u_h when restricted to element K_k can be written as

$$u_h\Big|_{K_k} = \sum_{i=1}^{3} U_i^{(k)} \psi_{k,i}$$

For each K_k we can write the element matrices A_k and vectors F_k as

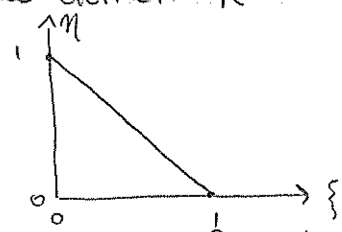
$$A_k = [a_{ij}^{(k)}] \in \mathbb{R}^{3\times 3}$$
$$F_k = [F_i^{(k)}] \in \mathbb{R}^3$$

$$a_{ij}^{(k)} = \int_{K_k} \nabla \psi_{ki} \cdot \nabla \psi_{kj} \, dx$$

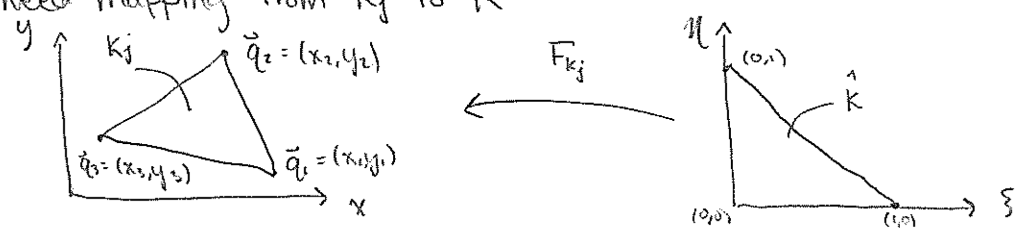
$$F_i^{(k)} = \int_{K_k} \psi_{k,i} f \, dx \qquad \text{(Dirichlet b.c. treated later)}$$

map these "physical" elements to a nice reference element

reference element $\hat{K}$



We need mapping from K_j to $\hat{K}$



$\vec{\xi} = (\xi, \eta)$ $\quad \vec{x} = (x, y)$ $\quad \vec{q}_i = (x_i, y_i)$

$\hookrightarrow$ local vertices of K_j $(i = 1,2,3)$

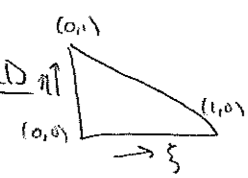
The mapping is given by

$$\vec{F}_{K_j} : \hat{K} \to K_j$$

$$\vec{\xi} \longmapsto \vec{x} = \vec{q}_1 \hat{\psi}_1(\vec{\xi}) + \vec{q}_2 \hat{\psi}_2(\vec{\xi}) + \vec{q}_3 \hat{\psi}_3(\vec{\xi})$$

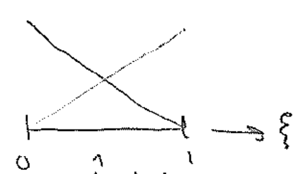
where $\hat{\psi}_i(\vec{\xi})$ are the <u>local</u> basis functions on $\hat{K}$

<u>2D</u>



$$\left.\begin{aligned} \hat{\psi}_1(\vec{\xi}) &= 1 - \xi - \eta \\ \hat{\psi}_2(\vec{\xi}) &= \xi \\ \hat{\psi}_3(\vec{\xi}) &= \eta \end{aligned}\right\} \text{these were permuted, I fixed them}$$

<u>1D</u>



$\hat{\psi}_1 = 1 - \xi$

$\hat{\psi}_2 = \xi$

$$a_{ij}^{(k)} = \int_{K_k} \nabla \psi_{ki} \cdot \nabla \psi_{kj} \, d\vec{x} = \int_{\hat{K}} \hat{\nabla} \hat{\psi}_i \cdot J_k^{-T} J_k^{-1} \hat{\nabla} \hat{\psi}_j \, |\det(J_k)| \, d\vec{\xi}$$

$$F_i^{(k)} = \int_{K_k} \psi_{k,i} f(\vec{x}) \, d\vec{x} = \int_{\hat{K}} \hat{\psi}_k f(\vec{F}_{K_j}(\vec{\xi})) |\det(J_k)| \, d\vec{\xi}$$

J_K is the Jacobian matrix such that

$$\hat{\nabla} h = J_K \nabla h$$

$\nabla = (\partial_x, \partial_y)$ $\hat{\nabla} = (\partial_\xi, \partial_\eta)$

J_K^{-1} inverse of J_K

$J_K^{-T} = (J_K^{-1})^T$

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We want to transform

$$a_{ij}^{(k)} = \int_{K_k} \nabla \psi_{k,i} \cdot \nabla \psi_{k,j} \, d\vec{x}$$

to an integral over $\hat{K}$.

Look at an integral over K_k. Let $g(\vec{x})$ be some function and $\vec{x} = \vec{x}(\vec{\xi})$.

Then

$$\int_K g(\vec{x}) d\vec{x} = \int_{\hat{K}} g(\vec{x}(\vec{\xi})) \, |\det(J_K)| \, d\vec{\xi}.$$

J_K is the Jacobian of Transformation. find later

$$a_{ij}^{(k)} = \int_{K_k} \nabla \psi_{k,i} \cdot \nabla \psi_{k,j} \, d\vec{x} = \int_{\hat{K}} (\ldots ? \ldots) \, |\det(J_K)| \, d\vec{\xi}.$$

We have to relate $\nabla \psi_{k,i}$ to $\hat{\nabla} \hat{\psi}_i$.

$\nabla \leftarrow (\partial_x, \partial_y)$ $\hat{\nabla} \leftarrow (\partial_\xi, \partial_\eta)$

Let $h(\vec{x})$ be some differentiable function and let $\vec{x} = \vec{x}(\vec{\xi})$.

$$\frac{\partial h}{\partial \xi} = \frac{\partial x}{\partial \xi} \frac{\partial h}{\partial x} + \frac{\partial y}{\partial \xi} \frac{\partial h}{\partial y}$$

$$\frac{\partial h}{\partial \eta} = \frac{\partial x}{\partial \eta} \frac{dh}{dx} + \frac{dy}{d\eta} \frac{dh}{dy}$$

$$\begin{bmatrix} \frac{\partial h}{\partial \xi} \\ \frac{\partial h}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{dy}{d\xi} \\ \frac{\partial x}{\partial \eta} & \frac{dy}{d\eta} \end{bmatrix} \begin{bmatrix} \frac{\partial h}{\partial x} \\ \frac{\partial h}{\partial y} \end{bmatrix} \longrightarrow \hat{\nabla} h = J_K \nabla h$$

$$\vec{F}_{K_j} : \hat{K} \to K_j \quad \vec{\xi} \mapsto \vec{x}$$

$$J_K = \begin{bmatrix} \frac{\partial (F_{K_j})_1}{\partial \xi} & \frac{\partial (F_{K_j})_2}{\partial \xi} \\ \frac{\partial (F_{K_j})_1}{\partial \eta} & \frac{\partial (F_{K_j})_2}{\partial \eta} \end{bmatrix}$$

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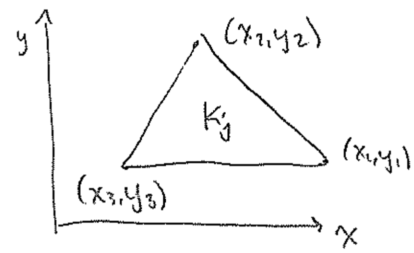
$(i=1,2)$

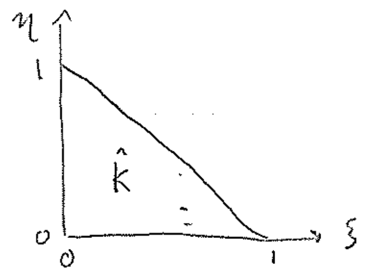
$$(F_{K_j})_i = (\vec{q}_1)_i\,\hat{\psi}_1(\vec{\xi}) + (\vec{q}_2)_i\,\hat{\psi}_2(\vec{\xi}) + (\vec{q}_3)_i\,\hat{\psi}_3(\vec{\xi})$$

$$(F_{K_j})_1 = x_1(1-\xi-\eta) + x_2\xi + x_3\eta$$

$$(F_{K_j})_2 = y_1(1-\xi-\eta) + y_2\xi + y_3\eta$$



$$J_k = \underbrace{\begin{bmatrix} x_2-x_1 & y_2-y_1 \\ x_3-x_1 & y_3-y_1 \end{bmatrix}}_{\text{known data}}$$



Determinant of J_K:

$$\det(J_K) = \begin{vmatrix} x_2-x_1 & y_2-y_1 \\ x_3-x_1 & y_3-y_1 \end{vmatrix}$$
$$= (x_2-x_1)(y_3-y_1) - (x_3-x_1)(y_2-y_1)$$

Given a triangle K_k with vertices $(x_1,y_1), (x_2,y_2), (x_3,y_3)$, the area of this triangle is given by

$$|K_k| = \frac{1}{2}\left|(x_2-x_1)(y_3-y_1) - (x_3-x_1)(y_2-y_1)\right|$$

$$\Rightarrow |\det(K_k)| = 2|K_k|.$$

$|K_k| \neq 0 \rightarrow \det(K_k) \neq 0 \longrightarrow J_k^{-1}$ exists

$$J_{K_k}^{-1} = \frac{1}{\det(J_{K_k})}\begin{bmatrix} y_3-y_1 & y_1-y_2 \\ x_1-x_3 & x_2-x_1 \end{bmatrix}$$

If $\hat{\nabla} h = J_{K_k}\nabla h$ then $\nabla h(\vec{x}) = J_{K_k}^{-1}\hat{\nabla} h(\vec{\xi}(\vec{x})) = J_{K_k}^{-1}\hat{\nabla}\hat{h}(\vec{\xi})$,
where $\hat{h}(\vec{\xi}) = h(\vec{x}(\vec{\xi}))$.

Using that $\nabla\psi_{k,i} = J_{K_k}^{-1}\hat{\nabla}\hat{\psi}_i(\vec{\xi})$, we have

$$a_{ij}^{(k)} = \int_{K_k} \nabla\psi_{k,i}\cdot\nabla\psi_{k_j}\,d\vec{x} = \int_{\hat{K}} (J_{K_k}^{-1}\hat{\nabla}\hat{\psi}_i)\cdot(J_{K_k}^{-1}\hat{\nabla}\hat{\psi}_j)|\det(J_k)|\,d\vec{\xi}.$$

$$= \int_{\hat{K}} \hat{\nabla}\hat{\psi}_i\cdot(J_{K_k}^{-T}J_{K_k}^{-1}\hat{\nabla}\hat{\psi}_j)|\det(J_k)|\,d\vec{\xi}$$

We have $\hat{\psi}_1 = 1-\xi-\eta$, $\hat{\psi}_2 = \xi$, $\hat{\psi}_3 = \eta$

$$\hat{\nabla}\hat{\psi}_1 = \begin{bmatrix}\partial_\xi\hat{\psi}_1 \\ \partial_\eta\hat{\psi}_1\end{bmatrix} = \begin{bmatrix}-1 \\ -1\end{bmatrix},\quad \hat{\nabla}\hat{\psi}_2 = \begin{bmatrix}1 \\ 0\end{bmatrix},\quad \hat{\nabla}\hat{\psi}_3 = \begin{bmatrix}0 \\ 1\end{bmatrix}$$

"let's say we have a linear line"

$$a_{ij}^{(k)} = \hat{\nabla}\hat{\psi}_i \cdot J_{K_k}^{-T} J_{K_k}^{-1} \hat{\nabla}\hat{\psi}_j \underbrace{|\det(J_{K_k})|}_{2|K_k|} \underbrace{\int_{\hat{K}} d\vec{\xi}}_{\text{area of reference triangle, } \frac{1}{2}}$$

$$= |K_k| \hat{\nabla}\hat{\psi}_i \cdot J_{K_k}^{-T} J_{K_k}^{-1} \hat{\nabla}\hat{\psi}_j$$

We still need

$$F_i^{(k)} = \int_{K_k} \psi_{k,i} f(\vec{x}) d\vec{x}$$

$$= \int_{\hat{K}} \psi_{k,i}(\vec{x}(\vec{\xi})) f(\vec{x}(\vec{\xi})) |\det(J_{K_k})| d\vec{\xi}$$

$$= \int_{\hat{K}} \hat{\psi}_i(\vec{\xi}) f(\vec{x}(\vec{\xi})) |\det(J_{K_k})| d\vec{\xi}$$

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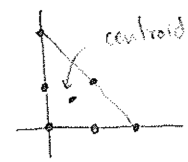
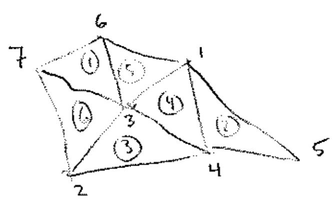
The integral of $h(\xi,\eta)$ over the unit triangle is given by

$$\int_{\hat{K}} h(\xi,\eta)\, d\xi d\eta = \int_0^1 \int_0^{1-\xi} h(\xi,\eta)\, d\eta d\xi$$

$$\approx \sum_{i=1}^{n} h(\xi_i,\eta_i) w_i .$$



$$F_i^{(k)} \approx |\det(J_k)| \sum_{j=1}^{n} \hat{\psi}_i(\xi_j,\eta_j) f(\vec{F}_k(\xi_j,\eta_j)) w_j$$

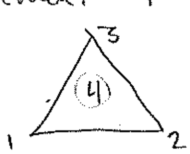


$$P^T = \begin{array}{c} \begin{matrix} (1) & (2) & (3) & (4) & (5) & (6) \end{matrix} \\ \begin{pmatrix} 3 & 1 & 2 & 1 & 1 & 2 \\ 6 & 4 & 3 & 3 & 3 & 3 \\ 7 & 5 & 4 & 4 & 6 & 7 \end{pmatrix} \end{array}$$

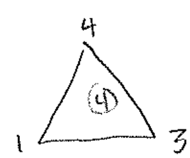
P is called a connectivity matrix

$r = P(k,i)$ specifies the global node of local node i of element k

Element 4



local vs global



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```
Aglobal = 0
Fglobal = 0
for  k = 1 : N
    for  i = 1 : 3
        for  j = 1 : 3
            Aglobal(P(k,i), P(k,j)) = Aglobal(P(k,i), P(k,j))
                                      + Alocal(k,i,j)
        end
        Fglobal(P(k,i)) = Fglobal(P(k,i)) + Flocal(k,i)
    end
end
```

elems

Alocal $(k,i,j) \Longleftrightarrow a_{ij}^{(k)}$

Flocal $(k,i) \Longleftrightarrow F_i^{(k)}$

Aglobal $\Longleftrightarrow A \in \mathbb{R}^{M \times M}$ nodes $\quad (M \neq N) \quad M = N + N_\partial$

Fglobal $\Longleftrightarrow F \in \mathbb{R}^M$

for $N+1 \le i \le N+N_\partial$

$\quad j =$ boundarynode(i)

$\quad F(j) = g_0$

$\quad A(j,j) = 1$

$\quad A(j,k) = 0, \quad k \neq j$

end

Higher-Order Approximation of 1D Poisson

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$$\begin{cases} -u'' = f(x), & x \in (0,1) \\ u(0) = 0 \\ u(1) = 0 \end{cases}$$

Assume v a smooth enough test function:

$$\int_0^1 f(x)v(x)\,dx = \int_0^1 v(x)(-u''(x))\,dx = \int_0^1 v'u'\,dx - \cancel{vu'}\Big|_0^1 \xrightarrow{} 0$$

Impose on test function that $v(0) = v(1) = 0$

Define function space

$$V_0 = \{v \in C^1(\overline{\Omega}) ; v(0) = v(1) = 0\}$$

Find $u \in V_0$ such that

$$a(u,v) = \ell(v) \quad \forall v \in V_0$$

where

$$a(u,v) = \int_0^1 v'u'\,dx$$
$$\ell(v) = \int_0^1 f(x)v(x)\,dx$$

Let $V_{h,0} \subset V_0$ be a finite-dimensional subspace of V_0. Find $u_h \in V_{h,0}$ such that

$$a(u_h, v_h) = \ell(v_h) \quad \forall v_h \in V_{h,0}.$$

$V_{h,0}$ is an N-dimensional vector space

$$V_{h,0} = \text{span}\{\varphi_1, \dots, \varphi_N\}$$
$$u \approx u_h = \sum_{j=1}^{N} U_j \varphi_j$$
$$v \approx v_h = \sum_{j=1}^{N} V_j \varphi_j$$

Find U_j $(j = 1, \dots, N)$ such that

$$\sum_{j=1}^{N} U_j a_{ij} = F_i,$$

where

$$a_{ij} = \int_0^1 \frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx}\,dx$$
$$F_i = \int_0^1 f(x)\varphi_i(x)\,dx$$

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Introduce mesh

$$0 = x_0 < x_1 < x_2 < \cdots < x_J = 1.$$

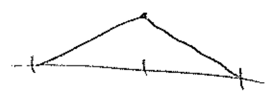
Define element $K_j = (x_{j-1}, x_j)$ with length $h_j = x_j - x_{j-1}$.

<u>What we did for linear FEM:</u>

$$N = J-1 \qquad u_h = \sum_{j=1}^{J-1} U_j \varphi_j + U_0 \varphi_0 + U_J \varphi_J$$

At each interior point we associate a basis function, $\varphi_j(x_i) = \delta_{ij}$, $i,j = 1, \dots, J-1$.

$$\varphi_j(x) = \begin{cases} \frac{x - x_j}{h_j} & x \in K_j \\ \frac{x_{j+1} - x}{h_{j+1}} & x \in K_{j+1} \\ 0 & \text{else} \end{cases}$$



Reference element $\hat{K} = (0,1)$. Mapping:

$$F_{K_j} : \hat{K} \to K_j : \xi \mapsto x = h_j \xi + x_{j-1}$$

$$\hat{\psi}_1(\xi) = 1 - \xi, \quad \hat{\psi}_2(\xi) = \xi$$

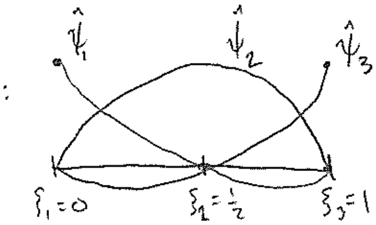
$$\hat{\psi}_1(0) = 1 \quad \hat{\psi}_1(1) = 0$$

$$\hat{\psi}_2(0) = 0 \quad \hat{\psi}_2(1) = 1$$

We saw that

$$\psi_{j,m}(x) = \hat{\psi}_m(F_{K_j}^{-1}(x)), \quad m = 1,2$$

What we are going to do in the quadratic case:



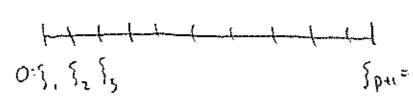
$$\hat{\psi}_1(\xi) = \frac{\xi - \xi_2}{\xi_1 - \xi_2} \frac{\xi - \xi_3}{\xi_1 - \xi_3} = 2(\xi - \tfrac{1}{2})(\xi - 1)$$

$$\hat{\psi}_2(\xi) = \frac{\xi - \xi_1}{\xi_2 - \xi_1} \frac{\xi - \xi_3}{\xi_2 - \xi_3} = 4\xi(1 - \xi)$$

$$\hat{\psi}_3(\xi) = \frac{\xi - \xi_1}{\xi_3 - \xi_1} \frac{\xi - \xi_2}{\xi_3 - \xi_2} = 2\xi(\xi - \tfrac{1}{2})$$

Polynomial of degree p:

$$\hat{\psi}_i(\xi) = \prod_{j=1, j \neq i}^{p+1} \frac{\xi - \xi_j}{\xi_i - \xi_j}$$



$p+1$ basis functions

I'm not gonna prove this.
This is mostly true, but not always.

polynomial approximation of degree $p \Rightarrow$ Error $\sim \mathcal{O}(h^{p+1})$

Finite Element Discretization of a Fourth-Order ODE

$$\begin{cases} -\frac{d^4u}{dx^4} + u = 0, \quad x \in (0, \pi) \\ u(0) = 1 \\ u(\pi) = 0 \\ \frac{du}{dx}(0) = 0 \\ \frac{du}{dx}(\pi) = g \end{cases}$$

Let w be smooth enough

$$0 = \int_0^\pi w(x)\left(\frac{-d^4u}{dx^4} + u\right)dx$$

$$= \int_0^\pi \left(\frac{dw}{dx}\frac{d^3u}{dx^3} + wu\right)dx - w\frac{d^3u}{dx^3}\Big|_0^\pi$$

$$= \int_0^\pi \left(-\frac{d^2w}{dx^2}\frac{d^2u}{dx^2} + wu\right)dx + \frac{dw}{dx}\frac{d^2u}{dx^2}\Big|_0^\pi - w\frac{d^3u}{dx^3}\Big|_0^\pi$$

Let

$$H_g^2(\Omega) = \left\{u, \frac{du}{dx}, \frac{d^2u}{dx^2} \in L^2(\Omega);\ u(0)=1, u(\pi)=0, \frac{du}{dx}(0)=0, \frac{du}{dx}(\pi)=g\right\}$$

$$H_0^2(\Omega) = \left\{u, u', u'' \in L^2(\Omega);\ u(0) = u(\pi) = \frac{du}{dx}(0) = \frac{du}{dx}(\pi) = 0\right\}$$

Weak formulation: Find $u \in H_g^2(\Omega)$ such that

$$\int_0^\pi \left(-\frac{d^2w}{dx^2}\frac{d^2u}{dx^2} + uw\right)dx = 0 \quad \forall w \in H_0^2(\Omega).$$

For $-u'' + u = 0$,

$$\int_0^\pi (u'w' + uw)\,dx = 0$$

$\hookrightarrow u \in H_g^1 = \{u, u' \in L^2; \ldots\}$

$u_h \in V_h \subset H_g^1$

$\uparrow$

$u_h \in C^0$

here we will need $u_h \in C^1$

Grid: $0 = x_0 < x_1 < \cdots < x_N = \pi$

Elements: $K_j = (x_{j-1}, x_j)$, $j = 1, \ldots, N$

Function spaces:

$$V_{g,h}^k = \{u \in C^1(\Omega);\ u \in \underbrace{P^k(K_j)}_{\text{polynomials of degree } k \text{ in element } K_j}\ \forall K_j \subset \Omega,\ u(0)=1,\ u(\pi) = u'(0) = 0,\ u'(\pi) = g\}$$

$$V_{0,h}^k = \{u \in C^1(\Omega);\ u \in P^k(K_j)\ \forall K_j \subset \Omega,\ u(0) = u(\pi) = u'(0) = u'(\pi) = 0\}.$$

To achieve that $u_h \in C^1(\Omega)$ we use Hermite polynomials. These are automatically in $C^1(\Omega)$. The simplest Hermite polynomials are the polynomials $p_3 \in P^3([0,1])$.

$$p_3(\xi) = \hat{\psi}_1(\xi) f(0) + \hat{\psi}_2(\xi) f(1) + \hat{\psi}_3(\xi) \frac{df}{d\xi}(0) + \hat{\psi}_4(\xi) \frac{df}{d\xi}(1)$$

Requirements:

$$p_3(0) = f(0),\quad p_3(1) = f(1),\quad \frac{dp_3}{p\xi}(0) = \frac{df}{d\xi}(0),\quad \frac{dp_3}{d\xi}(1) = \frac{df}{d\xi}(1)$$

$$\hat{\psi}_1(0) = 1,\quad \hat{\psi}_2(0) = 0,\quad \hat{\psi}_3(0) = 0,\quad \hat{\psi}_4(0) = 0$$

$$\hat{\psi}_1(1) = 0,\quad \hat{\psi}_2(1) = 1,\quad \hat{\psi}_3(1) = 0,\quad \hat{\psi}_4(1) = 0$$

$$\frac{d\hat{\psi}_1}{d\xi}(0) = 0,\quad \frac{d\hat{\psi}_2}{d\xi}(0) = 0,\quad \frac{d\hat{\psi}_3}{d\xi}(0) = 1,\quad \frac{d\hat{\psi}_4}{d\xi}(0) = 0$$

$$\frac{d\hat{\psi}_1}{d\xi}(1) = 0,\quad \frac{d\hat{\psi}_2}{d\xi}(1) = 0,\quad \frac{d\hat{\psi}_3}{d\xi}(1) = 0,\quad \frac{d\hat{\psi}_4}{d\xi}(1) = 1$$

$$\Longrightarrow \quad \hat{\psi}_1(\xi) = (\xi - 1)^2 (2\xi + 1)$$

$$\hat{\psi}_2(\xi) = \xi^2 (3 - 2\xi)$$

$$\hat{\psi}_3(\xi) = \xi(\xi - 1)^2$$

$$\hat{\psi}_4(\xi) = \xi^2(\xi - 1)$$

Let $\hat{K} = (0,1)$ be reference element and $F_{K_j} : \hat{K} \to K_j$ maps from $\hat{K}$ to K_j then

$$\psi_{j,m}(x) = \hat{\psi}_m(F_{K_j}^{-1}(x)) = \hat{\psi}_m(\xi), \quad m = 1,2,3,4$$

So u_h restricted to element K_j:

$$u_h|_{K_j} = U_{j-1} \psi_{j,1}(x) + U_j \psi_{j,2}(x) + \frac{dU_{j-1}}{dx} \psi_{j,3}(x) + \frac{dU_j}{dx} \psi_{j,4}(x) = \sum_{m=1}^{4} U_m^{(j)} \hat{\psi}_m(\xi)$$

$$U_1^{(j)} = U_{j-1}, \quad U_2^{(j)} = U_j, \quad U_3^{(j)} = \frac{dU_{j-1}}{dx}, \quad U_4^{(j)} = \frac{dU_j}{dx}$$

Global basis functions:

$$\left.\begin{aligned} \varphi_j^0 &= \psi_{j,2}(x), & x \in K_j \\ \varphi_{j-1}^0 &= \psi_{j,1}(x), & x \in K_j \\ \varphi_j^1 &= \psi_{j,4}(x), & x \in K_j \\ \varphi_{j-1}^1 &= \psi_{j,3}(x), & x \in K_j \end{aligned}\right\} \quad j = 1, \dots, N-1$$

On boundaries:

$$\begin{aligned} \varphi_0^0(x) &= \psi_{1,1}(x), & x \in K_1 \\ \varphi_0^1(x) &= \psi_{1,3}(x), & x \in K_1 \\ \varphi_N^0(x) &= \psi_{N,2}(x), & x \in K_N \\ \varphi_N^1(x) &= \psi_{N,4}(x), & x \in K_N \end{aligned}$$

We have

$$U_h(x) = \sum_{j=0}^{N} \left(U_j \varphi_j^0(x) + \frac{dU_j}{dx} \varphi_j^1(x) \right)$$

$$w_h(x) = \sum_{j=1}^{N-1} \left(W_j \varphi_j^0(x) + \frac{dW_j}{dx} \varphi_j^1(x) \right.$$

Look at internal node x_j $(j = 1, \dots, N)$: ← N-1?

- only non-zero basis functions at x_j: φ_j^0 and φ_j^1
- W_j and $\frac{dW_j}{dx}$ are arbitrary:

⇒ at each node we obtain two equations, 1 related to φ_j^0, 1 related to φ_j^1

$$\varphi_j^0 : \int_{K_j} \left(-\psi_2''(x) U_h'' + \psi_2(x) U_h(x) \right) dx + \int_{K_{j+1}} \left(-\psi_1''(x) U_h''(x) + \psi_1(x) U_h(x) \right) dx = 0$$

$$\varphi_j^1 : \int_{K_j} \left(-\psi_4''(x) U_h''(x) + \psi_4(x) U_h(x) \right) dx + \int_{K_{j+1}} \left(-\psi_3''(x) U_h''(x) + \psi_3(x) U_h(x) \right) dx = 0.$$

Transform to reference element $\hat{K}$, where $F_{K_j} : \hat{K} \to K_j : \xi \mapsto x = h_j \xi + x_{j-1}$.

$$\int_0^1 \left(-\frac{d^2 \hat{\psi}_2}{d\xi^2} \frac{d^2}{d\xi^2} \left(\sum_{n=1}^{4} U_n^{(k)} \hat{\psi}_n(\xi) \right) \left(\frac{d\xi}{dx} \right)_{K_j}^2 + \hat{\psi}_2(\xi) \left(\sum_{n=1}^{4} U_n^{(k)} \hat{\psi}_n(\xi) \right) \right) \left(\frac{dx}{d\xi} \right)_{K_k} d\xi$$

$$+ \int_0^1 \left(-\frac{d^2 \hat{\psi}_1}{d\xi^2} \frac{d^2}{d\xi^2} \left(\sum_{n=1}^{4} U_n^{(k+1)} \hat{\psi}_n(\xi) \right) \left(\frac{d\xi}{dx} \right)_{K_{j+1}}^2 + \hat{\psi}_1(\xi) \left(\sum_{n=1}^{4} U_n^{(k+1)} \hat{\psi}_n(\xi) \right) \right) \left(\frac{dx}{d\xi} \right)_{K_{k+1}} d\xi = 0$$

$$\left(\frac{dx}{d\xi}\right)_{K_k} = h_k \qquad \left(\frac{d\xi}{dx}\right)_{K_k} = \frac{1}{h_k}$$

$$A_{n,m}(K_j) = \int_0^1 \left(-\frac{1}{h_j} \frac{d^2\hat{\psi}_m}{d\xi^2} \frac{d^2\hat{\psi}_n}{d\xi^2} + h_j \hat{\psi}_n(\xi)\hat{\psi}_m(\xi) \right) d\xi$$

$$A_{21}(K_j)U_{j-1} + A_{23}(K_j)\frac{dU_{j-1}}{dx} + \Big(A_{22}(K_j) + A_{11}(K_{j+1})\Big) U_j$$

$$+ \Big(A_{24}(K_j) + A_{13}(K_{j+1})\Big) \frac{dU_j}{dx} + A_{12}(K_j)U_{j+1} + A_{14}(K_{j+1}) \frac{dU_{j+1}}{dx} = 0$$

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$$A_{41}^j U_{j-1} + A_{43}^j \frac{dU_{j-1}}{dx} + \left(A_{42}^j + A_{31}^{j+1}\right)U_j + \left(A_{44}^j + A_{33}^{j+1}\right)\frac{dU_j}{dx}$$

$$+ A_{32}^{j+1} U_{j+1} + A_{34}^{j+1} \frac{dU_{j+1}}{dx} = 0$$

at interior nodes x_j (where $A_{ik}^j = A_{ik}(K_j)$ apparently).

$$\underbrace{\begin{bmatrix} A_{21}^j & A_{23}^j \\ A_{41}^j & A_{43}^j \end{bmatrix}}_{M_j^1} \begin{bmatrix} U_{j-1} \\ \frac{dU_{j-1}}{dx} \end{bmatrix} + \underbrace{\begin{bmatrix} A_{22}^j + A_{11}^{j+1} & A_{24}^j + A_{13}^{j+1} \\ A_{42}^j + A_{31}^{j+1} & A_{44}^j + A_{33}^{j-1} \end{bmatrix}}_{M_j^2} \underbrace{\begin{bmatrix} U_j \\ \frac{dU_j}{dx} \end{bmatrix}}_{V_j}$$

$$+ \underbrace{\begin{bmatrix} A_{12}^j & A_{14}^{j+1} \\ A_{32}^{j+1} & A_{32}^{j+1} \end{bmatrix}}_{M_j^3} \begin{bmatrix} U_{j+1} \\ \frac{dU_{j+1}}{dx} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

M_j^1: $j = 2, \dots, N-1$ $\qquad$ M_j^3: $j = 1, \dots, N-2$ $\qquad$ M_j^2: $j = 1, \dots, N-1$

At interior node x_j:

$$M_j^1 V_{j-1} + M_j^2 V_j + M_j^3 V_{j+1} = 0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = R^j$$

At boundary nodes:

$$V_0 = \begin{bmatrix} U_0 \\ \frac{dU_0}{dx} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad V_N = \begin{bmatrix} U_N \\ \frac{dU_N}{dx} \end{bmatrix} = \begin{bmatrix} 0 \\ g \end{bmatrix}$$

- $M_1^1 V_0 + M_1^2 V_1 + M_1^3 V_2 = 0$

  $\rightarrow M_1^2 V_1 + M_1^3 V_2 = -M_1^1 V_0$

  $$= -\begin{bmatrix} A_{21}^1 & A_{23}^1 \\ A_{41}^1 & A_{43}^1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = -\begin{bmatrix} A_{21}^1 \\ A_{41}^1 \end{bmatrix} = R^1$$

- $M_{N-1}^1 V_{N-2} + M_{N-1}^2 V_{N-1} + M_{N-1}^3 V_N = 0$

  $\rightarrow M_{N-1}^1 V_{N-2} + M_{N-1}^2 V_{N-1} = -M_{N-1}^3 V_N$

  $$= -\begin{bmatrix} A_{14}^N g \\ A_{34}^N g \end{bmatrix} = R^{N-1}$$

Put into global matrix:

$$
\begin{bmatrix}
M_1^2 & M_1^3 & 0 & \cdots & & & & & 0 \\
M_2^1 & M_2^2 & M_2^3 & 0 & \cdots & & & & 0 \\
0 & \ddots & \ddots & \ddots & & & & & \vdots \\
\vdots & & & & & & & & \\
0 & \cdots & 0 & M_j^1 & M_j^2 & M_j^3 & 0 & \cdots & 0 \\
\vdots & & & & \ddots & \ddots & \ddots & & \vdots \\
 & & & & & & & & 0 \\
0 & \cdots & & & & 0 & M_{N-2}^1 & M_{N-2}^2 & M_{N-2}^3 \\
0 & \cdots & & & & & 0 & M_{N-1}^1 & M_{N-1}^2
\end{bmatrix}
\begin{bmatrix}
V_1 \\ V_2 \\ \vdots \\ \\ V_j \\ \vdots \\ \\ V_{N-2} \\ V_{N-1}
\end{bmatrix}
=
\begin{bmatrix}
R^1 \\ R^2 \\ \vdots \\ \\ R^j \\ \vdots \\ \\ R^{N-2} \\ R^{N-1}
\end{bmatrix}
$$

Discontinuous Galerkin FEM

$$\partial_t u + \partial_x f(u) = 0$$

Grid: $0 = x_0 < x_1 < \cdots < x_N = 1$

Elements: $K_j = (x_{j-1}, x_j)$

Function space: $V_h = \{ v \in L^2(0,1) ;\ v|_{K_k} \in \mathbb{P}^p(K_k),\ k = 1, \dots, N \}$

Mapping: $F_{K_j} : \hat{K} \to K_j :\ x = \frac{1}{2}(x_{j-1} + x_j) + \frac{1}{2}\xi h_j$

$\hat{K} = (-1, 1)$

Basis Functions $\hat{\psi}_0(\xi) = 1, \quad \hat{\psi}_m(\xi) = \xi^m$

$$U_h(x,t)\big|_{K_k} = \sum_{m=0}^{m} \hat{U}_m^k(t)\, \psi_{k,m}(x)$$

$$0 = \int_{K_k} v_h\left(\partial_t u_h + \partial_x f(u_h)\right) dx$$

$$= \int_{K_k} v_h\, \partial_t u_h\, dx - \int_{K_k} (\partial_x u_h)\, f(u_h)\, dx + V_h(x_k) f(u_h(x_k)) - V_h(x_{k-1}) f(u_h(x_{k-1}))$$

$$= \int_{K_k} v_h \partial_t u_h\, dx - \int_{K_k} (\partial_x v_h) f(u_h)\, dx + v_h(x_k)\hat{f}\left(U_n^k(x_k), U_n^{k+1}(x_k)\right)$$

$$- V_h(x_{k-1})\hat{f}\left(U_n^{k-1}(x_k), U_n^k(x_k)\right)$$

Finite Volume Method

Choose $\hat{\psi}_0(\xi) = 1$

then $u_h(x,t)|_{K_k} = \hat{u}_0^{(k)}(t)$

$v_h(x,t)|_{K_k} = \hat{V}_0^{(k)}$

$$0 = \int_{K_k} \hat{V}_0^{(k)} \partial_t \hat{U}_0^{(k)} dx + \hat{V}_0^{(k)} \hat{f}\left(\hat{U}_0^k, \hat{U}_0^{k+1}\right) - \cancel{\hat{V}_0^{(k)}} \hat{f}\left(\hat{U}_0^{k-1}, \hat{U}_0^k\right)$$

$$\partial_t \hat{U}_0^{(k)} \Delta x \approx \left(\frac{\hat{U}_0^{(k),n+1} - \hat{U}_0^{(k),n}}{\Delta t}\right)\Delta x$$

$$\hat{U}_0^{(k),n+1} = \hat{U}_0^{(k),n} - \frac{\Delta t}{\Delta x}\left(\hat{f}\left(\hat{U}_0^{(k),n}, \hat{U}_0^{(k+1),n}\right) - \hat{f}\left(\hat{U}_0^{(k-1),n}, \hat{U}_0^{(k),n}\right)\right).$$