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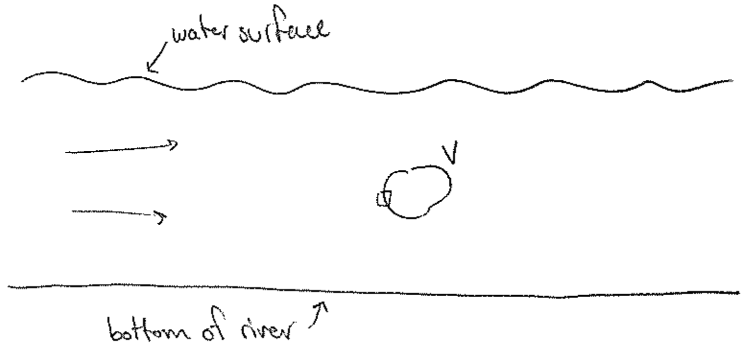
→ <u>Conservation Laws</u> ⟵ closely related ⟶ Finite Volume Methods

hyperbolic PDEs

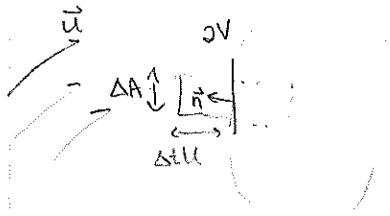


Mass of water in volume V

$$M(t+\Delta t) = M(t) + \text{Mass added} - \text{Mass removed}$$

$$M(t) = \iiint_V \rho(\vec{x}, t)\, dV$$

Zoom in on ∂V



$\rho \Delta A \Delta t U$

$$U = -\vec{n}\cdot\vec{u}$$

net flow of (mass) water out of V through ∂V:

$$\Delta t \iint_{\partial V} \rho \vec{u}\cdot\vec{n}\, dS$$

dA if you want

$$M(t+\Delta t) - M(t) = -\Delta t \iint_{\partial V} \rho \vec{u}\cdot\vec{n}\, dS$$

$$\underbrace{\frac{M(t+\Delta t) - M(t)}{\Delta t}}_{\lim_{\Delta t\to 0}\ \uparrow\ = \frac{\partial M}{\partial t}} = -\iint_{\partial V} \rho \vec{u}\cdot\vec{n}\, dS$$

Use the definition of $M(t)$:

$$\frac{\partial}{\partial t}\iiint_V \rho(\vec{x},t)\, dV + \iint_{\partial V} \rho \vec{u}\cdot\vec{n}\, dS = 0$$

Mass conservation in integral form: ~~ ~~

$$\frac{\partial}{\partial t}\iiint_V \rho(\vec{x},t)\,dV + \iint_{\partial V} \rho\vec{u}\cdot\vec{n}\,dS = 0 \qquad -(1)$$

Alternative: integrate in time from t_1 to t_2 $(t_1 < t_2)$

$$\iiint_V \rho(\vec{x},t_2)\,dV = \iiint_V \rho(\vec{x},t_1)\,dV - \int_{t_1}^{t_2}\left(\iint_{\partial V}\rho\vec{u}\cdot\vec{n}\,dS\right)dt \qquad -(2)$$

*V is fixed in time

$$\frac{\partial}{\partial t}\iiint_V \rho\,dV = \iiint_V \partial_t\rho\,dV$$

* Divergence theorem

$\vec{a}$ is vector

$$\iint_{\partial V} \vec{a}\cdot\vec{n}\,dS = \iiint_V \nabla\cdot\vec{a}\,dV.$$

$$\iint_{\partial V} \rho\vec{u}\cdot\vec{n}\,dS = \iiint_V \nabla\cdot(\rho\vec{u})\,dV$$

$$\iiint_V (\partial_t\rho + \nabla\cdot\rho\vec{u})\,dV = 0$$

Because V is arbitrary integrand has to be zero

$$\boxed{\partial_t\rho + \nabla\cdot\rho\vec{u} = 0}$$

<u>Momentum Conservation</u>

Newton's second law:

$$\vec{F} = \frac{d}{dt}(m\vec{v})$$

force

m = mass
$\vec{v}$ = velocity
$m\vec{v}$ = momentum

What forces do we have?

1/ Body forces: gravity, electromagnetic,

2/ Surface forces: pressure, stress, $\longrightarrow \vec{F}_{viscous}$

Denote all body forces per unit mass acting on fluid inside V by $\vec{f}$

$\iiint_V \rho\vec{f}\,dV$ total force inside V

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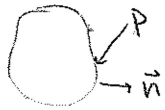

pressure forces

$$-\iint_{\partial V} p\vec{n}\,dS$$

Total force

$$\vec{F} = \underbrace{\iiint_V \rho \vec{f}\,dV}_{\text{body forces}} - \iint_{\partial V} \underset{\substack{\uparrow \\ \text{pressure}}}{p}\,\vec{n}\,dS + \vec{F}_{\text{viscous}}$$

time rate of change of momentum

* Net flow of momentum out of V through ∂V

$$\iint_{\partial V} \rho\, \vec{v}\, \vec{u}\cdot\vec{n}\,dS$$

* Momentum in V at time t

$$\iiint_V \rho\vec{v}\,dV \xrightarrow[\text{change}]{\text{rate of}} \frac{\partial}{\partial t}\iiint_V \rho\vec{v}\,dV$$

Total rate of change of momentum

$$\frac{\partial}{\partial t}\iiint_V \rho\vec{v}\,dV + \iint_{\partial V}\rho\vec{v}\,\vec{u}\cdot\vec{n}\,dS$$

Apply Newton's second law

$$(\ast) - \quad \frac{\partial}{\partial t}\iiint_V \rho\vec{v}\,dV + \iint_{\partial V}\rho\vec{v}\,\vec{u}\cdot\vec{n}\,dS = -\iint_{\partial V} p\,\vec{n}\,dS + \iiint_V \rho\vec{f}\,dV + \vec{F}_{\text{viscous}}$$

Momentum conservation in integral form

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PDE form: Assume that V is fixed in time

$$\frac{d}{dt}\iiint_V \rho\vec{v}\,dV = \iiint_V \frac{\partial}{\partial t}(\rho\vec{v})\,dV$$

Gradient theorems

let f be some scalar

$$\iint_{\partial V} f\vec{n}\,dS = \iiint_V \nabla f\,dV$$

So

$$\iint_{\partial V} p\,\vec{n}\,dS = \iiint_V \nabla p\,dV$$

Rewrite $(\ast)$:

$$\iiint_V \left(\frac{\partial}{\partial t}(\rho\vec{v}) + \nabla p\right) dV + \iint_{\partial V} \rho\vec{v}\,\vec{u}\cdot\vec{n}\, dS = \iiint_V \rho\vec{f}\, dV + \vec{F}_{viscous}$$

$\vec{v} = [v_1, v_2, v_3]$, $\vec{f} = [f_1, f_2, f_3]$, $\vec{F}_{viscous} = [F_1, F_2, F_3]$

$\nabla p = \left[\frac{\partial p}{\partial x_1}, \frac{\partial p}{\partial x_2}, \frac{\partial p}{\partial x_3}\right]$

$$\iiint_V \left(\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial p}{\partial x_i}\right) dV + \underbrace{\iint_{\partial V} \rho v_i \vec{u}\cdot\vec{n}\, dS}_{= \iiint_V \nabla\cdot(\rho v_i \vec{u})\, dV \text{ by div. thm}} = \iiint_V \rho f_i\, dV + F_i \quad \text{for } i=1,2,3$$

Introduce $\mathcal{F}_i$ st $\iiint_V \mathcal{F}_i\, dV = F_i$. So

$$\iiint_V \left\{\frac{\partial \rho v_i}{\partial t} + \nabla\cdot(\rho v_i \vec{u}) + \frac{\partial p}{\partial x_i} - \rho f_i - \mathcal{F}_i\right\} dV = 0$$

The integrand must be 0 because V is arbitrary.

$$\frac{\partial \rho v_i}{\partial t} + \nabla\cdot(\rho v_i \vec{u}) + \frac{\partial p}{\partial x_i} = \rho f_i + \mathcal{F}_i \quad , \; i=1,2,3$$

$$\left.\begin{aligned} &\partial_t \rho + \nabla\cdot\rho\vec{u} = 0 \\ &\partial_t \rho v_i + \nabla\cdot\rho v_i\vec{u} + \partial p/\partial x_i = \rho f_i + \cancel{\mathcal{F}_i} \quad i=1,2,3 \end{aligned}\right\} \text{Navier-Stokes equations}$$

if $\mathcal{F}_i = 0 \rightarrow$ Euler equations

<u>Def</u>) All conservation Laws can be expressed as

PDE $\quad \partial_t \vec{U} + \nabla\cdot\vec{F}(\vec{U}) = 0$

integrals

(1) $\displaystyle \frac{d}{dt}\iiint_V \vec{U}\, dV = -\iint_{\partial V} \vec{F}(\vec{U})\cdot\vec{n}\, dS$

(2) $\displaystyle \iiint_V \vec{U}(\vec{x}, t_2)\, dV = \iiint_V \vec{U}(\vec{x}, t_1)\, dV - \int_{t_1}^{t_2}\left(\iint_{\partial V} \vec{F}(\vec{U})\,\vec{n}\, dS\right) dt$

Euler:

$$\vec{U} = \begin{bmatrix} \rho \\ \rho v_1 \\ \rho v_2 \\ \rho v_3 \end{bmatrix} \qquad \vec{F}(\vec{U}) = \begin{bmatrix} \rho\vec{u} \\ \rho v_1\vec{u} + [p,0,0] \\ \rho v_2\vec{u} + [0,p,0] \\ \rho v_3\vec{u} + [0,0,p] \end{bmatrix}$$

<u>Simplifications</u>

① assume we know $\rho, \vec{v}$

$$\partial_t \rho + \nabla\cdot\rho\vec{v} = 0$$

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assume that $\vec{v} = \vec{a}$ is given and satisfies

$$\nabla \cdot \vec{a} = 0$$

$$\partial_t \rho + \nabla \cdot \rho \vec{a} = \partial_t \rho + \vec{a} \cdot \nabla \rho + \cancel{\rho \nabla \cdot \vec{a}} = 0$$

assume 1D flow

$$\boxed{\partial_t \rho + a \partial_x \rho = 0}$$ Linear advection equation

② assume that ρ = constant, p = constant, $f_i = 0$

$$\cancel{\partial_t \rho} + \nabla \cdot \rho \vec{v} = \rho \nabla \cdot \vec{v} = 0 \iff \boxed{\nabla \cdot \vec{v} = 0}$$

$$\rho \partial_t v_i + \cancel{v_i \partial_t \rho} + \cancel{\rho v_i \nabla \cdot \vec{v}} + \vec{v} \cdot \nabla \rho v_i + \cancel{\partial p / \partial x_i} = 0$$

$$\boxed{\partial_t v_i + \vec{v} \cdot \nabla v_i = 0}$$

1D flow:

$$\boxed{\partial_t v + v \partial_x v = 0}$$ Burgers equation

$$\partial_t v + \partial_x \left(\tfrac{1}{2} v^2 \right) = 0$$

↑ nonlinear

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$$\begin{cases} \partial_t u + a \partial_x u = 0, & -\infty < x < \infty \\ u(x,0) = u_0(x) & t > 0 \end{cases}$$

Exact solution: $u(x,t) = u_0(x - at)$, $t \geq 0$

Let $s = x - at$.

$$\partial_t u + a \partial_x u = \partial s / \partial t \; u_0'(s) + a \, \partial s / \partial x \; u_0'(s) = -a u_0' + a u_0' = 0$$

$\underline{a > 0}$

Consider a curve in x-t plane

$$x = x(t) = at - x_0$$

Differentiate wrt t:

$$\frac{dx}{dt} = a$$

Differentiate $u(x,t)$ along $x = at + x_0$

$$\frac{d}{dt} u(x(t), t) = \left(\frac{\partial}{\partial x} u \right) \frac{dx}{dt} + \frac{\partial}{\partial t} u$$

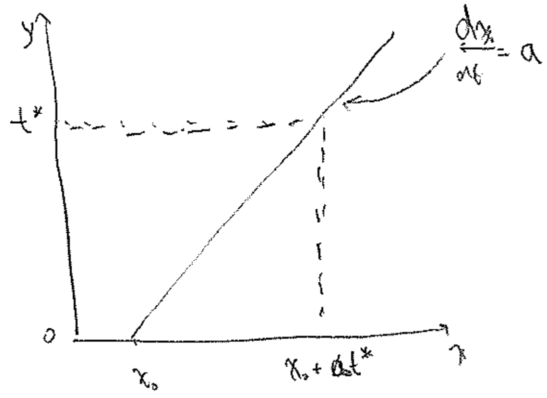
$$= \frac{\partial u}{\partial x} a + \frac{\partial}{\partial t} u = 0$$

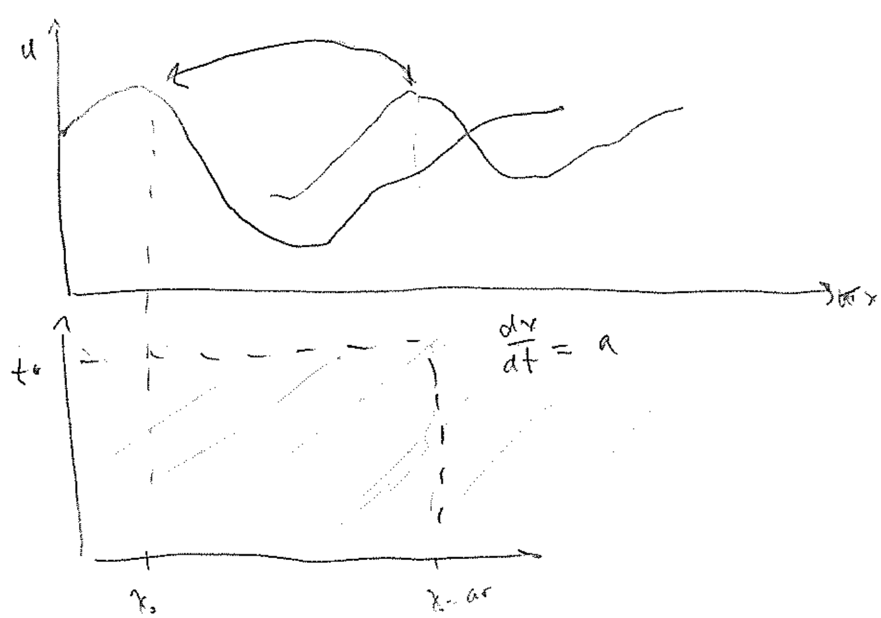
$du/dt = 0$ along rays that satisfy $dx/dt = a$.

Definition Rays that satisfy $dx/dt = a$ are characteristics.

Important: $u =$ constant along a characteristic

Example





The solution $u(x,t)$ at (x^*,t^*) depends only on the initial data at a single point. Which point? that point x_0 st (x^*,t^*) lies on the characteristic through x_0.

Domain of dependence of the advection equation at a point (x^*,t^*) is the set

$$\mathcal{D}(x^*,t^*) = \{x_0\}$$

General scalar hyperbolic PDE

$$\partial_t u + \partial_x f(u) = 0$$

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If u is smooth, then any curve $x(t)$ satisfying
$$x'(t) = f'(u(x(t),t))$$
is called a characteristic curve.
The solution $u(x,t)$ is constant on char. curves.
Proof: To show $du/dt\,(x(t),t) = 0$
$$\frac{du}{dt}(x(t),t) \underset{x'=f'}{=} x'(t)\partial_x u + \partial_t u = \partial_t u + \partial_x f(u) = 0$$ □

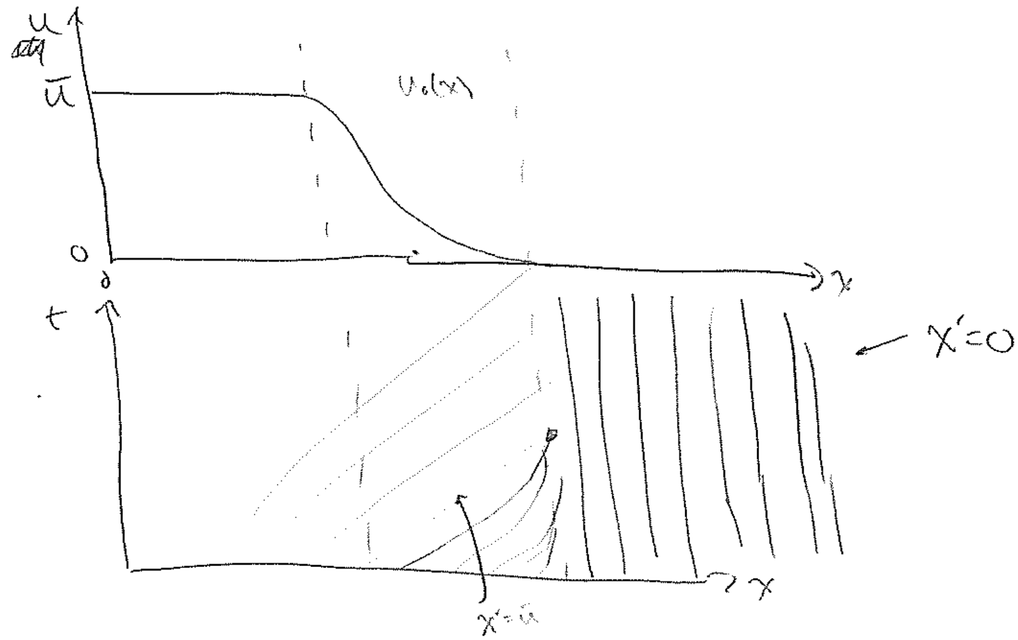
u=constant on $x(t)$, $f'(u)$ constant on $x(t)$
$\to x'(t)$ = constant $\to x(t)$ is a straight line / only if u is smooth

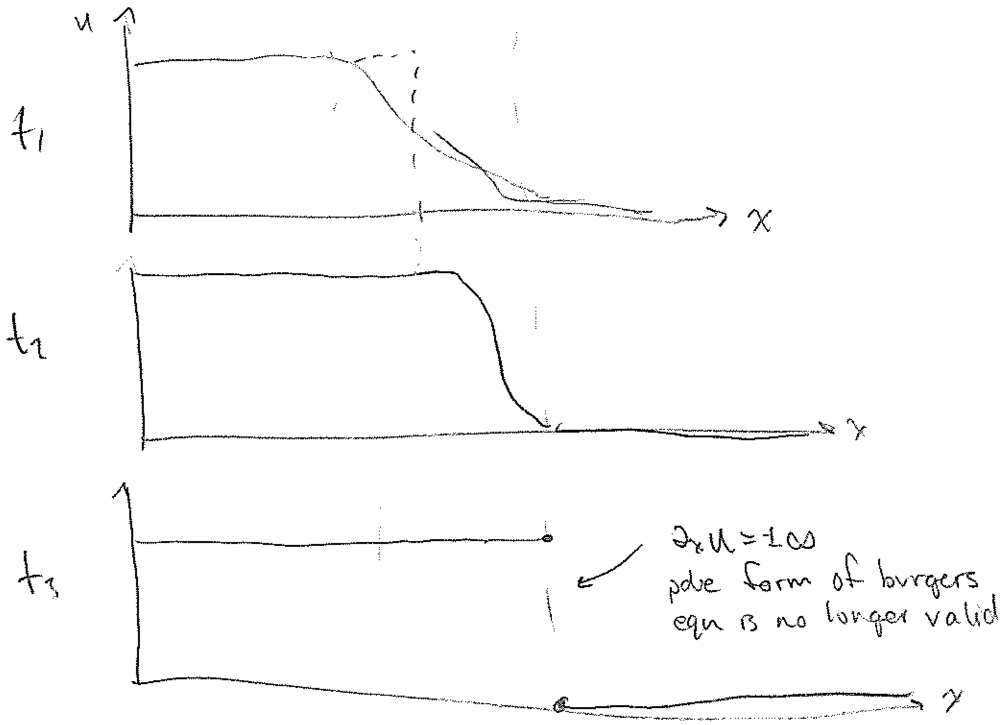
Burgers equation

$$\partial_t u + u\partial_x u = 0 \longleftrightarrow \partial_t u + \partial_x f(u) = 0, \quad f(u) = \tfrac{1}{2}u^2$$

The characteristics are given by
$$x'(t) = f'(u(x(t),t)) = u(x(t),t)$$
- u is constant on curves that satisfy $x'(t) = f'(u(x(t),t))$
- char.s are straight lines (determined by $u_0(x)$)

Example





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Bergers $\quad \partial_t u + u\partial_x u = 0$

$$\partial_t u + \partial_x f(u) = 0, \quad f(u) = \tfrac{1}{2}u^2$$

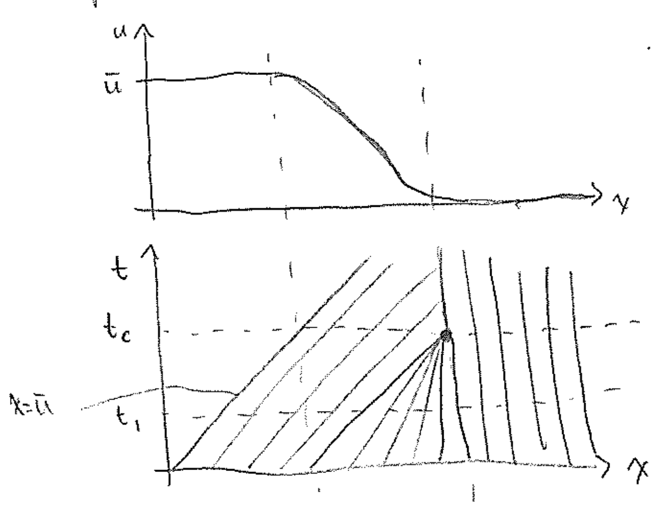
Characteristics

$$x'(t) = f'(u(x(t),t)) = u(x(t),t)$$

We know:

- u is constant on a characteristic
- characteristics are straight lines

Example $\quad -\infty < x < \infty$



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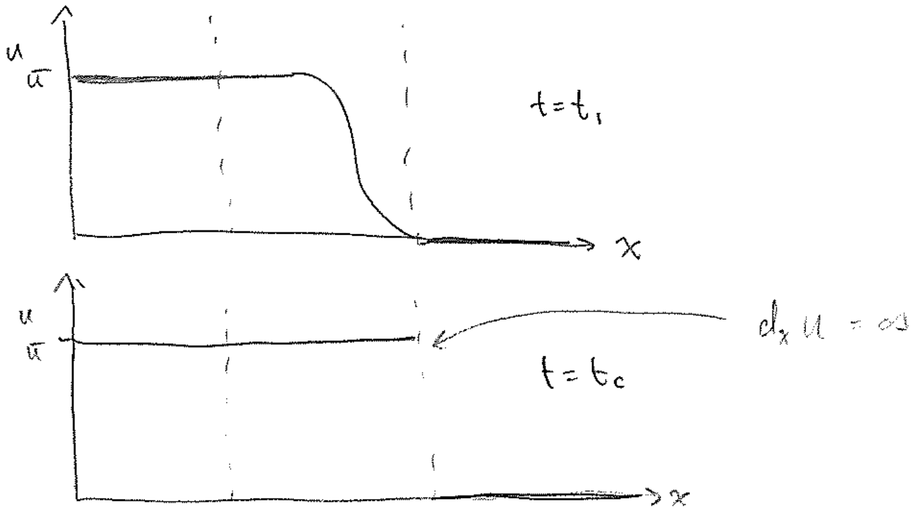
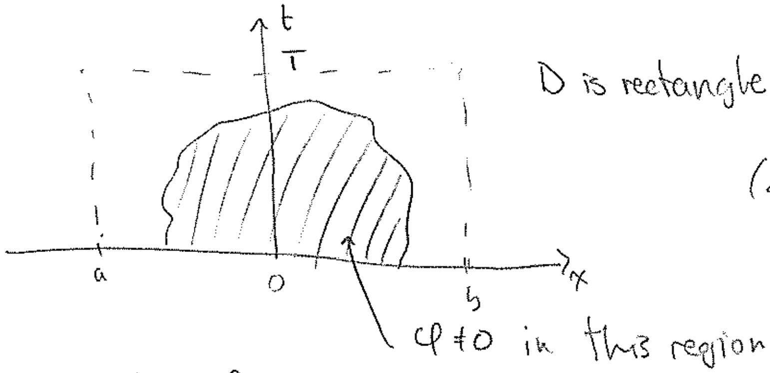


What to do if u is discontinuous?

→ Weak formulation

Assume a $\varphi \in C_0^1(\mathbb{R}\times\mathbb{R})$

C_0^1 is the space of functions that are continuously differentiable with compact support



D is rectangle $0 \le t \le T$, $a \le x \le b$

$(\operatorname{supp}\varphi) \cap (t \ge 0) \subseteq D$

- $\varphi = 0$ outside of D and on $t=T$, $x=a$, $x=b$
- φ is continuously differentiable in D

$$\partial_t u + \partial_x f(u) = 0 \qquad \text{multiply by } \varphi$$

$$(\varphi)(\partial_t u + \partial_x f(u)) = 0 \qquad \text{integrate over space and time}$$

$$\int_0^\infty \int_{-\infty}^{\infty} \varphi \cdot (\partial_t u + \partial_x f(u))\, dx\, dt = 0$$

$$\int_0^T \int_a^b \varphi \cdot (\partial_t u + \partial_x f(u))\, dx\, dt = 0 \qquad \text{because } \varphi \text{ has compact support}$$

($\varphi = 0$ outside D)

Integrate by parts

$$0 = \int_a^b \left(\int_0^T \varphi\, \partial_t u\, dt \right) dx + \int_0^T \left(\int_a^b \varphi\, \partial_x f(u)\, dx \right) dt$$

$$= \int_a^b \left(-\int_0^T (\partial_t \varphi) u \, dt + \underbrace{\varphi u \Big|_{t=0}^{t=T}}_{-\varphi(x,0) u_0(x)} \right) dx + \int_0^T \left(-\int_a^b (\partial_x \varphi) f(u) \, dx - \overset{0}{\cancel{\varphi f(u) \Big|_{x=a}^{x=b}}} \right) dt$$

($\varphi = 0$ on $x = a$, $x = b$)

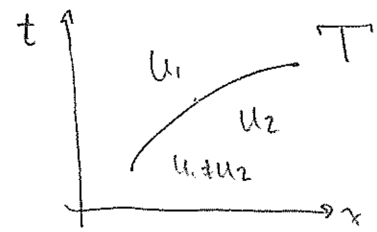
$$= -\int_0^T \int_a^b \Big((\partial_t \varphi) u + (\partial_x \varphi) f(u) \Big) dx\, dt - \int_a^b \varphi(x,0) u_0(x) \, dx$$

$$\boxed{\int_0^T \int_a^b \left(u \partial_t \varphi + f(u) \partial_x \varphi \right) dx\, dt = -\int_a^b \varphi(x,0) u_0(x) \, dx}$$

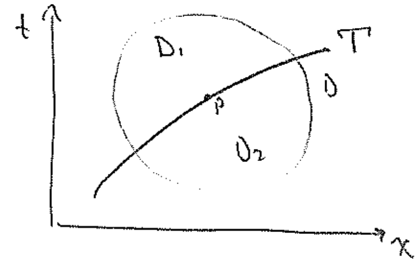
The <u>weak formulation</u> of $\partial_t u + \partial_x f(u) = 0$

Any function u that satisfies the weak formulation is called a <u>weak solution</u> of $\partial_t u + \partial_x f(u) = 0$.

We can use the weak formulation to compute the shock speed.



- T is a smooth curve given by $x = x(t)$
- u discontinuous across this curve T, smooth otherwise



D is small ball around p

Let $\varphi \in C_0^1(D)$

$$0 = \iint_D (u \partial_t \varphi + f(u) \partial_x \varphi) \, dx\, dt = \iint_{D_1} (u \partial_t \varphi + f(u) \partial_x \varphi) \, dx\, dt + \iint_{D_2} \text{''} \, dx\, dt$$

$u \in C^1$ in D_i, $i = 1, 2$

$$\iint_{D_i} (u \partial_t \varphi + f(u) \partial_x \varphi) \, dx\, dt = \iint_{D_i} \partial_t (\varphi u) + \partial_x (\varphi f(u)) \, dx\, dt$$

$$- \iint_{D_i} \varphi \, (\cancelto{0}{\partial_t u + \partial_x f(u)}) \, dx\, dt$$

$$= \iint_{D_i} \partial_t (\varphi u) + \partial_x (\varphi f(u)) \, dx\, dt$$

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Scalar hyperbolic conservation law

$$\partial_t u + \partial_x f(u) = 0, \qquad -\infty < x < \infty$$
$$u(x,0) = u_0(x), \qquad t > 0$$

Weak formulation

$$\int_0^T \int_a^b \left(u \partial_t \varphi + f(u)\partial_x \varphi\right) dx\,dt = -\int_a^b \varphi(x,0)u_0(x)\,dx$$

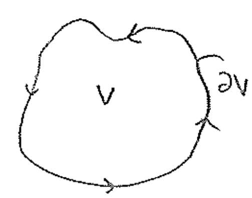
useful to determine shock speed

We showed last time

$$0 = \int_0^T \int_a^b \left(u\partial_t \varphi + f(u)\partial_x \varphi\right) dx\,dt$$
$$= \sum_{i=1}^{2} \iint_{D_i} \left(u\partial_t \varphi + f(u)\partial_x \varphi\right) dx\,dt$$
$$= \sum_{i=1}^{2} \iint_{D_i} \left(\partial_t(\varphi u) + \partial_x(\varphi f(u))\right) dx\,dt$$

Green's Theorem

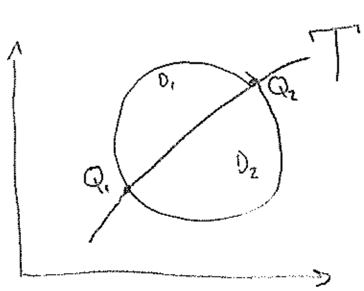
Let ∂V be a closed curve and let f and g have continuous partial derivatives on V, then



$$\iint_V \left(\frac{\partial g}{\partial t} + \frac{\partial f}{\partial x}\right) dx\,dt = \int_{\partial V} -g\,dx + f\,dt$$

So

$$\iint_{D_i} \partial_t(\varphi u) + \partial_x(\varphi f(u))\,dx\,dt = \int_{\partial D_i} \varphi(-u\,dx + f(u)\,dt)$$
$$= \int_T \varphi(-u\,dx + f(u)\,dt$$



$$u_L = u(x(t)-0, t)$$
$$u_R = u(x(t)+0, t)$$

$$\int_{\partial D_1} \varphi(-u\,dx + f(u)\,dt) = \int_{Q_1}^{Q_2} \varphi(-u_L dx + f(u_L)\,dt)$$
$$\int_{\partial D_2} \varphi(-u\,dx + f(u)\,dt) = -\int_{Q_1}^{Q_2} \varphi(-u_R dx + f(u_R)\,dt)$$

$$\int_T \varphi\left(-(u_L-u_R)\,dx+(f(u_L)-f(u_R))\,dt\right)=0$$

Introduce notation

$$[\![u]\!]=u_L-u_R,\qquad [\![f(u)]\!]=f(u_L)-f(u_R)$$

$$\int_T \varphi\left(-[\![u]\!]\,dx+[\![f(u)]\!]\,dt\right)=0.$$

Since φ is arbitrary function:

$$-[\![u]\!]\,dx+[\![f(u)]\!]\,dt=0$$

$$\Longleftrightarrow$$

$$[\![u]\!]\frac{dx}{dt}=[\![f(u)]\!] \quad \text{in each point of } T$$

$\frac{dx}{dt}$ is the shock speed

Let $s=dx/dt$.

$$\boxed{s=\frac{[\![f(u)]\!]}{[\![u]\!]}=\frac{f(u_L)-f(u_R)}{u_L-u_R}} \quad \text{Rankine-Hugoniot condition}$$

The only/unique solution that we are interested in has to satisfy

$$\boxed{f'(u_L)>s>f'(u_R)} \quad \text{(Entropy Condition)}$$

Burgers $\quad \partial_t u+\partial_x\left(\frac{1}{2}u^2\right)=0$

If u is discontinuous then according to Rankine-Hugoniot the shock speed is:

$$s=\frac{[\![f]\!]}{[\![u]\!]}=\frac{\frac{1}{2}u_L^2-\frac{1}{2}u_R^2}{u_L-u_R}=\frac{1}{2}(u_L+u_R)$$

$$\partial_t u+u\partial_x u=0$$

$$u\partial_t u+u^2\partial_x u=0$$

$$\partial_t\left(\tfrac{1}{2}u^2\right)+\partial_x\left(\tfrac{1}{3}u^3\right)=0 \longrightarrow s=\frac{\frac{1}{3}u_L^3-\frac{1}{3}u_R^3}{\frac{1}{2}u_L^2-\frac{1}{2}u_R^2}$$

Riemann Problem

$$\partial_t u+\partial_x f(u)=0 \qquad -\infty<x<\infty$$

$$u(x,0)=\begin{cases}u_L & x<0\\ u_R & x>0\end{cases}$$

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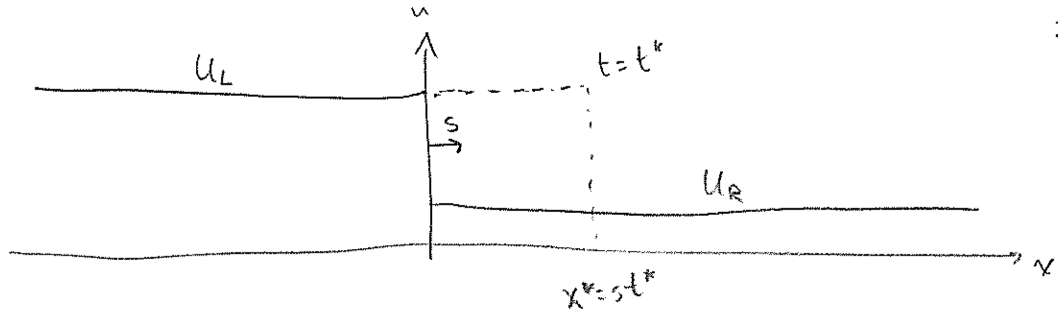


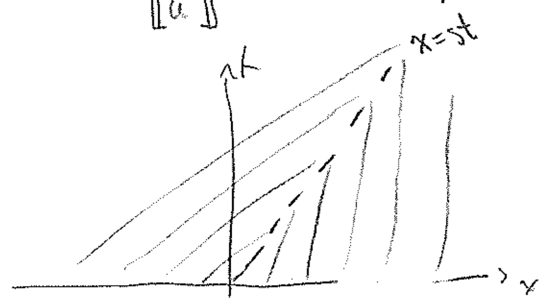
Two cases: 1/ $u_L > u_R$ 2/ $u_L < u_R$

Case 1: $u(x,t) = \begin{cases} u_L & \text{if } x < st \\ u_R & \text{if } x > st \end{cases}$ $s = \dfrac{[\![f]\!]}{[\![u]\!]} = \frac{1}{2}(u_L + u_R)$

$f'(u_L) > s > f'(u_R)$

$\Longleftrightarrow$

$u_L > s > u_R$ ✓



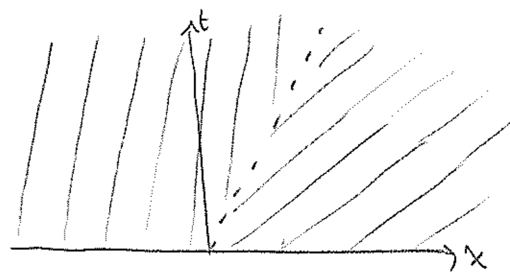
Case 2: $u_L < u_R$

Check: $f'(u_L) > s > f'(u_R)$

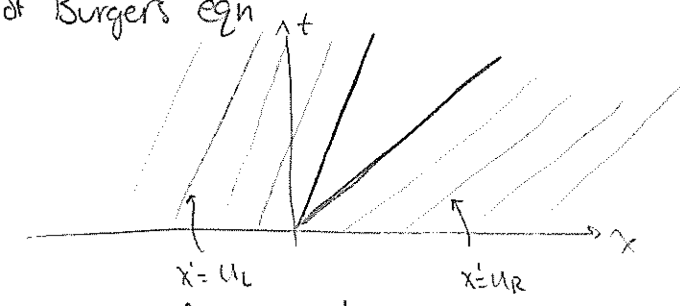
$u_L > \frac{1}{2}(u_L + u_R) > u_R$ ✗

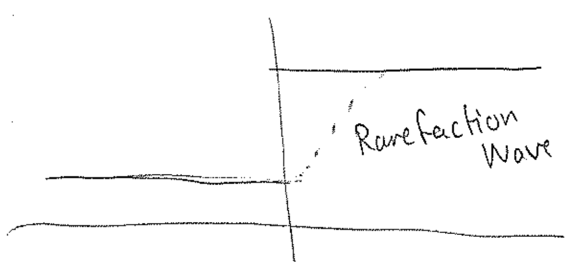
No discontinuities allowed!

→ must have continuous solution



$x'(t) = u$ are characteristics of Burgers eqn





$u = u_L$ if $x < u_L t$

$u = u_R$ if $x > u_R t$

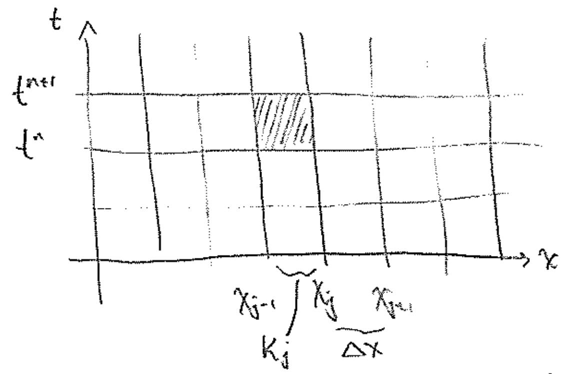
$u_L t < x < u_R t$? $u = \frac{x}{t}$

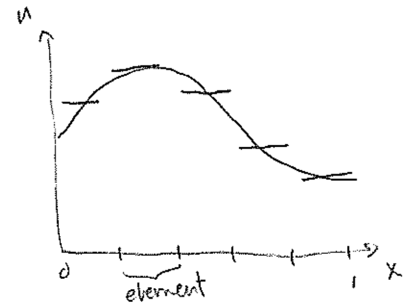
2015 10 28

Finite Volume Method

Conservation Laws

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 \\ u(x,0) = u_0(x) \end{cases}$$





$$\Delta x_j = x_j - x_{j-1}, \quad \Delta t_n = t_n - t_{n-1}$$

Simplicity: $\Delta x_j = \Delta x \ \ \forall j, \quad \Delta t^n = \Delta t \ \ \forall n$

<u>Element</u> $K_j = [x_{j-1}, x_j]$

FV in $K_j \times [t^n, t^{n+1}]$

The integral form of $\partial_t u + \partial_x f(u) = 0$ on $K_j \times [t^n, t^{n+1}]$

$$\int_{K_j} u(x, t^{n+1})\,dx = \int_{K_j} u(x, t^n)\,dx - \int_{t^n}^{t^{n+1}} \Big(f(u(x_j, t)) - f(u(x_{j-1}, t)) \Big)\,dt$$

Define the averages

$$U_j^n = \frac{1}{\Delta x} \int_{K_j} u(x, t_n)\,dx, \quad F_j^n = \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} f(u(x_j, t))\,dt.$$

$$\int_{K_j} u(x, t_{n+1})\,dx = U_j^{n+1} \Delta x, \quad \int_{K_j} u(x, t_n)\,dx = U_j^n \Delta x, \ \&$$

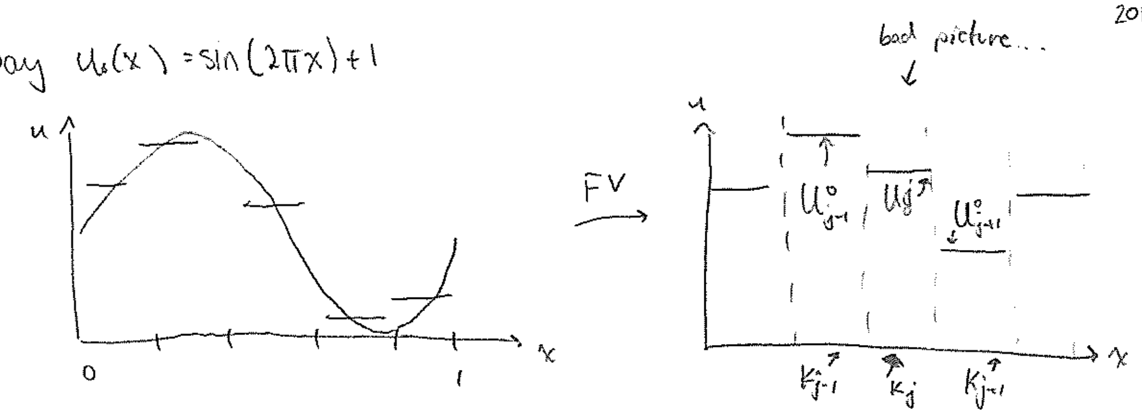
$$\int_{t_n}^{t_{n+1}} f(u(x_j, t))\,dt = F_j^n \Delta t, \quad \int_{t_n}^{t_{n+1}} f(u(x_{j-1}, t))\,dt = F_{j-1}^n \Delta t$$

$$\Delta x U_j^{n+1} = \Delta x U_j^n - \Delta t \left(F_j^n - F_{j-1}^n \right)$$

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x} \left(F_j^n - F_{j-1}^n \right)$$

2015 10 28

Say $u_0(x) = \sin(2\pi x) + 1$



$$U_j^1 = U_j^0 - \frac{\Delta t}{\Delta x}\left(F_j^0 - F_{j-1}^0\right)$$

evaluated at the boundary of K_j

Problem: How do we evaluate F_j^n?

Solution: Introduce a function $\hat{F}$ that approximates F_j^n and is a function of U_j^n and U_{j+1}^n.

$$\hat{F}(U_j^n, U_{j+1}^n) \approx F_j^n$$

numerical flux

Options for Numerical flux

- $\hat{F}(U_j^n, U_{j+1}^n) = \frac{1}{2}\left(f(U_j^n) + f(U_{j+1}^n)\right)$ $\times$
- $\hat{F}(U_j^n, U_{j+1}^n) = f(U^*)$

Call U^* the solution to the Riemann problem:

(Godunov Method)

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 \\ u_0(x) = \begin{cases} U_j^n & x_j < 0 \\ U_{j+1}^n & x_j > 0 \end{cases} \end{cases}$$

number?

Burgers

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, \quad f(u) = \frac{1}{2}u^2 \\ u(x,0) = u_0(x) \end{cases}$$

FV: $$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x}\left(\hat{F}(U_j^n, U_{j+1}^n) - \hat{F}(U_{j-1}^n, U_j^n)\right)$$

What is $\hat{F}(U_L, U_R)$? Riemann problem of Burgers eqn.

$$\partial_t u + \partial_{\tilde{x}} f(u) = 0$$

$$u(\tilde{x}, 0) = \begin{cases} u_L & \tilde{x} < 0 \\ u_R & \tilde{x} > 0 \end{cases}$$

Solution to the Riemann problem (Burgers):

if $u_L \le u_R$ $\quad u(\tilde{x},t) = \begin{cases} u_L & \tilde{x}/t < u_L \\ \tilde{x}/t & u_L \le \tilde{x}/t \le u_R \\ u_R & u_R < \tilde{x}/t \end{cases}$

if $u_L > u_R$ $\quad u(\tilde{x},t) = \begin{cases} u_L & \tilde{x}/t < S \\ u_R & \tilde{x}/t > S \end{cases}$ $\qquad S = \dfrac{[\![f]\!]}{[\![u]\!]} = \frac{1}{2}(u_L + u_R)$

Combined:

$$u^*(\tilde{x},t) = \begin{cases} u_L & \tilde{x}/t \le S_L \\ \tilde{x}/t & S_L < \tilde{x}/t < S_R \\ u_R & \tilde{x}/t) \ge S_R \end{cases}$$

$$S_L = \min\left(\tfrac{1}{2}(u_L + u_R), u_L\right)$$
$$S_R = \max\left(\tfrac{1}{2}(u_L + u_R), u_R\right)$$

$$\hat{F}(u_L, u_R) = \begin{cases} \frac{1}{2}u_L^2 & 0 \le S_L \\ 0 & S_L < 0 < S_R \\ \frac{1}{2}u_R^2 & S_R \le 0 \end{cases}$$

2015 10 30

Numerical Flux Options

- $\hat{F}(u_L, u_R) = \frac{1}{2}\left(f(u_L) + f(u_R)\right)$ $\quad$ X
- Riemann Problem

$$\hat{F}(u_L,u_R) = \begin{cases} \frac{1}{2}u_L^2 & 0 \le S_L \\ 0 & S_L \le 0 < S_R \\ \frac{1}{2}u_R^2 & S_R \le 0 \end{cases} \qquad \begin{aligned} S_L &= \min\left(\tfrac{1}{2}(u_L+u_R), u_L\right) \\ S_R &= \min\left(\tfrac{1}{2}(u_L+u_R), u_R\right) \end{aligned}$$

- Approximate Riemann Problem

$$\partial_t u + \partial_x\left(\tfrac{1}{2}u^2\right) = 0$$
$$u(x,0) = \begin{cases} u_L & x < 0 \\ u_R & x > 0 \end{cases}$$

Approximation: assume the solution is always a flux

$$u(x,t) = \begin{cases} u_L & x/t < S \\ u_R & x/t > S \end{cases} \qquad S = \tfrac{1}{2}(u_L + u_R)$$

$$\hat{F}(u_L, u_R) = \begin{cases} \frac{1}{2}u_L^2 & 0 < S \\ \frac{1}{2}u_R^2 & S < 0 \end{cases}$$

approximate Riemann solver

2015 10 30(2)

Properties of numerical flux

- $\hat{F}$ has to be consistent
  c is constant solution then $\hat{F}(c,c) = f(c)$
- $\hat{F}$ has to be conservative (unclear)
- higher order FV
  $\hat{F}(U_{j-2}^n, U_{j-1}^n, U_j^n, U_{j+1}^n, U_{j+2}^n) = \cdots$ for example
- Implicit FV method:

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x}\left(\hat{F}(U_j^{n+1}, U_{j+1}^{n+1}) - \hat{F}(U_{j-1}^{n+1}, U_j^{n+1})\right)$$

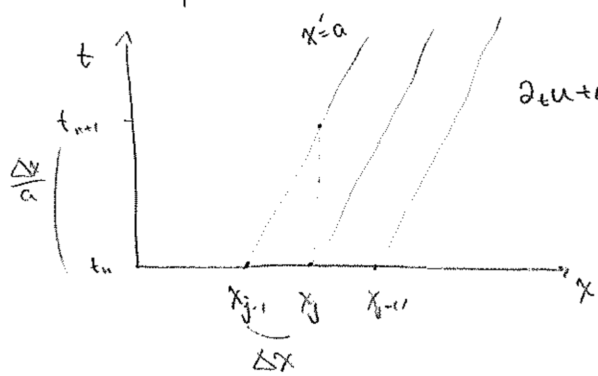
The CFL-condition

↳ states a necessary condition for stability
↳ says information from a point x_j cannot travel past one of its neighbouring points within one timestep

$$t^{n+1} < t^c \iff \Delta t < t^c - t^n$$

$$\Delta t \le \frac{\Delta x}{a}$$

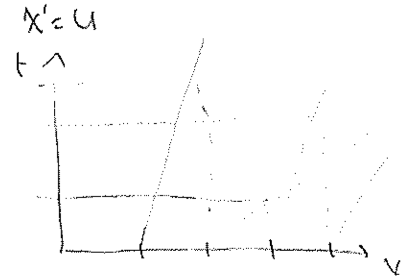
$$\Rightarrow \Delta t = \frac{\nu \Delta x}{a} \qquad 0 < \underset{\boxed{\text{CFL-number}}}{\nu} \le 1$$



$\partial_t u + a\partial_x u = 0$

For Burgers, characteristics are $x' = u$

$$\Delta t = \frac{\nu \Delta x}{\max_j |u|} = \min_j \frac{\nu \Delta x}{|u|}$$



Convergence

Remember: stability + consistency = convergence
CFL ✓ ✓

(Entropy condition $f'(u_L) > s > f'(u_R)$)

Theorem: The solution of a monotone conservative scheme converges to an entropy solution.

- Conservative scheme

a scheme of the form

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x}\left(F_j^n - F_{j-1}^n\right)$$

list at top of prev. page

where the fluxes can be written in terms of a "proper" numerical flux:

$$F_j^n = \hat{F}(U_j^n, U_{j+1}^n)$$

- Monotone scheme

A scheme is monotone when any two solutions u and v that satisfy

$$U_j^n \geq V_j^n \ \forall j \longrightarrow U_j^{n+1} \geq V_j^{n+1} \ \forall j$$

Monotonicity check

write FV as

$$U_j^{n+1} = H(U_{j-1}^n, U_j^n, U_{j+1}^n)$$

check whether $H(u,v,w)$ is increasing in all of its arguments:

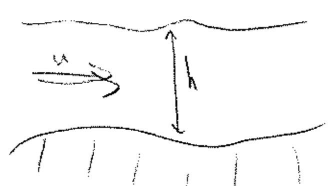
$$\frac{\partial H}{\partial u} > 0, \quad \frac{\partial H}{\partial v} > 0, \quad \frac{\partial H}{\partial w} > 0$$

Systems of Conservation Laws

The Shallow Water Equations (1D) (SWE)

$$\partial_t h + \partial_x (hu) = 0 \quad \text{(conservation of mass)}$$

$$\partial_t hu + \partial_x (hu^2 + \tfrac{1}{2}gh^2) = 0 \quad \text{(conservation of momentum)}$$



g = accelleration due to gravity constant

2015 11 02

$$\partial_t U + \partial_x F(U) = 0, \qquad U = \begin{bmatrix} h \\ hu \end{bmatrix} \quad F(U) = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \end{bmatrix}$$

FV: $$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x}\left(\hat{F}(U_j^n, U_{j+1}^n) - \hat{F}(U_{j-1}^n, U_j^n)\right)$$

time step. CFL-condition $\quad \Delta t \leq \frac{\nu \Delta x}{|\lambda|} \quad 0 < \nu \leq 1$

$|\lambda| \longleftarrow$ slope of characteristics

scalar: $\partial_t u + \partial_x f(u) = 0$

$$x' = f(u)$$

2015 11 02 (2)

<u>system</u>: $\partial_t U + \partial_x F(U) = 0$

$$\partial_t U + \frac{\partial F}{\partial U}\partial_x U = 0$$

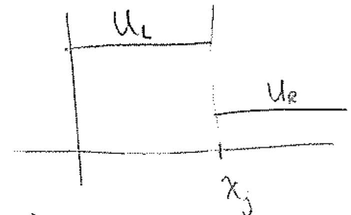
↪ compute eigenvalues

$$\frac{\partial F}{\partial U} = \begin{bmatrix} 0 & 1 \\ -u^2+gh & 2u \end{bmatrix}$$

$$0 = \det\left(\frac{\partial F}{\partial U} - \lambda I\right) = \begin{vmatrix} -\lambda & 1 \\ -u^2+gh & 2u-\lambda \end{vmatrix} = (-\lambda)(2u-\lambda) - (1)(-u^2+gh)$$

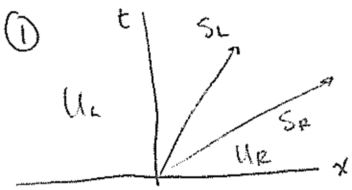
$$= \lambda^2 - 2u\lambda + (u^2 - gh)$$

$$\lambda^{\pm} = \frac{2u \pm \sqrt{4u^2 - 4(u^2-gh)}}{2} = u \pm \sqrt{gh}$$



$$S_L = \min\left(u_L - \sqrt{gh_L},\ u_R - \sqrt{gh_R}\right)$$
$$S_R = \max\left(u_L + \sqrt{gh_L},\ u_R + \sqrt{gh_R}\right)$$

①



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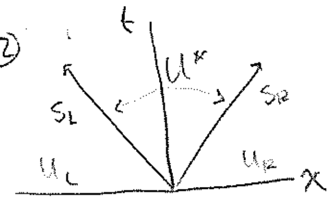

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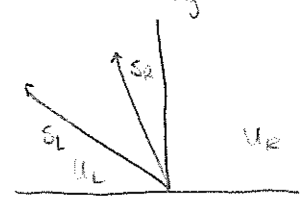


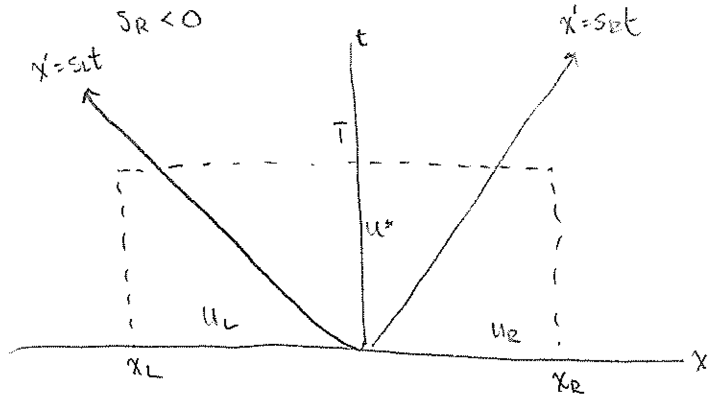
$$\Delta t = \min_{K_j} \frac{\nu \Delta x}{\max(|S_L|, |S_R|)}$$

<u>Numerical Flux</u>

$$\hat{F}(U_L, U_R) = \begin{cases} F(U_L) & S_L > 0 \\ F^* & S_L < 0 < S_R \\ F(U_R) & S_R < 0 \end{cases}$$

Define $\overline{U}^* = \dfrac{1}{TS_R - TS_L}\displaystyle\int_{TS_L}^{TS_R} U(x,T)\,dx$

$$\partial_t U + \partial_x F(U) = 0$$



$$\int_{x_L}^{0} U(T,x)\,dx = \int_{x_L}^{0} U(0,x) - \int_{0}^{T}\Big(F(U(t,0)) - F(U(t,x_L))\Big)dt$$

$$\int_{x_L}^{0} U(T,x)\,dx = \int_{x_L}^{S_L T} U_L\,dx - \int_{S_L T}^{0} U(x,T)\,dx$$

$$\int_{x_L}^{0} U(0,x)\,dx = \int_{x_L}^{0} U_L\,dx$$

$$\int_0^T F(U(t,x_L))\,dt = \int_0^T F_L\,dt \qquad \left(F_L = F(U_L)\right)$$

$$(S_L T - x_L)U_L + (0 - S_L T)\overline{U}^* = (0 - x_L)U_L + TF_L - TF^*$$

$$\Leftrightarrow \boxed{S_L U_L - S_L \overline{U}^* = F_L - F^*}$$

Look now at $[0, x_R] \times [0, T]$

$$\boxed{S_R \overline{U}^* - S_R U_R = F^* - F_R}$$

two equations for $\overline{U}^*$, F^*

$$F^* = \tfrac{1}{2}(F_L + F_R) + \tfrac{1}{2}\left(S_R \overline{U}^* + S_L \overline{U}^* - S_R U_R - S_L U_L\right)$$

$$\left(\overline{U}^* = \frac{S_R U_R - S_L U_L + F_L + F_R}{S_R - S_L}\right)$$

$$F^* = \frac{S_R F_L - S_L F_R + S_L S_R (U_R - U_L)}{S_R - S_L}$$

$$\hat{F}(U_L, U_R) = \begin{cases} F_L & S_L > 0 \\ F^* & S_L < 0 < S_R \\ F_R & S_R < 0 \end{cases}$$

HLL-flux

2015 11 04

2D SWE

$\vec{u} = (u, v)$ (velocity vector)

$\partial_t h + \partial_x hu + \partial_y hv = 0$ (mass conservation)

$\partial_t hu + \partial_x (hu^2 + \tfrac{1}{2} g h^2) + \partial_y (huv) = 0$ (x-momentum)

$\partial_t hv + \partial_x (huv) + \partial_y (hv^2 + \tfrac{1}{2} g h^2) = 0$ (y-momentum)

Conservation form

$$\partial_t U + \nabla \cdot \overline{F} = 0$$

$$U = \begin{bmatrix} h \\ hu \\ hv \end{bmatrix} \quad \overline{F} = [F \; G] \quad F = \begin{bmatrix} hu \\ hu^2 + \tfrac{1}{2} g h^2 \\ huv \end{bmatrix} \quad G = \begin{bmatrix} hu \\ huv \\ hv^2 + \tfrac{1}{2} g h^2 \end{bmatrix}$$

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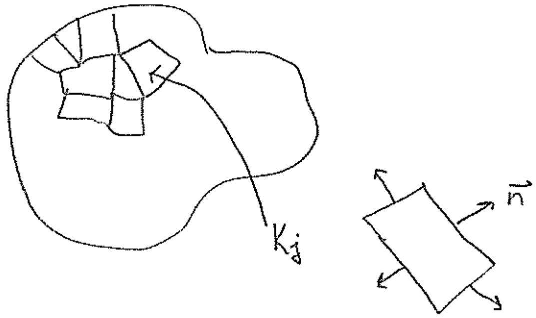


Restrict pde to K_j and $[t_n, t_{n+1}]$, then the integral form of pde:

$$\int_{K_j} U(t_{n+1}, \vec x)\,d\vec x = \int_{K_j} U(t_n, \vec x)\,d\vec x - \int_{t_n}^{t_{n+1}} \left(\int_{\partial K_j} \bar F(U(t,\vec x))\cdot \vec n\, ds \right) dt$$

Denote the area of K_j by $|K_j|$.

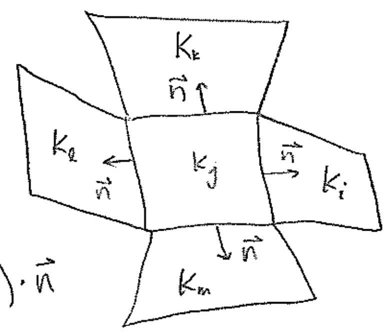
Introduce the average:

$$U_j^n = \frac{1}{|K_j|} \int_{K_j} U(t_n, \vec x)\, d\vec x \qquad \left(1D: \ \frac{1}{\Delta x} \int_{x_{j-1}}^{x_j} u(t_n, x)\,dx \right)$$

Approximate the flux term by evaluating in $t = t_n$:

$$\int_{t_n}^{t_{n+1}} \left(\int_{\partial K_j} \bar F(U(t,\vec x))\cdot \vec n\, ds \right) dt \approx \Delta t \int_{\partial K_j} \bar F(U(t_n, \vec x))\cdot \vec n\, ds$$

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{|K_j|} \underbrace{\int_{\partial K_j} \bar F(U(t_n,\vec x))\cdot \vec n\, ds}_{\text{look at this}}$$



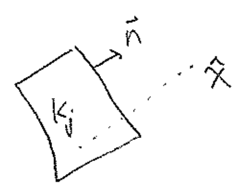
We need a numerical flux!

Introduce

$$\hat{\bar F}\left(U^n_{interior}, U^n_{exterior}, \vec n\right) \approx \bar F(U(t_n, \vec x))\cdot \vec n$$

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{|K_j|} \int_{\partial K_j} \hat{\bar F}(U_{int}, U_{ext}, \vec n)\, ds$$

What is $\hat{\bar F}$?



in coordinate system $\hat x$, use 1D numerical flux

$$\partial_t U + \partial_{\hat x} H = 0$$

$$H = \bar F\cdot \vec n = F n_x + F n_y = \begin{bmatrix} h(un_x + vn_y) \\ hu(un_x+vn_y) + n_x(\frac12 gh^2) \\ hv(un_x+vn_y) + n_y(\frac12 gh^2) \end{bmatrix}$$

$$\begin{bmatrix} hq \\ huq + \frac12 gh^2 n_x \\ hvq + \frac12 gh^2 n_y \end{bmatrix} =$$

$q = un_x + vn_y$

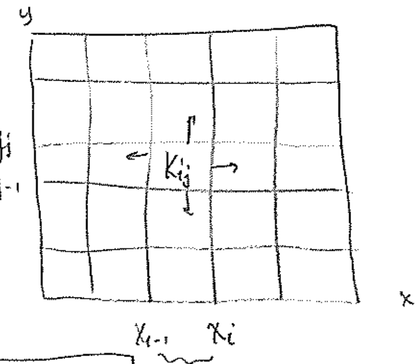
Use HLL flow:

$$\hat{\bar{F}}(u_L, u_R, \vec{n}) = \begin{cases} \bar{F}^L \cdot \vec{n} & S_L \geq 0 \\ F^* & S_L < 0 < S_R \\ \bar{F}^R \cdot \vec{n} & S_R \leq 0 \end{cases}$$

$$F^* = \frac{S_R \bar{F}_L \cdot \vec{n} - S_L \bar{F}_R \cdot \vec{n} + S_L S_R (U_R - U_L)}{S_R - S_L}$$

$$S_L = \min\left(q_L - \sqrt{g h_L},\ q_R - \sqrt{g h_R}\right)$$

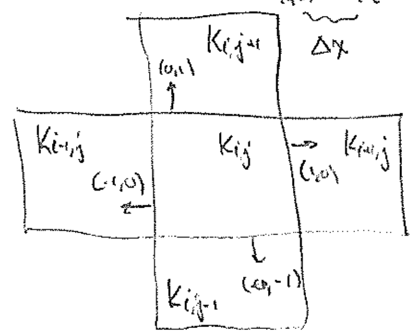
$$S_R = \max\left(q_L + \sqrt{g h_R},\ q_R + \sqrt{g h_R}\right)$$



normals: $(1,0), (-1,0), (0,1), (0,-1)$

Solution in K_{ij} as U_{ij}^n.

$$U_{ij}^{n+1} = U_{ij}^n - \frac{\Delta t}{|K_{ij}|} \int_{\partial K_{ij}} \hat{\bar{F}}(U_{int}^n, U_{ext}^n, \vec{n})\, ds$$



$$\int_{\partial K_{ij}} \hat{\bar{F}}(U_{int}^n, U_{ext}^n, \vec{n})\, ds$$

$$= \int_{x_{i-1}}^{x_i} \hat{\bar{F}}(U_{ij}^n, U_{i,j-1}^n, (0,-1))\, ds$$

$$+ \int_{y_{j-1}}^{y_j} \hat{\bar{F}}(U_{ij}^n, U_{i+1,j}^n, (1,0))\, ds$$

$$+ \int_{x_{i-1}}^{x_i} \hat{\bar{F}}(U_{ij}^n, U_{i,j+1}, (0,1))\, ds + \int_{y_{j-1}}^{y_j} \hat{\bar{F}}(U_{ij}^n, U_{i-1,j}^n, (-1,0))\, ds$$

$$\int_{x_{i-1}}^{x_i} \hat{\bar{F}}(U_{ij}^n, U_{i,j-1}^n, (0,-1))\, ds = \Delta x\, \hat{G}(U_{ij}^n, U_{i,j-1}^n, (0,-1))$$

$$\int_{y_{j-1}}^{y_j} \hat{\bar{F}}(U_{ij}^n, U_{i+1,j}^n, (1,0))\, ds = \Delta y\, \hat{F}(U_{ij}^n, U_{i+1,j}^n, (1,0))$$

(3rd term) $= \Delta x\, \hat{G}(U_{ij}^n, U_{i,j+1}^n, (0,1))$

(4th term) $= \Delta y\, \hat{F}(U_{ij}^n, U_{i-1,j}^n, (-1,0))$

$$|K_{ij}| = \Delta x \Delta y$$

2015 11 04(3)

$$U_{ij}^{n+1} = U_{ij}^{n} - \frac{\Delta t}{\Delta x}\left(\hat{F}(U_{ij}^{n}, U_{i+1,j}^{n}, (1,0)) + \hat{F}(U_{ij}^{n}, U_{i-1,j}^{n}, (-1,0))\right)$$
$$- \frac{\Delta t}{\Delta y}\left(\hat{F}(U_{ij}^{n}, U_{i,j+1}^{n}, (0,1)) + \hat{F}(U_{ij}^{n}, U_{i,j-1}^{n}, (0,-1))\right)$$