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Chapter 4

$$\vec{y}' = \Lambda\vec{y}, \quad \vec{y}(0) = \vec{y}_0, \qquad \Lambda = \begin{bmatrix} -100 & 1 \\ 0 & -1/10 \end{bmatrix}$$

Euler:

$$\vec{y}_1 = \vec{y}_0 + h\Lambda\vec{y}_0 = (I + h\Lambda)\vec{y}_0$$

$$\vec{y}_2 = \vec{y}_1 + h\Lambda\vec{y}_1 = (I + h\Lambda)\vec{y}_1 = (I + h\Lambda)^2\vec{y}_0$$

$$\vdots$$

$$\vec{y}_n = (I + h\Lambda)^n \vec{y}_0$$

Eigenvalues and eigenvectors of Λ:

$$\lambda_1 = -100 \qquad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\lambda_2 = -1/10 \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ 999/10 \end{bmatrix}$$

We can write $\Lambda = VDV^{-1}$ where

$$V = [\vec{v}_1 \ \vec{v}_2] = \begin{bmatrix} 1 & 1 \\ 0 & \frac{999}{10} \end{bmatrix}, \quad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -100 & 0 \\ 0 & -1/10 \end{bmatrix}.$$

The exact solution is $\vec{y}(t) = \exp(\Lambda t)\vec{y}_0$. By definition,

$$\exp(\Lambda t) = \sum_{k=0}^{\infty} \frac{1}{k!} t^k \Lambda^k = \sum_{k=0}^{\infty} \frac{1}{k!} t^k \underbrace{(VDV^{-1})^k}_{VD^kV^{-1}} = V\left(\sum_{k=0}^{\infty} \frac{1}{k!} t^k D^k\right)V^{-1} = V\exp(Dt)V^{-1}.$$

Thus $\vec{y}(t) = V\exp(Dt)V^{-1}\vec{y}_0$. Note

$$(I + hVDV^{-1})^2 = I + 2hVDV^{-1} + h^2VD^2V^{-1} = V(I + hD)^2V^{-1}$$

and so induction can show

$$(I + hVDV^{-1})^n = V(I + hD)^nV^{-1}.$$

So $\vec{y}_n = (I + hVDV^{-1})^n\vec{y}_0 = V(I + hD)^nV^{-1}\vec{y}_0$.

Exact: $\vec{y}(t) = V\exp(tD)V^{-1}\vec{y}_0$

Numerical: $\vec{y}_n = V(I + hD)^nV^{-1}\vec{y}_0$

$$\exp(tD) = \begin{bmatrix} \exp(-100t) & 0 \\ 0 & \exp(t/10) \end{bmatrix}, \quad (I + hD)^n = \begin{bmatrix} (1-100h)^n & 0 \\ 0 & (1 - \frac{1}{10}h)^n \end{bmatrix}$$

Let $\underline{x} = V^{-1}\vec{y}_0$. Then

$$V\begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}V^{-1}y_0 = V\begin{bmatrix} d_1x_1 \\ \vdots \\ d_nx_n \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n V_{1i}d_ix_i \\ \vdots \\ \sum_{i=1}^n V_{ni}d_ix_i \end{bmatrix} = \sum_{i=1}^n d_i \begin{bmatrix} V_{1i}x_i \\ \vdots \\ V_{ni}x_i \end{bmatrix} = \sum_{i=1}^n d_i\vec{z}_i.$$

So

$$\vec{y}(t) = \exp(-100t)\vec{z}_1 + \exp(-t/10)\vec{z}_2$$

$$\vec{y}_n = (1-100h)^n\vec{z}_1 + (1 - h/10)^n\vec{z}_2$$

Exact: $\lim_{t\to\infty} \vec{y}(t) = 0$

Numerical: $\lim_{n\to\infty} \left((1-100h)^n \vec{z}_1 + (1-\frac{1}{10}h)^n \vec{z}_2\right) = 0$ if and only if $|1-\frac{1}{10}h| < 1$ and $|1-100h| < 1$, that is, $h < \frac{1}{50}$.

So for Euler's method we need $h < \frac{1}{50}$ for stability.

What about trapezoidal rule?

$$\vec{y}_{n+1} = \vec{y}_n + \tfrac{1}{2}h\Lambda(\vec{y}_n + \vec{y}_{n+1})$$
$$(I - \tfrac{1}{2}h\Lambda)\vec{y}_{n+1} = (I + \tfrac{1}{2}h\Lambda)\vec{y}_n$$
$$(I - \tfrac{1}{2}hVDV^{-1})\vec{y}_{n+1} = (I + \tfrac{1}{2}hVDV^{-1})\vec{y}_n$$
$$V(I - \tfrac{1}{2}hD)V^{-1}\vec{y}_{n+1} = V(I + \tfrac{1}{2}hD)V^{-1}\vec{y}_n$$
$$\vec{y}_{n+1} = V(I - \tfrac{1}{2}hD)^{-1}(I + \tfrac{1}{2}hD)V^{-1}\vec{y}_n$$

By induction

$$\vec{y}_n = V\underbrace{\left((I - \tfrac{1}{2}hD)^{-1}(I + \tfrac{1}{2}hD)\right)^n}_{\begin{bmatrix} \left(\frac{1+\frac{1}{2}h\lambda_1}{1-\frac{1}{2}h\lambda_1}\right)^n & 0 \\ 0 & \left(\frac{1+\frac{1}{2}h\lambda_2}{1-\frac{1}{2}h\lambda_2}\right)^n \end{bmatrix}} V^{-1}\vec{y}_0$$

$$\vec{y}_n = \left(\frac{1+\frac{1}{2}h\lambda_1}{1-\frac{1}{2}h\lambda_1}\right)^n \vec{x}_1 + \left(\frac{1+\frac{1}{2}h\lambda_2}{1-\frac{1}{2}h\lambda_2}\right)^n \vec{x}_2$$
$$= \underbrace{\left(\frac{1-50h}{1+50h}\right)^n}_{|\cdot|<1} \vec{x}_1 + \underbrace{\left(\frac{1-\frac{1}{20}h}{1+\frac{1}{20}h}\right)^n}_{|\cdot|<1} \vec{x}_2$$

$$\vec{y}' = \begin{bmatrix} -100 & 0 \\ 0 & -\frac{1}{10} \end{bmatrix}$$

is a stiff ODE: there is a numerical method where $h \ll 1$ to be stable.

$$\text{stiffness ratio} = \frac{|\lambda_{max}|}{|\lambda_{min}|} = \frac{|-100|}{|-\frac{1}{10}|} = 1000$$

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§4.2 - Linear stability domain and A-stability

Linear scalar equation

$$y' = \lambda y, \quad t \geq 0, \quad y(0) = 1$$

Exact solution: $y(t) = \exp(\lambda t)$. We have $\lim_{t\to\infty} y(t) = 0$ if and only if $\mathrm{Re}(\lambda) < 0$.

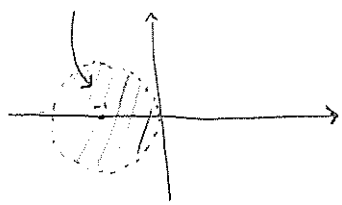
Def] The linear stability domain $\mathcal{D}$ of a numerical method is the set of all $h\lambda \in \mathbb{C}$ such that $\lim_{n\to\infty} y_n = 0$.

Euler's method for $y' = \lambda y$, $y(0) = 1$:

$$y_n = (1+h\lambda)y_{n-1} = (1+h\lambda)^n y_0 = (1+h\lambda)^n$$

So $\lim_{n\to\infty} y_n = 0$ if and only if $|1+h\lambda| < 1$. The linear stability Domain of Euler's method is

$$\mathcal{D} = \{z \in \mathbb{C};\ |1+z| < 1\}$$



General linear ODE:

$$\vec{y}' = \Lambda \vec{y}, \quad \vec{y}(0) = \vec{y}_0, \quad \Lambda \in \mathbb{R}^{d\times d}$$

Assume that $\Lambda = VDV^{-1}$ where $D = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_d)$ and $V = [\vec{v}_1 \cdots \vec{v}_d]$.
We can write Euler's method using spectral factorization as

$$\vec{y}_n = \sum_{k=1}^{d} (1+h\lambda_k)^n \vec{x}_k, \quad n = 0,1,2,\dots$$

where $\vec{x}_k \in \mathbb{C}^d$. For the scheme to be stable, we need $|1+h\lambda| < 1 \ \forall k$.

For the rest of this chapter we look only at the linear scalar ODE $y' = \lambda y$.
Euler: $\mathcal{D} = \{z \in \mathbb{C},\ |1+z| < 1\}$
Trapezoidal rule:

$$y_n = y_{n-1} + \tfrac{1}{2}h\lambda y_{n-1} + \tfrac{1}{2}h\lambda y_n$$
$$\Rightarrow (1 - \tfrac{1}{2}h\lambda)y_n = (1 + \tfrac{1}{2}h\lambda)y_{n-1}$$
$$\Rightarrow y_n = \left(\frac{1+\frac{1}{2}h\lambda}{1-\frac{1}{2}h\lambda}\right)y_{n-1} = \left(\frac{1+\frac{1}{2}h\lambda}{1-\frac{1}{2}h\lambda}\right)^n$$

For the trapezoidal rule to be stable:

$$\left|\frac{1+\frac{1}{2}h\lambda}{1-\frac{1}{2}h\lambda}\right| < 1.$$

Linear stability domain:

$$\mathcal{D} = \left\{ z \in \mathbb{C};\ \left|\frac{1+\frac{1}{2}z}{1-\frac{1}{2}z}\right| < 1 \right\}$$

We can simplify this:

$$\left|\frac{1+\frac{1}{2}z}{1-\frac{1}{2}z}\right|<1 \quad \text{if and only if } |1+\tfrac{1}{2}z| < |1-\tfrac{1}{2}z|$$

Switch to polar coordinates: $z = \rho\exp(i\theta)$, $\rho > 0$.

$$|1+\tfrac{1}{2}\rho\exp(i\theta)|^2 < |1-\tfrac{1}{2}\rho\exp(i\theta)|^2$$

$$|1+\tfrac{1}{2}\rho\cos(\theta)+\tfrac{1}{2}\rho i\sin(\theta)|^2 < |1-\tfrac{1}{2}\rho\cos(\theta)-\tfrac{1}{2}\rho i\sin(\theta)|$$

$$(1+\tfrac{1}{2}\rho\cos(\theta))^2 + \cancel{(\tfrac{1}{2}\rho\sin(\theta))^2} < (1-\tfrac{1}{2}\rho\cos(\theta))^2 + \cancel{(\tfrac{1}{2}\rho\sin(\theta))^2}$$

$$\cancel{1}+\rho\cos(\theta)+\cancel{\tfrac{1}{4}\rho^2\cos^2(\theta)} < \cancel{1}-\rho\cos(\theta)+\cancel{\tfrac{1}{4}\rho^2\cos^2(\theta)}$$

$$\cos(\theta) < -\cos(\theta)$$

$$\Rightarrow \theta \in \left(-\tfrac{3\pi}{2}, -\tfrac{\pi}{2}\right)$$

$$\Rightarrow \mathrm{Re}(z) < 0$$

So

$$\mathcal{D} = \{z\in\mathbb{C};\ \mathrm{Re}(z)<0\}.$$

So given that $\lambda<0$, we can choose any $h>0$.

<u>Def</u> | A method is A-stable if

$$\mathbb{C}^- := \{z\in\mathbb{C};\ \mathrm{Re}(z)<0\} \subseteq \mathcal{D}.$$

<u>§4.3 - A-stability of Runge-Kutta methods</u>

Problem:

$$y' = \lambda y,\quad t\geq 0,\quad y(0)=1$$

RK method:

$$k_j = y_n + h\sum_{i=1}^{\nu} a_{ji}\lambda k_i, \quad j=1,\dots,\nu$$

$$\vec{k} = \begin{bmatrix} k_1\\ k_2\\ \vdots\\ k_\nu\end{bmatrix},\quad \vec{1} = \begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix},\quad A = \begin{bmatrix} a_{11} & \cdots & a_{1\nu}\\ \vdots & & \vdots\\ a_{\nu 1} & \cdots & a_{\nu\nu}\end{bmatrix}$$

$$\vec{k} = \vec{1}y_n + h\lambda A\vec{k}$$

$$(I-h\lambda A)\vec{k} = \vec{1}y_n \qquad I\in\mathbb{R}^{\nu\times\nu} \text{ identity matrix}$$

$$\vec{k} = (I-\lambda hA)^{-1}\vec{1}y_n$$

and

$$y_{n+1} = y_n + h\sum_{j=1}^{\nu} b_j\lambda k_j \qquad \vec{b} = \begin{bmatrix} b_1\\ \vdots\\ b_\nu\end{bmatrix}$$

$$= y_n + h\lambda\vec{b}^T\vec{k}$$

$$= y_n + h\lambda\vec{b}^T(I-h\lambda A)^{-1}\vec{1}y_n$$

$$= (1+h\lambda\vec{b}^T(I-h\lambda A)^{-1}\vec{1})y_n$$

Notation: $\mathbb{P}_{\alpha/\beta}$ is the set of all rational functions $\hat{p}/\hat{q}$ where $\hat{p} \in \mathbb{P}_\alpha$ and $\hat{q} \in \mathbb{P}_\beta$.

Lemma 4.1: For every Runge-Kutta method there is an $r \in \mathbb{P}_{\nu/\nu}$ such that

$$y_n = (r(\lambda h))^n, \quad n = 0, 1, \ldots,$$

and if the RK method is explicit, then $r \in \mathbb{P}_\nu$.

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Proof: Recall

$$y_{n+1} = (1 + h\lambda \vec{b}^T (I - h\lambda A)^{-1} \vec{1}) y_n$$

By induction, since $y_0 = 1$, we obtain

$$y_n = (r(h\lambda))^n,$$

where

$$r(z) = 1 + z\vec{b}^T (I - zA)^{-1} \vec{1}, \quad z \in \mathbb{C}.$$

We need to show $r(z) \in \mathbb{P}_{\nu/\nu}$. We need tools from linear algebra:

- C is a square matrix
- the cofactor matrix of C is the matrix C^{co} where the (i,j) entry is (i,j) cofactor of C
- example:

$$C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \qquad C^{co} = \begin{bmatrix} \begin{vmatrix} c_{22} & c_{23} \\ c_{32} & c_{33} \end{vmatrix} & -\begin{vmatrix} c_{21} & c_{23} \\ c_{31} & c_{33} \end{vmatrix} & \cdots \\ -\begin{vmatrix} c_{12} & c_{13} \\ c_{32} & c_{33} \end{vmatrix} & \begin{vmatrix} c_{11} & c_{13} \\ c_{31} & c_{33} \end{vmatrix} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

- the adjugate of C is the transpose of C^{co}, $\operatorname{adj}(C) = (C^{co})^T$
- $C \operatorname{adj}(C) = \det(C) I$, ie $C^{-1} = \frac{1}{\det(C)} \operatorname{adj}(C)$.

Now,

$$(I - zA)^{-1} = \frac{\operatorname{adj}(I - zA)}{\det(I - zA)}.$$

Each entry of $I - zA$ is linear in z, and each entry of $\operatorname{adj}(I - zA)$ is the determinant of a $(\nu-1)\times(\nu-1)$ matrix, so each entry is in $\mathbb{P}_{\nu-1}$. Also note $\det(I - zA) \in \mathbb{P}_\nu$. So

$$z\vec{b}^T (I - zA)\vec{1} = \frac{z\vec{b}^T \operatorname{adj}(I - zA)\vec{1}}{\det(I - zA)} \begin{matrix} \leftarrow \in \mathbb{P}_\nu \\ \leftarrow \in \mathbb{P}_\nu \end{matrix} \in \mathbb{P}_{\nu/\nu}.$$

Now suppose we have an explicit RK method. We show $r \in \mathbb{P}_\nu$. Then

$$A = \begin{bmatrix} 0 & \cdots & & 0 \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \\ a_{\nu 1} & \cdots & a_{\nu \nu-1} & 0 \end{bmatrix}.$$

Hence $I - zA$ has ones on the diagonal. Thus $\det(I-zA) = 1$. Therefore

$$\frac{z\vec{b}^T \operatorname{adj}(I-zA)\vec{1}}{\det(I-zA)} \in \mathbb{P}_\nu.$$

■

Lemma 4.2: If the numerical method can be written as

$$y_n = (r(h\lambda))^n$$

with r an arbitrary function, then the linear stability domain is

$$D = \{z \in \mathbb{C};\ |r(z)| < 1\}.$$

Proof: Immediate by definition of linear stability domain. ■

Corollary: No explicit RK method can be A-stable.

Proof: By lemma 4.1, we know $r \in \mathbb{P}_\nu$, and

$$r(z) = 1 + z\vec{b}^T (I - zA)^{-1}\vec{1}.$$

So $r(0) = 1$. Hence

$$r(z) = 1 + \sum_{i=1}^{\nu} r_i z^i.$$

By lemma 4.2, to be A-stable we need $|r(z)| < 1\ \forall z \in \mathbb{C}^-$. This is not possible. ■

Example: implicit RK method.

$$\begin{array}{c|cc} 0 & \frac{1}{4} & -\frac{1}{4} \\ \frac{2}{3} & \frac{1}{4} & \frac{5}{12} \\ \hline & \frac{1}{4} & \frac{3}{4} \end{array} \qquad r(z) = 1 + z\vec{b}^T (I - zA)^{-1}\vec{1}$$

$$Iz - A = \begin{bmatrix} 1 - \frac{1}{4}z & \frac{1}{4}z \\ -\frac{1}{4}z & 1 - \frac{5}{12}z \end{bmatrix} \qquad (I - zA)^{-1} = \frac{1}{1 - \frac{2}{3}z + \frac{1}{6}z^2} \begin{bmatrix} 1 - \frac{5}{12}z & -\frac{1}{4}z \\ \frac{1}{4}z & 1 - \frac{1}{4}z \end{bmatrix}$$

$$\vec{b}^T (I - zA)^{-1}\vec{1} = \frac{1 - \frac{1}{6}z}{1 - \frac{2}{3}z + \frac{1}{6}z^2}, \quad r(z) = \frac{1 + \frac{1}{3}z}{1 - \frac{2}{3}z + \frac{1}{6}z^2}.$$

We have A-stability if $|r(z)|<1$ $\forall z \in \mathbb{C}^-$. Use polar coordinates: $z=\rho\exp(i\theta)$, $\rho>0$, $|\theta+\pi|<\pi/2$. Check:

$$\left|1+\tfrac{1}{3}\rho\exp(i\theta)\right|^2 < \left|1-\tfrac{2}{3}\rho\exp(i\theta)+\tfrac{1}{6}\rho^2\exp(2i\theta)\right|^2.$$

Expanding and using $\exp(i\theta)=\cos(\theta)+i\sin(\theta)$, we get

$$2\rho\left(1+\tfrac{1}{9}\rho^2\right)\cos(\theta) < \tfrac{2}{3}\rho^2\cos^2(\theta)+\tfrac{1}{36}\rho^2,$$

which is true as the right is positive and $\cos(\theta)<0$ by $|\theta-\pi|<\pi/2$.

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Lemma 4.3: Let r be an arbitrary rational function that is not constant then $|r(z)|<1$ for all $z\in\mathbb{C}^-$ if and only if

(i) all the poles of r have positive real part and

(ii) $|r(it)|\le 1$ $\forall t\in\mathbb{R}$.

Proof: ($\Rightarrow$) If $|r(z)|<$ $\forall z\in\mathbb{C}^-$ then $|r(z)|\le 1$ $\forall \in \overline{\mathbb{C}^-}$. This means there can be no poles in $\overline{\mathbb{C}^-}$. Therefore all poles have to have positive real part. Furthermore, if $\mathrm{Re}(z)=0$ then

$$|r(z)|=|r(i\mathrm{Im}(z))|=|r(it)|\le \quad \forall t\in\mathbb{R}.$$

($\Leftarrow$): If all the poles of $r(z)$ are such that they have positive real part then $r(z)$ is analytic (holomorphic) $\forall z\in\overline{\mathbb{C}^-}$. Holomorphic functions don't have local maxima so $r(z)$ has its maximum on the boundary of $\mathbb{C}^-$, which is $\{it\,;t\in\mathbb{R}\}$. So $|r(it)|\le 1$ $\forall t\in\mathbb{R}$ implies that $|r(z)|<1$ $\forall z\in\mathbb{C}^-$. ∎

Example

$$\begin{array}{c|cc} 0 & \frac{1}{4} & -\frac{1}{4} \\ \frac{2}{3} & \frac{1}{4} & \frac{5}{12} \\ \hline & \frac{1}{4} & \frac{3}{4} \end{array} \longrightarrow r(z)=\frac{1+\frac{1}{3}z}{1-\frac{2}{3}z+\frac{1}{6}z^2}$$

The poles of $r(z)$ are the roots of $1-\frac{2}{3}z+\frac{1}{6}z^2$, which are $z=2\pm\sqrt{-2}=2\pm i\sqrt{2}$. These have positive real part. To check:

$$|r(it)|\le 1 \;\forall t\in\mathbb{R} \iff \left|1+\tfrac{1}{3}it\right|^2\le\left|1-\tfrac{2}{3}it+\tfrac{1}{6}(it)^2\right|^2$$
$$\iff 1+\tfrac{1}{9}t^2 \le 1+\tfrac{1}{9}t^2+\tfrac{1}{36}t^4$$

which is true $\forall t\in\mathbb{R}$. By lemma 4.3, $|r(z)|<1$ $\forall z\in\mathbb{C}^-$. By lemma 4.2, $D=\{z\in\mathbb{C}\,;|r(z)|<1\}$. Thus $\mathbb{C}^-\subseteq D$ so we have A-stability.

We prove next that all Gauss-Legendre IRK methods are A-stable.

Lemma 4.4: Consider $y' = \lambda y$, $t \geq 0$, $y(0) = 1$. Assume a p^{th} order method is applied to the ODE and can be written as $y_n = (r(h\lambda))^n$. Then $r(z) = \exp(z) + \mathcal{O}(z^{p+1})$, $z \to 0$.

Proof: Note that $y_{n+1} = r(h\lambda) y_n$, $y(t_{n+1}) = \exp(h\lambda) y_n$ (exact solution to $y' = \lambda y$, $y(t_n) = y_n$). The method is of p^{th} order means

$$y(t_{n+1}) - y_{n+1} = \exp(h\lambda) y_n - r(h\lambda) y_n = \mathcal{O}(h^{p+1})$$

$$\Rightarrow r(z) = \exp(z) + \mathcal{O}(z^{p+1}).$$ ∎

Def | A function that obeys $r(z) = \exp(z) + \mathcal{O}(z^{p+1})$ is said to be of order p.

Taylor series of exp:

$$\exp(x) = 1 + x + \tfrac{1}{2}x^2 + \cdots$$

There is also the Padé approximation:

Theorem 4.5: Given any two integers $\alpha, \beta \geq 0$, there exists a unique function $\hat{r}_{\alpha/\beta} \in \mathbb{P}_{\alpha/\beta}$ such that

proof omitted

$$\hat{r}_{\alpha/\beta} = \frac{\hat{p}_{\alpha/\beta}}{\hat{q}_{\alpha/\beta}}, \quad \hat{q}_{\alpha/\beta}(0) = 1$$

and $\hat{r}_{\alpha/\beta}$ is of order $\alpha + \beta$.

$$\hat{p}_{\alpha/\beta}(z) = \sum_{k=0}^{\alpha} \binom{\alpha}{k} \frac{(\alpha+\beta-k)!}{(\alpha+\beta)!} z^k$$

$$\hat{q}_{\alpha/\beta}(z) = \sum_{k=0}^{\beta} \binom{\beta}{k} \frac{(\alpha+\beta-k)!}{(\alpha+\beta)!} (-z)^k = \hat{p}_{\beta/\alpha}(-z)$$

Furthermore, $\hat{r}_{\alpha/\beta}$ is the only member of $\mathbb{P}_{\alpha/\beta}$ of order $\alpha + \beta$.

Examples

$$\hat{r}_{1/0}(z) = 1 + z \quad \text{(Euler)}$$

$$\hat{r}_{1/1}(z) = \frac{1 + \frac{1}{2}z}{1 - \frac{1}{2}z} \quad \text{(Trapezoidal rule)}$$

$$\hat{r}_{1/2}(z) = \frac{1 + \frac{2}{3}z}{1 - \frac{2}{3}z + \frac{1}{6}z^2} \quad \text{(IRK method from previous example)}$$

proof omitted

Theorem 4.6: The Padé approximation $\hat{r}_{\alpha/\beta}$ results in A-stable methods if $\alpha \leq \beta \leq \alpha + 2$.

All Gauss-Legendre IRK methods are A-stable $\forall \nu \geq 1$:

- A ν-stage GL method is of order 2ν (§3.4)
- Lemma 4.1: $\exists r \in \mathbb{P}_{\nu/\nu}$ such that $y_n = (r(h\lambda))^n$
- Lemma 4.4: $r(z) = \exp(z) + \mathcal{O}(z^{2\nu+1}) \Rightarrow r$ is of order 2ν.
- Theorem 4.5: $r = \hat{r}_{\nu/\nu}$
- Theorem 4.6: $\nu \leq \nu \leq \nu + 2 \Rightarrow$ A-stable