UHF algebras by an example

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1 Introduction and preliminaries

In his PhD thesis in 1960 ([2]), James Glimm introduced a class of C*-algebras called *uniformly hyperfinite* (UHF); these are C*-algebras that can be written as the closed union of an increasing chain of matrix subalgebras. There turn out to be interesting examples of these: the CAR algebra is a UHF algebra of interest in quantum mechanics. A particularly remarkable property of UHF algebras is that there is an invariant that completely captures the isomorphism class of a UHF algebra, called the *supernatural number*.

In this paper, I give an exposition of UHF algebras, the CAR algebra, and the supernatural number of a UHF algebra. My presentation is a simplified version of [1, Chapter III], which develops the more general class of approximately finite C*-algebras, and views UHF algebras as examples of these.

Ideals are closed and two-sided; $0 \in \mathbb{N}$.

A preliminary definition and theorem:

Definition 1.1. Suppose $\mathfrak{A}$ is a C*-algebra; suppose $F_{ij} \in \mathfrak{A}$ for $i, j \in \{1, \ldots, n\}$ satisfy:

1. $F_{ij}F_{k\ell} = \delta_{jk}F_{i\ell}$.
2. $F_{ij}^* = F_{ji}$.
3. $\sum_{i=1}^{n} F_{ii} = 1$.

Then the F_{ij} are $n \times n$ *matrix units* $\mathfrak{A}$.

Lemma 1.2. *Suppose $\mathfrak{A}$ is a C*-algebra; suppose $F_{ij} \in \mathfrak{A}$ are $n \times n$ matrix units for $\mathfrak{A}$. Then $C^*(\{F_{ij} : i, j \in \{1, \ldots, n\}\})$ is *-isomorphic to $M_n(\mathbb{C})$.*

Proof. It is clear from the given conditions that $\operatorname{span}\{F_{ij} : i, j \in \{1, \ldots, n\}\}$ is a unital *-subalgebra of $\mathfrak{A}$ of dimension n^2; hence since it is finite-dimensional it is closed, and is thus $C^*(\{F_{ij} : i, j \in \{1, \ldots, n\}\})$. Furthermore, an identical proof to the case of $M_\ell(\mathbb{C})$ shows that $C^*(\{F_{ij} : i, j \in \{1, \ldots, n\}\})$ is simple. But up to isomorphism the only simple finite-dimensional C*-algebra of dimension n^2 is $M_n(\mathbb{C})$; the result follows. □ Lemma 1.2

2 A motivating example: the CAR algebra

Before defining UHF algebras, we first study a prototype: the CAR algebra. This is a C*-algebra that arises naturally in the context of quantum mechanics (as noted in [1, Example III.5.4]). We will show that it is the closed union of a chain of matrix subalgebras; generalizing this property will yield the definition of a UHF algebra.

Fix a separable Hilbert space $\mathcal{H}$ and some continuous linear map $\alpha\colon \mathcal{H} \to \mathcal{B}(\mathcal{H})$ such that for all $x, y \in \mathcal{H}$ we have:

$$\alpha(x)\alpha(y) + \alpha(y)\alpha(x) = 0 \qquad \text{(AC1)}$$
$$\alpha(x)\alpha(y)^* + \alpha(y)^*\alpha(x) = \langle x, y\rangle I \quad \text{(AC2)}$$

(These are called the *canonical anticommutation relations*; hence the name "CAR algebra".)

Fact 2.1 ([3, Lemma 6.5])**.** *Such $\mathcal{H}$ and α exist.*

The *CAR algebra* is defined to be $\mathfrak{A} = C^*(\alpha(\mathcal{H}))$. (In principle this might depend on α and $\mathcal{H}$; we will see in Corollary 4.11 that it does not.)

Fix an orthonormal basis $\{ e_n : n \in \mathbb{N} \}$ for $\mathcal{H}$. For $n \in \mathbb{N}$ define $\mathfrak{A}_n = C^*(\alpha(e_0), \ldots, \alpha(e_{n-1}))$.

Proposition 2.2. *There is a *-isomorphism $\mathfrak{A}_n \cong M_{2^n}(\mathbb{C})$ (and hence a continuous *-isomorphism, since they're finite dimensional).*

The following proof appears in [1, Example III.5.4], though I have added some (perhaps unnecessary) details. Said details are somewhat unpleasant; the uninterested reader is encouraged to skip the proofs of the claims.

Proof. For $n \in \mathbb{N}$ we let $W_n = I - 2\alpha(e_n)^*\alpha(e_n)$. Note that AC1 implies $\alpha(x)^2 = 0$ for all $x \in \mathcal{H}$; hence by AC2 we have

$$(\alpha(e_n)^*\alpha(e_n))^2 = \alpha(e_n)^*\alpha(e_n)(I - \alpha(e_n)\alpha(e_n)^*) = \alpha(e_n)^*\alpha(e_n) - \alpha(e_n)^* \underbrace{\alpha(e_n)^2}_{=0} \alpha(e_n)^* = \alpha(e_n)^*\alpha(e_n)$$

Claim 2.3. *We have*

$$W_n\alpha(e_m) = (-1)^{\delta_{mn}+1}\alpha(e_m)W_n$$
$$W_n\alpha(e_m)^* = (-1)^{\delta_{mn}+1}\alpha(e_m)W_n$$

and W_n is self-adjoint with $W_n^2 = I$.

Proof. Note by AC2 that $W_n = I - 2(\langle e_n, e_n\rangle I - \alpha(e_n)\alpha(e_n)^*) = 2\alpha(e_n)\alpha(e_n)^* - I$. Thus

$$\begin{aligned}
W_n\alpha(e_n) &= (I - 2\alpha(e_n)^*\alpha(e_n))\alpha(e_n) \\
&= \alpha(e_n) - 2\alpha(e_n)^*\alpha(e_n)^2 \\
&= \alpha(e_n) \\
\alpha(e_n)W_n &= \alpha(e_n)(2\alpha(e_n)\alpha(e_n)^* - I) \\
&= -\alpha(e_n) \\
&= -W_n\alpha(e_n)
\end{aligned}$$

Also if $m \neq n$ then $\langle e_m, e_n\rangle = 0$; so

$$\begin{aligned}
W_n\alpha(e_m) &= (I - 2\alpha(e_n)^*\alpha(e_n))\alpha(e_m) \\
&= \alpha(e_m) - 2\alpha(e_n)^*\alpha(e_n)\alpha(e_m) \\
&= \alpha(e_m) + 2\alpha(e_n)^*\alpha(e_m)\alpha(e_n) \text{ (AC1)} \\
&= \alpha(e_m) + 2(\langle e_n, e_m\rangle I - \alpha(e_m)\alpha(e_n)^*)\alpha(e_n) \text{ (AC2)} \\
&= \alpha(e_m) - 2\alpha(e_m)\alpha(e_n)^*\alpha(e_n) \\
&= \alpha(e_m)(I - 2\alpha(e_n)^*\alpha(e_n)) \\
&= \alpha(e_m)W_n
\end{aligned}$$

Also it is clear that $W_n^* = W_n$; hence

$$W_n \alpha(e_m)^* = (-1)^{\delta_{mn}+1} \alpha(e_m)^* W_n$$

follows by taking adjoints. Finally we have

$$W_n^2 = (I - 2\alpha(e_n)^*\alpha(e_n))(I - 2\alpha(e_n)^*\alpha(e_n)) = I - 4\alpha(e_n)^*\alpha(e_n) + 4(\alpha(e_n)^*\alpha(e_n))^2 = I$$

as desired. □ Claim 2.3

For $n \in \mathbb{N}$ let $V_n = W_0 \cdots W_{n-1} \in C^*(\alpha(e_1), \ldots, \alpha(e_{n-1}))$.

Claim 2.4. *The elements*

$$\begin{aligned} E_{11}^{(n)} &= \alpha(e_n)^*\alpha(e_n) \\ E_{12}^{(n)} &= V_n\alpha(e_n) \\ E_{21}^{(n)} &= V_n\alpha(e_n)^* \\ E_{22}^{(n)} &= \alpha(e_n)\alpha(e_n)^* \end{aligned}$$

are 2×2 matrix units for $\mathfrak{A}$; furthermore if $m \neq n$ then $E_{ij}^{(m)}$ commutes with $E_{k\ell}^{(n)}$ for all $i, j, k, \ell \in \{1, 2\}$.

Proof. Note by Claim 2.3 that V_n commutes with $\alpha(e_n)$ and $\alpha(e_n)^*$, and that $V_n^*V_n = V_nV_n^* = I$. We now verify the properties of matrix units:

1. This is routine; we do two sample computations and leave the rest to the interested reader.

$$\begin{aligned} E_{11}^{(n)}E_{21}^{(n)} &= \alpha(e_n)^*\alpha(e_n)V_n\alpha(e_n)^* \\ &= V_n(I - \alpha(e_n)\alpha(e_n)^*)\alpha(e_n)^* \\ &= V_n\alpha(e_n)^* \\ &= E_{21}^{(n)} \\ E_{11}^{(n)}E_{12}^{(n)} &= \alpha(e_n)^*\alpha(e_n)V_n\alpha(e_n) \\ &= V_n\alpha(e_n)^*\alpha(e_n)^2 \\ &= 0 \end{aligned}$$

2. We have $(E_{12}^{(n)})^* = (V_n\alpha(e_n))^* = V_n\alpha(e_n)^* = E_{21}^{(n)}$, and we already saw that $(E_{11}^{(n)})^2 = (\alpha(e_n)^*\alpha(e_n))^2 = \alpha(e_n)^*\alpha(e_n) = E_{11}^{(n)}$; a similar proof yields that $(E_{22}^{(n)})^2 = E_{22}^{(n)}$.

3. This is just AC2.

For the last claim, we show that $V_n\alpha(e_n)$ commutes with $C^*(\alpha(e_0), \ldots, \alpha(e_{n-1}))$; this will suffice, since $E_{ij}^{(n)} \in C^*(V_n\alpha(e_n))$ and $E_{ij}^{(k)} \in C^*(\alpha(e_0), \ldots, \alpha(e_{n-1}))$ if $k < n$. But by Claim 2.3 we get for $k < n$ that

$$\begin{aligned} V_n\alpha(e_n)\alpha(e_k) &= -W_0 \cdots W_{n-1}\alpha(e_k)\alpha(e_n) \\ &= \alpha(e_k)W_0 \cdots W_{n-1}\alpha(e_n) \\ &= \alpha(e_k)V_n\alpha(e_n) \end{aligned}$$

and similarly $V_n\alpha(e_n)\alpha(e_k)^* = \alpha(e_k)^*V_n\alpha(e_n)$. □ Claim 2.4

Claim 2.5. *The $E_{i_0j_0}^{(0)} \cdots E_{i_{n-1}j_{n-1}}^{(n-1)}$ form $2^n \times 2^n$ matrix units for $\mathfrak{A}$.*

Proof. We verify the properties of matrix units.

1. By Claim 2.4 we have

$$\begin{aligned} E^{(0)}_{i_0 j_0} \cdots E^{(n-1)}_{i_{n-1} j_{n-1}} \cdot E^{(0)}_{k_0 \ell_0} \cdots E^{(n-1)}_{k_{n-1} \ell_{n-1}} &= E^{(0)}_{i_0 j_0} E^{(0)}_{k_0 \ell_0} \cdots E^{(n-1)}_{i_{n-1} j_{n-1}} E^{(n-1)}_{k_{n-1} \ell_{n-1}} \\ &= \delta_{j_0 k_0} \cdots \delta_{j_{n-1} k_{n-1}} E^{(0)}_{i_0 \ell_0} \cdots E^{(n-1)}_{i_{n-1} \ell_{n-1}} \\ &= \delta_{(j_0, \ldots, j_{n-1}), (k_0, \ldots, k_{n-1})} E^{(0)}_{i_0 \ell_0} \cdots E^{(n-1)}_{i_{n-1} \ell_{n-1}} \end{aligned}$$

as desired.

2. This follows directly from Claim 2.4.

3. We do this by induction on n. The base case is Claim 2.4; for the induction step notice that

$$\sum_{i_0=1}^{2} \cdots \sum_{i_n=1}^{2} E^{(0)}_{i_0 i_0} \cdots E^{(n)}_{i_n i_n} = \left(\sum_{i_0=1}^{2} \cdots \sum_{i_{n-1}=1}^{2} E^{(0)}_{i_0 i_0} \cdots E^{(n-1)}_{i_{n-1} i_{n-1}} \right) \sum_{i_n=1}^{2} E^{(n)}_{i_n i_n} = \sum_{i_n=1}^{2} E^{(n)}_{i_n i_n} = I$$

by the induction hypothesis. □ Claim 2.5

Thus by Lemma 1.2 we get that $C^*(\{ E^{(0)}_{i_0 j_0} \cdots E^{(n-1)}_{i_{n-1} j_{n-1}} \})$ is *-isomorphic to $M_{2^n}(\mathbb{C})$. But $E^{(k)}_{ij} \in C^*(\alpha(e_0), \ldots, \alpha(e_k))$; also

$$\alpha(e_\ell) = V_\ell^* V_\ell \alpha(e_\ell) = (I - 2E^{(\ell)}_{11}) E^{(\ell)}_{12}$$

and

$$E^{(\ell)}_{ij} = \sum_{i_0=1}^{2} \cdots \sum_{i_{\ell-1}=1}^{2} \sum_{i_{\ell+1}=1}^{2} \cdots \sum_{i_{n-1}=1}^{2} E^{(0)}_{i_0 i_0} \cdots E^{(\ell-1)}_{i_{\ell-1} i_{\ell-1}} E^{(\ell)}_{ij} E^{(\ell+1)}_{i_{\ell+1} i_{\ell+1}} \cdots E^{(n-1)}_{i_{n-1} i_{n-1}} \in C^*(\{ E^{(0)}_{i_0 j_0} \cdots E^{(n-1)}_{i_{n-1} j_{n-1}} \})$$

So $C^*(\{ \alpha(e_0), \ldots, \alpha(e_{n-1}) \}) = C^*(\{ E^{(0)}_{i_0 j_0} \cdots E^{(n-1)}_{i_{n-1} j_{n-1}} \})$; so $\mathfrak{A}_n = C^*(\{ \alpha(e_0), \ldots, \alpha(e_{n-1}) \})$ is *-isomorphic to $M_{2^n}(\mathbb{C})$, as desired. □ Proposition 2.2

Note, however, that

$$\mathfrak{A} = C^*(\alpha(\mathcal{H})) = \overline{\bigcup_{n \in \mathbb{N}} \operatorname{alg}\{ \alpha(e_0), \ldots, \alpha(e_{n-1}) \}} \subseteq \overline{\bigcup_{n \in \mathbb{N}} C^*(\alpha(e_0), \ldots, \alpha(e_{n-1}))} = \overline{\bigcup_{n \in \mathbb{N}} \mathfrak{A}_n}$$

We have thus shown that there is a chain $\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \cdots \subseteq \mathfrak{A}$ of matrix subalgebras whose union is dense; this is the condition we will generalize when defining UHF algebras. This is a strong finiteness condition; we will see in Theorem 4.7 that the behaviour of algebras with such a chain is tightly controlled by the behaviour within the chain.

3 UHF algebras: definitions and first properties

Definition 3.1. A *UHF algebra* is a unital C*-algebra $\mathfrak{A}$ for which there exists a chain $\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \cdots \subseteq \mathfrak{A}$ of unital subalgebras (i.e. subalgebras containing the unit of $\mathfrak{A}$) such that

- each $\mathfrak{A}_k$ is *-isomorphic to a matrix algebra $M_{m_k}(\mathbb{C})$, and
- $\mathfrak{A} = \overline{\bigcup_{k \in \mathbb{N}} \mathfrak{A}_k}$.

So in particular the CAR algebra is a UHF algebra.

Some properties:

Remark 3.2. Since linear maps, and in particular *-homomorphisms, between finite dimensional spaces are continuous, we get that the isomorphism $\mathfrak{A}_k \cong M_{m_k}(\mathbb{C})$ is continuous; so we may assume that topological properties are preserved as well.

Remark 3.3. UHF algebras are separable. Indeed, suppose

$$\mathfrak{A} = \overline{\bigcup_{k \in \mathbb{N}} \mathfrak{A}_k}$$

is a UHF algebra. Let $A_k \subseteq \mathfrak{A}_k$ be the image of $M_{m_k}(\mathbb{Q})$ under the given isomorphism $M_{m_k}(\mathbb{C}) \to \mathfrak{A}_k$; so A_k is countable and dense in $\mathfrak{A}_k$. Then

$$A = \bigcup_{k \in \mathbb{N}} A_k$$

is dense in

$$\bigcup_{k \in \mathbb{N}} \mathfrak{A}_k$$

and hence also in its closure $\mathfrak{A}$. But A is countable; so $\mathfrak{A}$ is separable.

Proposition 3.4. *UHF algebras are simple.*

Proof. Suppose

$$\mathfrak{A} = \bigcup_{k \in \mathbb{N}} \mathfrak{A}_k$$

is a UHF algebra; suppose $\mathfrak{J} \subsetneqq \mathfrak{A}$ is an ideal.

Claim 3.5. $\mathfrak{J} \cap \mathfrak{A}_k = \{ 0 \}$ *for all* $k \in \mathbb{N}$.

Proof. We first check that $\mathfrak{J} \cap \mathfrak{A}_k$ is an ideal of $\mathfrak{A}_k$. It is clear that $\mathfrak{J} \cap \mathfrak{A}_k$ is a vector subspace of $\mathfrak{A}_k$ that is closed in the relative topology. Suppose $a, b \in \mathfrak{A}_k$ and $c \in \mathfrak{J} \cap \mathfrak{A}_k$. Then $acb \in \mathfrak{A}_k$ since $\mathfrak{A}_k$ is a subalgebra and $acb \in \mathfrak{J}$ since $\mathfrak{J}$ is an ideal; so $acb \in \mathfrak{J} \cap \mathfrak{A}_k$, and $\mathfrak{J} \cap \mathfrak{A}_k$ is an ideal of $\mathfrak{A}_k$.

But $\mathfrak{A}_k \cong M_{m_k}(\mathbb{C})$, and $M_{m_k}(\mathbb{C})$ is simple. So $\mathfrak{J} \cap \mathfrak{A}_k$ is either $\{ 0 \}$ or all of $\mathfrak{A}_k$.

Case 1. Suppose there is some $k \in \mathbb{N}$ such that $\mathfrak{J} \cap \mathfrak{A}_k = \{ 0 \}$. Then for all $\ell < k$ we immediately get that $\mathfrak{J} \cap \mathfrak{A}_\ell \subseteq \mathfrak{J} \cap \mathfrak{A}_k = \{ 0 \}$; also for $\ell > k$ with $\mathfrak{A}_\ell \supsetneqq \mathfrak{A}_k$ we get that $\mathfrak{J} \cap \mathfrak{A}_\ell \subseteq (\mathfrak{A}_\ell \setminus \mathfrak{A}_k) \cup \{ 0 \} \subsetneqq \mathfrak{A}_\ell$, and thus $\mathfrak{J} \cap \mathfrak{A}_\ell = \{ 0 \}$. The claim then follows.

Case 2. Suppose $\mathfrak{J} \cap \mathfrak{A}_k = \mathfrak{A}_k$ for all $k \in \mathbb{N}$. Then $\mathfrak{J}$ is a closed set containing the dense set

$$\bigcup_{k \in \mathbb{N}} \mathfrak{A}_k$$

So $\mathfrak{J} = \mathfrak{A}$, contradicting our assumption that $\mathfrak{J}$ was proper. □ Claim 3.5

Consider the quotient map $q \colon \mathfrak{A} \to \mathfrak{A}/\mathfrak{J}$; we will show that q is injective. Note that $q \restriction \mathfrak{A}_k$ is injective: indeed, $\ker(q \restriction \mathfrak{A}_k) = \ker(q) \cap \mathfrak{A}_k = \mathfrak{J} \cap \mathfrak{A}_k = \{ 0 \}$. We can thus define a continuous linear map $\varphi_k \colon q(\mathfrak{A}_k) \to \mathfrak{A}_k$ such that $\varphi_k \circ q = \mathrm{id}_{\mathfrak{A}_k}$. This φ_k is uniquely determined, and thus the φ_k form a chain; so we can define

$$\varphi_\omega = \bigcup_{k \in \mathbb{N}} \varphi_k \colon \bigcup_{k \in \mathbb{N}} q(\mathfrak{A}_k) \to \bigcup_{k \in \mathbb{N}} \mathfrak{A}_k$$

But this φ_ω is a continuous linear map, and thus extends to

$$\varphi \colon \overline{\bigcup_{k \in \mathbb{N}} q(\mathfrak{A}_k)} \to \mathfrak{A}$$

But if $q(a) \in \mathfrak{A}/\mathfrak{J}$ and $\varepsilon > 0$ then there is some $k \in \mathbb{N}$ and $a_0 \in \mathfrak{A}_k$ such that $\|a - a_0\| < \varepsilon$; so $\|q(a) - q(a_0)\| \leq \|a - a_0\| < \varepsilon$. So

$$\bigcup_{k \in \mathbb{N}} q(\mathfrak{A}_k)$$

is dense in $\mathfrak{A}/\mathfrak{J}$; so $\varphi \colon \mathfrak{A}/\mathfrak{J} \to \mathfrak{A}$. But by construction $\varphi \circ q \restriction \mathfrak{A}_k = \mathrm{id}_{\mathfrak{A}_k}$; so by continuity of $\varphi \circ q$ we get that $\varphi \circ q = \mathrm{id}_{\mathfrak{A}}$. So q is injective, and $\mathfrak{J} = \ker(q) = \{ 0 \}$. □ Proposition 3.4

4 The supernatural number of a UHF algebra

In this section we will justify our earlier claim that the behaviour of a UHF algebra is tightly controlled by the behaviour of its matrix subalgebras; in fact in Theorem 4.7 we will give an invariant that completely classifies the isomorphism type of a UHF algebra.

To understand the structure of a UHF algebra, it behooves us to study the structure of unital *-embeddings between matrix algebras: the embeddings $M_{m_k}(\mathbb{C}) \to M_{m_{k+1}}(\mathbb{C})$ induced by the inclusions $\mathfrak{A}_k \hookrightarrow \mathfrak{A}_{k+1}$ will determine how the $M_{m_k}(\mathbb{C})$ "fit together" in $\mathfrak{A}$. Since unital *-embeddings between matrix algebras can be viewed as non-degenerate representations, we first study the representation theory of matrix algebras.

Proposition 4.1. *All irreducible representations of $M_n(\mathbb{C})$ are unitarily equivalent to the identity representation* $\mathrm{id}\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$.

Proof. We know from assignment 3 that an irreducible representation is a surjective homomorphism $M_n(\mathbb{C}) \to M_m(\mathbb{C})$, and the kernel is a maximal ideal of $M_n(\mathbb{C})$, and thus 0. So the only irreducible representations are *-automorphisms $\rho\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$.

Note that the E_{ii} must get sent to pairwise orthogonal rank 1 projections; we thus get an orthonormal basis $f_1, \ldots, f_n$ for $\mathbb{C}^n$ such that $\rho(E_{ii}) = f_i f_i^* = F_{ii}$. But then also $\rho(E_{ij}) = \rho(E_{ii}E_{ij}E_{jj}) = F_{ii}\rho(E_{ij})F_{jj}$, so since ρ preserves spectra we get $\rho(E_{ij}) = f_i f_j^* = F_{ij}$. Then the map $Ue_i = f_i$ is unitary, and $\rho(E_{ij}) = F_{ij} = f_i f_j^* = U e_i e_j^* U^* = U E_{ij} U^*$ for all i, j; so $\rho(A) = UAU^*$ for all A. So ρ is unitarily equivalent to the identity. □ Proposition 4.1

We can use this to classify all unital *-embeddings between matrix algebras.

Corollary 4.2. *Suppose $\varphi\colon M_m(\mathbb{C}) \to M_n(\mathbb{C})$ is a unital *-homomorphism. Then $m \mid n$ and there is a unitary $U \in M_n(\mathbb{C})$ such that $\varphi(A) = U^*(A \oplus \cdots \oplus A)U$ for all $A \in M_m(\mathbb{C})$.*

Proof. Since φ is unital we get that φ defines a non-degenerate representation of a finite-dimensional C*-algebra, and is thus completely reducible. (Indeed, one checks that the orthogonal complement of a subrepresentation is also a subrepresentation.) Hence by previous proposition we get that $\varphi = \rho_1 \oplus \cdots \oplus \rho_k$ with each ρ_k unitarily equivalent to $\mathrm{id}\colon M_m(\mathbb{C}) \to M_m(\mathbb{C})$; so $n = km$ and $m \mid n$. If $U_1, \ldots, U_k$ are unitaries in $M_m(\mathbb{C})$ such that $\rho_i(A) = U_i A U_i^*$, then $U = U_1 \oplus \cdots \oplus U_k \in M_n(\mathbb{C})$ is the desired unitary. □ Corollary 4.2

Suppose now that

$$\mathfrak{A} = \overline{\bigcup_{k \in \mathbb{N}} \mathfrak{A}_k}$$

is a UHF algebra. Then the inclusions $\mathfrak{A}_k \to \mathfrak{A}_{k+1}$ induce unital *-homomorphisms $M_{m_k}(\mathbb{C}) \to M_{m_{k+1}}(\mathbb{C})$; so $m_0 \mid m_1 \mid \cdots$. Consider the prime factorizations of the m_k: let

$$m_k = \prod_{p \text{ prime}} p^{v_p(m_k)}$$

Then since the $m_k \mid m_{k+1}$ we get that the $(v_p(m_k))_k$ are increasing sequences of natural numbers; so each one is either eventually constant or diverges to infinity. Let $e_p = \sup_k v_p(m_k) \in \mathbb{N} \cup \{\infty\}$.

Definition 4.3. The *supernatural number* associated to $\mathfrak{A}$ is the formal product

$$\delta(\mathfrak{A}) = \prod_{p \text{ prime}} p^{e_p}$$

(Of course $\delta(\mathfrak{A})$ is only a true natural number if $\mathfrak{A} \cong M_{\delta(\mathfrak{A})}(\mathbb{C})$.)

In principle our definition of $\delta(\mathfrak{A})$ may depend on the choice of chain $\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \cdots$; in fact it depends only on $\mathfrak{A}$.

Theorem 4.4. $\delta(\mathfrak{A})$ *is well-defined.*

We will need a lemma whose proof I omit; I direct the interested reader to [1, Lemma III.3.2], of which the following is a weakening:

Lemma 4.5. *Suppose $\mathfrak{D}$ is a C*-algebra with a finite-dimensional C*-subalgebra $\mathfrak{A}$. Then there is $\delta > 0$ such that if $\mathfrak{B} \subseteq \mathfrak{D}$ is a C*-subalgebra with $\operatorname{dist}(a, \mathfrak{B}) < \delta$ for all $a \in b_1(\mathfrak{B})$ then there is a unitary $u \in \mathfrak{D}$ such that $u^*\mathfrak{A}u \subseteq \mathfrak{B}$.*

Proof of Theorem 4.4. Suppose

$$\mathfrak{A} = \overline{\bigcup_{k\in\mathbb{N}} \mathfrak{A}_k} = \overline{\bigcup_{\ell\in\mathbb{N}} \mathfrak{B}_\ell}$$

with

$$\mathfrak{A}_k \cong M_{m_k}(\mathbb{C})$$
$$\mathfrak{B}_\ell \cong M_{n_\ell}(\mathbb{C})$$

Claim 4.6. *For all k there is ℓ such that $m_k \mid n_\ell$. (And vice-versa.)*

Proof. Fix k; let δ be as in Lemma 4.5. Since $\mathfrak{A}_k$ is finite-dimensional and contained in

$$\overline{\bigcup_{\ell\in\mathbb{N}} \mathfrak{B}_\ell}$$

there is ℓ such that $\operatorname{dist}(a, \mathfrak{B}_\ell) < \delta$ for all $a \in \mathfrak{A}_k$. (Indeed, there is ℓ such that $\operatorname{dist}(a, \mathfrak{B}_\ell) < m_k^{-2}\delta$ whenever $a \in \mathfrak{A}_k$ is the image of one of the standard matrix units in $M_{m_k}(\mathbb{C})$; the triangle inequality then yields the desired bound.) So by Lemma 4.5 there is unitary $u \in \mathfrak{A}$ such that $u^*\mathfrak{A}_k u \subseteq \mathfrak{B}_\ell$. But then the map $a \mapsto u^*au$ is a unital *-homomorphism $\mathfrak{A}_k \to \mathfrak{B}_\ell$, and thus induces a unital *-homomorphism $M_{m_k}(\mathbb{C}) \to M_{n_\ell}(\mathbb{C})$; so by Corollary 4.2 we get that $m_k \mid n_\ell$. The "vice-versa" follows by symmetry. □ Claim 4.6

So repeatedly applying the above claim we get subsequences $k_0, k_1, \ldots$ and $\ell_0, \ell_1, \ldots$ such that $m_{k_0} \mid n_{\ell_0} \mid m_{k_1} \mid n_{\ell_1} \mid \cdots$. Thus for any prime p we have

$$v_p(m_{k_0}) \le v_p(n_{\ell_0}) \le v_p(m_{k_1}) \le v_p(n_{\ell_1}) \le \cdots$$

Thus

$$\sup_k v_p(m_k) = \sup\{\, v_p(m_{k_0}), v_p(m_{k_1}), \ldots\,\} \le \sup\{\, v_p(n_{\ell_0}), v_p(n_{\ell_1}), \ldots\,\} = \sup_\ell v_p(n_\ell)$$

(where the equalities follow because $(v_p(m_k))_k$ and $(v_p(n_\ell))_\ell$ are increasing sequences). We likewise get $\sup_\ell v_p(n_\ell) \le \sup_k v_p(m_k)$. So $\delta(\mathfrak{A})$ computed with respect to the $\mathfrak{A}_k$ agrees with $\delta(\mathfrak{A})$ computed with respect to the $\mathfrak{B}_\ell$. □ Theorem 4.4

We have shown that $\delta(\mathfrak{A})$ is an invariant depending only on $\mathfrak{A}$; in fact it completely characterizes the isomorphism class of $\mathfrak{A}$.

Theorem 4.7. *If $\mathfrak{A}, \mathfrak{B}$ are UHF algebras with $\delta(\mathfrak{A}) = \delta(\mathfrak{B})$ then there is a continuous *-isomorphism $\mathfrak{A} \to \mathfrak{B}$.*

Theorem 4.7 may seem unremarkable: it seems intuitive that the unions of "similar" chains of subalgebras should be isomorphic. This intuition hides the complexity that can arise from the arrangement of the subalgebras; indeed, the following fact shows that this intuition fails quite badly in even a slightly more general class of C*-algebras.

Fact 4.8 ([1, Example III.3.7])**.** *There exist C*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ with chains of unital subalgebras*

$$\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \cdots \subseteq \mathfrak{A}$$

and

$$\mathfrak{B}_0 \subseteq \mathfrak{B}_1 \subseteq \cdots \subseteq \mathfrak{B}$$

such that

- *each $\mathfrak{A}_k$ and $\mathfrak{B}_\ell$ is finite-dimensional (and thus isomorphic to a direct sum of matrix algebras), and*

- $\mathfrak{A} = \overline{\bigcup_{k\in\mathbb{N}} \mathfrak{A}_k}$ *and* $\mathfrak{B} = \overline{\bigcup_{\ell\in\mathbb{N}} \mathfrak{B}_\ell}$

with each $\mathfrak{A}_k \cong \mathfrak{B}_k$ *but* $\mathfrak{A} \not\cong \mathfrak{B}$.

Having hopefully convinced the reader that this is indeed a remarkable theorem, we proceed to its proof. This proof appears in [1, Theorem III.5.2].

Proof of Theorem 4.7. Write

$$\mathfrak{A} = \overline{\bigcup_{k\in\mathbb{N}} \mathfrak{A}_k} \qquad \mathfrak{B} = \overline{\bigcup_{\ell\in\mathbb{N}} \mathfrak{B}_\ell}$$

with each $\mathfrak{A}_k \cong M_{m_k}(\mathbb{C})$ and each $\mathfrak{B}_\ell \cong M_{n_\ell}(\mathbb{C})$.

Claim 4.9. *For all k there is ℓ such that $m_k \mid n_\ell$. (And vice-versa.)*

Proof. Suppose p is prime with $v_p(m_k) \neq 0$. Then since $\delta(\mathfrak{A}) = \delta(\mathfrak{B})$ we get

$$v_p(m_k) \leq \sup_k v_p(m_k) = \sup_\ell v_p(n_\ell)$$

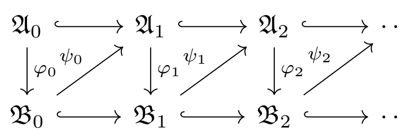
So there is ℓ_p such that $v_p(m_k) \leq v_p(n_{\ell_p})$. Let ℓ be the maximum over such p of ℓ_p. Then for such p we have

$$v_p(m_k) \leq v_p(n_{\ell_p}) \leq v_p(n_\ell)$$

So $m_k \mid n_\ell$. The "vice versa" again follows by symmetry. □ Claim 4.9

Thus using the above claim we can drop to a subsequence of the $\mathfrak{A}_k$ and a subsequence of the $\mathfrak{B}_\ell$ such that $m_0 \mid n_0 \mid m_1 \mid n_1 \mid \cdots$. (Note that since the $\mathfrak{A}_k$ form a chain dropping to a subsequence won't change the union; likewise with the $\mathfrak{B}_\ell$.)

Claim 4.10. *There exists unital *-homomorphisms $\varphi_k \colon \mathfrak{A}_k \to \mathfrak{B}_k$ and $\psi_k \colon \mathfrak{B}_k \to \mathfrak{A}_{k+1}$ such that $\psi_k \circ \varphi_k \colon \mathfrak{A}_k \to \mathfrak{A}_{k+1}$ and $\varphi_{k+1} \circ \psi_k \colon \mathfrak{B}_k \to \mathfrak{B}_{k+1}$ are the inclusion mappings. i.e. we require that the following diagram commute:*

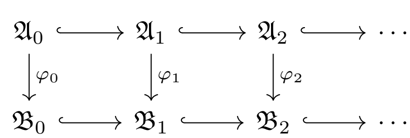


Proof. Since $m_0 \mid n_0$ there is a natural unital *-homomorphism $M_{m_0}(\mathbb{C}) \to M_{n_0}(\mathbb{C})$, namely $A \mapsto A \oplus \cdots \oplus A$; let $\varphi_0 \colon \mathfrak{A}_0 \to \mathfrak{B}_0$ be induced by this map.

Suppose we have defined φ_k. Since $n_k \mid m_{k+1}$ there is a natural unital *-homomorphism $M_{n_k}(\mathbb{C}) \to M_{m_{k+1}}(\mathbb{C})$; let ψ'_k be induced by this map. Then $\psi'_k \circ \varphi_k$ is a unital *-homomorphism $\mathfrak{A}_k \to \mathfrak{A}_{k+1}$, as is the inclusion map. So by Corollary 4.2 there is a unitary $u \in \mathfrak{A}_{k+1}$ such that $u^* \psi'_k(\varphi_k(a))u = a$ for all $a \in \mathfrak{A}_k$. Define $\psi_k \colon \mathfrak{B}_k \to \mathfrak{A}_{k+1}$ by $\psi_k(b) = u^*\psi'_k(b)u$; then by construction we have $\psi_k \circ \varphi_k$ is the inclusion $\mathfrak{A}_k \to \mathfrak{A}_{k+1}$.

The definition of φ_{k+1} assuming ψ_k has been defined is identical. □ Claim 4.10

Hence since the following diagram commutes:



we get a well-defined continuous *-homomorphism

$$\bigcup_{k\in\mathbb{N}} \varphi_k \colon \bigcup_{k\in\mathbb{N}} \mathfrak{A}_k \to \mathfrak{B}$$

Since this is continuous and linear, it extends to a map

$$\varphi \colon \overline{\bigcup_{k\in\mathbb{N}} \mathfrak{A}_k} = \mathfrak{A} \to \mathfrak{B}$$

Similarly we get $\psi \colon \mathfrak{B} \to \mathfrak{A}$ extending the ψ_k.

But for $a \in \mathfrak{A}_k$ we have $\psi(\varphi(a)) = \psi(\varphi_k(a)) = \psi_k(\varphi_k(a)) = a$; so $\psi \circ \varphi$ agrees with $\mathrm{id}_{\mathfrak{A}}$ on a dense subset, and by continuity we get $\psi \circ \varphi = \mathrm{id}_{\mathfrak{A}}$. We likewise get $\varphi \circ \psi = \mathrm{id}_{\mathfrak{B}}$. So φ is a continuous *-isomorphism $\mathfrak{A} \to \mathfrak{B}$. ☐ Theorem 4.7

Corollary 4.11. *The CAR algebra is independent of the choice of $\mathcal{H}$ and α.*

Proof. If $\mathfrak{A}$ is a CAR algebra (i.e. constructed from some $\mathcal{H}$ and α as detailed in Section 2) then recall from Section 2 that

$$\mathfrak{A} = \overline{\bigcup_{k\in\mathbb{N}} \mathfrak{A}_k}$$

where $\mathfrak{A}_k \cong M_{2^k}(\mathbb{C})$; so $\delta(\mathfrak{A}) = 2^\infty$. So any two constructions of the CAR algebra have the same supernatural number; so by Theorem 4.7 we get that there is a continuous *-isomorphism between any two constructions of the CAR algebra. ☐ Corollary 4.11

Remark 4.12. Suppose we are given a supernatural number

$$\mathbf{n} = \prod_{p \text{ prime}} p^{e_p}$$

We construct a UHF algebra $\mathfrak{A}$ with $\delta(\mathfrak{A}) = \mathbf{n}$. Pick a bijection $\Phi \colon \mathbb{N} \to P \times \mathbb{N}$, where P is the set of primes. Let $m_0 = 1$. Suppose we have chosen m_k; write $\Phi(k) = (p, e)$. If $e > e_p$, we let $m_{k+1} = m_k$; else let $m_{k+1} = \operatorname{lcm}(m_k, p^e)$. Note in particular that $m_k \mid m_{k+1}$ for all k.

Fix a separable Hilbert space $\mathcal{H}$ with orthonormal basis $\{ e_0, e_1, \dots \}$. For $n \geq 1$ and $0 \leq j < n$ let $P_j^{(n)} \in \mathcal{B}(\mathcal{H})$ be the orthogonal projection with range $\overline{\operatorname{span}\{ e_\ell : \ell \equiv j \pmod{n} \}}$; for $\delta \in \mathbb{Z}$ let $S_\delta \in \mathcal{B}(\mathcal{H})$ be

$$S_\delta e_k = \begin{cases} e_{k+\delta} & \text{if } k + \delta \geq 0 \\ 0 & \text{else} \end{cases}$$

Then for $i, j \in \{ 0, \dots, n-1 \}$ the map $E_{ij}^{(n)} = S_{i-j} P_j^{(n)}$ is given by

$$e_k \mapsto \begin{cases} e_{qn+i} & \text{if } k = qn + j \text{ for some } q \\ 0 & \text{else} \end{cases}$$

In fact the $E_{ij}^{(n)}$ act as $n \times n$ matrix units: we have $E_{ij}^{(n)} E_{i'j'}^{(n)} = \delta_{ji'} E_{ij'}$ and $(E_{ij}^{(n)})^* = E_{ji}^{(n)}$ and

$$\sum_{i=0}^{n-1} E_{ii}^{(n)} = I$$

Thus by Lemma 1.2 we get that $\mathfrak{M}_n = C^*(\{ E_{ij}^{(n)} : i, j \in \{ 0, \dots, n-1 \} \})$ is *-isomorphic to $M_n(\mathbb{C})$. Note also that if $n \mid m$ then

$$P_i^{(n)} = \sum_{j=0}^{\frac{m}{n}-1} P_{i+jn}^{(m)}$$

We thus get that $E_{ij}^{(n)} = S_{i-j} P_j^{(n)} \in \mathfrak{M}_m$ (by expanding using the above sum and distributing), and hence $\mathfrak{M}_n \subseteq \mathfrak{M}_m$.

Now if we let $\mathfrak{A}_k = \mathfrak{M}_{m_k} \subseteq \mathcal{B}(\mathcal{H})$ then since $m_k \mid m_{k+1}$ we get that $\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \cdots$; so if

$$\mathfrak{A} = \overline{\bigcup_{k\in\mathbb{N}} \mathfrak{A}_k}$$

then $\mathfrak{A}$ is a UHF algebra.

Claim 4.13. $\delta(\mathfrak{A}) = \mathbf{n}$.

Proof. Suppose p is prime, and recall that

$$\mathbf{n} = \prod_{p \text{ prime}} p^{e_p}$$

We must then check that $e_p = \sup_k v_p(m_k)$.

($\leq$) Suppose $e \in \mathbb{N}$ has $e \leq e_p$. Let $k = \Phi^{-1}(p, e)$; then by construction we have $p^e \mid m_{k+1}$, and hence $e \leq v_p(m_{k+1}) \leq \sup_k v_p(m_k)$. Since this holds for all natural $e \leq e_p$, we get that $e_p \leq \sup_k v_p(m_k)$.

($\geq$) One checks by induction on k that if $e_p < e$ then $v_p(m_k) < e$; this is simply the nature of the construction of the m_k. Hence in particular e_p is an upper bound for the $v_p(m_k)$, and $e_p \geq \sup_k v_p(m_k)$. ☐ Claim 4.13

So for every supernatural $\mathbf{n}$ there is a UHF algebra $\mathfrak{A}$ with $\delta(\mathfrak{A}) = \mathbf{n}$. So the continuous isomorphism classes of UHF algebras are in bijection with the supernatural numbers.

References

[1] Kenneth R. Davidson. *C*-Algebras by Example.* Hindustan Book Agency, 1996 (cit. on pp. 1, 2, 6–8).

[2] James G. Glimm. “On a Certain Class of Operator Algebras”. In: *Transactions of the American Mathematical Society* 95.2 (1960), pp. 318–340 (cit. on p. 1).

[3] Paul Skoufranis. *Separable Exact C*-Algebras Embed Into the Cuntz Algebra.* July 8, 2016. URL: http://pskoufra.info.yorku.ca/files/2016/07/Separable-Exact-C-Algebras-Embed-Into-the-Cuntz-Algebras.pdf (cit. on p. 2).