A brief introduction to trace class operators

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1 Introduction

The trace is a useful number we can attach to an operator on a finite-dimensional space; a natural question when studying operators on an arbitrary Hilbert space is whether we can formulate a meaningful generalization of the trace. The trace class operators provide an appropriate setting for generalizing the finite-dimensional trace. They are of further interest as the dual of the compact operators and a predual of the bounded operators.

In this paper, I introduce the trace class operators and verify the above claim about their dual and predual. This paper is largely based on the exposition in [2]; I have, however, avoided defining and using Hilbert-Schmidt operators, and thus deviated somewhat from my primary source. Where possible, I adapted proofs involving Hilbert-Schmidt operators to work solely with the trace class operators; elsewhere, I found unrelated proofs.

Section 2 covers some prerequisite definitions and results. Section 3 defines the trace class operators, covers their elementary properties, and relates them back to the original goal of finding an appropriate generalization of the finite-dimensional trace. Section 4 proves that the trace class operators are the dual of the compact operators and a predual of the bounded operators.

2 Preliminaries

In this section I present some standard definitions and results that we will make use of.

Unless otherwise stated, $\mathcal{H}$ refers to an arbitrary Hilbert space over $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$.

The following theorem is usually proven using the functional calculus or spectral theorem, as in [1]. A proof by approximations can be found in [3, Section 104].

Theorem 2.1. *Suppose $A \in \mathcal{B}(\mathcal{H})$ with $A \geq 0$. Then there is a unique $B \in \mathcal{B}(\mathcal{H})$ with $B \geq 0$ such that $B^2 = A$.*

Definition 2.2. Suppose $A \in \mathcal{B}(\mathcal{H})$. We define $|A|$ to be the unique $B \in \mathcal{B}(\mathcal{H})$ with $B \geq 0$ such that $B^2 = A^*A$.

Remark 2.3. $|A|$ is the unique positive $B \in \mathcal{B}(\mathcal{H})$ such that $\left\| |A|g \right\| = \|Ag\|$ for all $g \in \mathcal{H}$.

We would like an analogue of the polar decomposition of a matrix. Unfortunately, outside of the finite-dimensional context, there does not always exist a true isometry U such that $A = U|A|$. (Consider, for example, the backwards shift $A \in \mathcal{B}(\ell^2)$ given by $A(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots)$.) All is not lost, however; we simply need to relax our notion of isometry to the following:

Definition 2.4. We say $W \in \mathcal{B}(\mathcal{H})$ is a *partial isometry* if $W \upharpoonright \ker(W)^\perp$ is an isometry.

Fact 2.5. *If $W \in \mathcal{B}(\mathcal{H})$ is a partial isometry, then so is W^*.*

We then get the following generalization of the polar decomposition of a matrix:

Theorem 2.6 (Polar decomposition)**.** *Suppose $A \in \mathcal{B}(\mathcal{H})$. Then there is a partial isometry $W \in \mathcal{B}(\mathcal{H})$ with $\ker(W) = \ker(|A|)$ and $A = W|A|$.*

The proof is similar to the finite-dimensional case.

Remark 2.7. We then have that $\ker(W)^\perp = \ker(|A|)^\perp = \overline{\operatorname{Ran}(|A|^*)} = \overline{\operatorname{Ran}(|A|)}$ (since $A \geq 0$), and $W \upharpoonright \operatorname{Ran}(A)$ is an isometry. In particular, we get $W^* A = W^* W|A| = |A|$.

TODO 1. *Missing $|A|$ in* Ran

Finally, a notational convention:

Definition 2.8. For $g, h \in \mathcal{H}$, we define $g \otimes h^* \in \mathcal{B}_{00}(\mathcal{H})$ be $(g \otimes h^*)v = \langle v, h\rangle g$.

Remark 2.9. $\|g \otimes h^*\| = \|g\|\|h\|$.

It is a straightforward exercise in linear algebra to show that $\mathcal{B}_{00}(\mathcal{H})$ is spanned by operators of the form $g \otimes h^*$.

3 Trace class operators

In this section, I introduce the trace class operators and their properties. [2, Section 18] introduces the Hilbert-Schmidt operators early in the section, so the order and content of the proofs for the most part differs pretty strongly from that of [2].

Recall that in finite dimensions, we can recover the trace of a matrix A as

$$\sum_{i=1}^{n} \langle Ae_i, e_i\rangle$$

where $\{ e_1, \dots, e_n \}$ is an orthonormal basis. We might hope to extend this to an arbitrary Hilbert space $\mathcal{H}$ by setting

$$\operatorname{tr}(A) = \sum_{e \in \mathcal{E}} \langle Ae, e\rangle$$

for $A \in \mathcal{B}(\mathcal{H})$ (where $\mathcal{E}$ is some orthonormal basis for $\mathcal{H}$). We run into two problems: the sum may diverge, or it may differ depending on the choice of $\mathcal{E}$.

Example 3.1. If $\mathcal{H}$ is infinite-dimensional and $I \in \mathcal{B}(\mathcal{H})$ is the identity map, then for any orthonormal basis $\mathcal{E}$ of $\mathcal{H}$ we have

$$\sum_{e \in \mathcal{E}} \langle Ie, e\rangle = \sum_{e \in \mathcal{E}} 1 = \infty$$

Example 3.2. Let $\mathcal{H} = \ell^2_{\mathbb{R}}$. Consider $A \in \mathcal{B}(\mathcal{H})$ given by

$$Ae_i = \begin{cases} e_{2n+1} & i = 2n \\ e_{2n} & i = 2n+1 \end{cases}$$

Then $\langle Ae_n, e_n\rangle = 0$ for all $n \in \mathbb{N}$; so

$$\sum_{n=0}^{\infty} \langle Ae, e\rangle = 0$$

converges absolutely. But $\mathcal{F}_0 = \{\, 2^{-\frac{1}{2}}(e_{2n} + e_{2n+1}) : n \in \mathbb{N} \,\}$ is an orthonormal set, and $Af = f$ for all $f \in \mathcal{F}_0$; so

$$\begin{aligned}\sum_{f\in\mathcal{F}_0} \langle Af, f\rangle &= \sum_{n=0}^{\infty} \langle A(2^{-\frac{1}{2}}(e_{2n} + e_{2n+1})), 2^{-\frac{1}{2}}(e_{2n} + e_{2n+1})\rangle \\ &= \sum_{n=0}^{\infty} 2^{-\frac{1}{2}} \|e_{2n} + e_{2n+1}\|^2 \\ &= \sum_{n=0}^{\infty} 1 \\ &= \infty\end{aligned}$$

So

$$\sum_{f\in\mathcal{F}} \langle Af, f\rangle$$

does not converge absolutely for any orthonormal basis $\mathcal{F}$ extending $\mathcal{F}_0$. The Riemann rearrangement theorem then yields that, by reordering the basis, we can make the sum converge to any real number (or diverge).

If we want our notion of trace to make sense, we need to restrict our attention to a class of operators for which we will be able to prove the trace is finite and independent of the chosen orthonormal basis.

Definition 3.3. $T \in \mathcal{B}(\mathcal{H})$ is *trace class* if

$$\sum_{e\in\mathcal{E}} \langle |T|e, e\rangle < \infty$$

for some orthonormal basis $\mathcal{E}$ of $\mathcal{H}$. We write $\mathcal{B}_1(\mathcal{H})$ for the set of trace class operators on $\mathcal{H}$.

The following theorem follows quickly from Parseval's identity; see [2, Corollary 18.2].

Theorem 3.4. *Suppose $T \in \mathcal{B}_1$. Then*

$$\sum_{e\in\mathcal{E}} \langle |T|e, e\rangle = \sum_{f\in\mathcal{F}} \langle |T|f, f\rangle < \infty$$

for all orthonormal bases $\mathcal{E}$ and $\mathcal{F}$ of $\mathcal{H}$.

This justifies the following definition:

Definition 3.5. Suppose $T \in \mathcal{B}_1(\mathcal{H})$. We define

$$\|T\|_1 = \sum_{e\in\mathcal{E}} \langle |T|e, e\rangle$$

where $\mathcal{E}$ is some orthonormal basis for $\mathcal{H}$.

For now, this is just a notational convenience; however, we will see in Theorem 3.15 that $(\mathcal{B}_1(\mathcal{H}), \|\cdot\|_1)$ is a normed linear space.

The following theorem provides some justification for my implicit assertion that $\mathcal{B}_1(\mathcal{H})$ is the class of operators we wish to work in if we want a sensible notion of trace.

Theorem 3.6. *Suppose $T \in \mathcal{B}_1(\mathcal{H})$. Then*

$$\sum_{e\in\mathcal{E}} \langle Te, e\rangle$$

is finite and independent of the choice of orthonormal basis $\mathcal{E}$.

We need two more results before we can prove this.

I am indebted to [4, Lemma 6.3.1] for an idea in proving the following lemma and corollary.

Lemma 3.7. *Suppose $T \in \mathcal{B}_1(\mathcal{H})$. Then $|T|^{\frac{1}{2}} \in \mathcal{K}(\mathcal{H})$.*

Proof. Suppose $T \in \mathcal{B}_1(\mathcal{H})$. We show that $|T|^{\frac{1}{2}}$ can be approximated in $\|\cdot\|$ by finite rank operators.
Suppose $\varepsilon > 0$.

Claim 3.8. *There do not exist infinitely many linearly independent $g \in \mathcal{H}$ such that $\left\| |T|^{\frac{1}{2}} g \right\| \geq \varepsilon$.*

Proof. Suppose otherwise; then we may let $\mathcal{E}_0$ be an infinite orthonormal set such that $\left\| |T|^{\frac{1}{2}} e \right\| \geq \varepsilon$ for all $e \in \mathcal{E}_0$. Let $\mathcal{E}$ be an orthonormal basis extending $\mathcal{E}_0$. Then

$$\begin{aligned} \|T\|_1 &= \sum_{e \in \mathcal{E}} \langle |T| e, e \rangle \\ &\geq \sum_{e \in \mathcal{E}_0} \langle |T| e, e \rangle \\ &= \sum_{e \in \mathcal{E}_0} \left\| |T|^{\frac{1}{2}} e \right\|^2 \\ &\geq \sum_{e \in \mathcal{E}_0} \varepsilon^2 \\ &= \infty \end{aligned}$$

contradicting our assumption that $T \in \mathcal{B}_1(\mathcal{H})$. $\square$ Claim 3.8

We may then choose a finite orthonormal $\mathcal{E}_0$ such that if $\left\| |T|^{\frac{1}{2}} g \right\| \geq \varepsilon$, then $g \in \operatorname{span} \mathcal{E}_0$. Extend $\mathcal{E}_0$ to an orthonormal basis $\mathcal{E}$ of $\mathcal{H}$. Define $F \in \mathcal{B}_{00}(\mathcal{H})$ by

$$Fe = \begin{cases} |T|^{\frac{1}{2}} e & e \in \mathcal{E}_0 \\ 0 & e \in \mathcal{E} \setminus \mathcal{E}_0 \end{cases}$$

Then if $g + h \in (\operatorname{span} \mathcal{E}_0) + (\operatorname{span} \mathcal{E}_0)^\perp = \mathcal{H}$, then

$$\left(|T|^{\frac{1}{2}} - F\right)(g+h) = \left(|T|^{\frac{1}{2}} - F\right)g + \left(|T|^{\frac{1}{2}} - F\right)h = |T|^{\frac{1}{2}} h < \varepsilon$$

So $|T|^{\frac{1}{2}}$ can be approximated in $\|\cdot\|$ by finite rank operators. So $|T|^{\frac{1}{2}} \in \mathcal{K}(\mathcal{H})$. $\square$ Lemma 3.7

Corollary 3.9. $\mathcal{B}_1(\mathcal{H}) \subseteq \mathcal{K}(\mathcal{H})$.

Proof. Suppose $T \in \mathcal{B}_1(\mathcal{H})$. Let $T = W|T|$ be the polar decomposition. Then $T = W|T|^{\frac{1}{2}}|T|^{\frac{1}{2}}$. But $|T|^{\frac{1}{2}} \in \mathcal{K}(\mathcal{H})$, and $\mathcal{K}(\mathcal{H})$ is a two-sided ideal. So $T \in \mathcal{K}(\mathcal{H})$. $\square$ Corollary 3.9

We are now ready to prove Theorem 3.6:

Proof of Theorem 3.6. Suppose $T \in \mathcal{B}_1(\mathcal{H})$. Then by Corollary 3.9 we have that $|T|$ is compact. So there is an orthonormal basis $\mathcal{E}$ diagonalizing $|T|$; i.e. there are $s_e \geq 0$ such that $|T|e = s_e e$ for all $e \in \mathcal{E}$. Suppose $\mathcal{F}$ is any orthonormal basis. Let $T = W|T|$ be the polar decomposition of T. Then

$$\sum_{e \in \mathcal{E}} |s_e \langle We, e \rangle| \leq \sum_{e \in \mathcal{E}} |s_e| \|We\| \|e\| \leq \sum_{e \in \mathcal{E}} |s_e| = \|T\|_1 < \infty$$

so

$$\sum_{e \in \mathcal{E}} s_e \langle We, e \rangle$$

converges absolutely, and we need not worry about rearranging terms. Then

$$\begin{aligned}
\sum_{e\in\mathcal{E}} s_e\langle We, e\rangle &= \sum_{e\in\mathcal{E}} s_e\left\langle We, \sum_{f\in\mathcal{F}}\langle e,f\rangle f\right\rangle \\
&= \sum_{e\in\mathcal{E}}\sum_{f\in\mathcal{F}}\langle f,e\rangle\langle W|T|e,f\rangle \\
&= \sum_{f\in\mathcal{F}}\sum_{e\in\mathcal{E}}\langle W|T|\langle f,e\rangle e,f\rangle \\
&= \sum_{f\in\mathcal{F}}\left\langle W|T|\sum_{e\in\mathcal{E}}\langle f,e\rangle e, f\right\rangle \\
&= \sum_{f\in\mathcal{F}}\langle W|T|f,f\rangle \\
&= \sum_{f\in\mathcal{F}}\langle Tf,f\rangle
\end{aligned}$$

□ Theorem 3.6

This allows us to make the following definition:

Definition 3.10. Suppose $T \in \mathcal{B}_1(\mathcal{H})$. We define the *trace* of T to be

$$\operatorname{tr}(T) = \sum_{e\in\mathcal{E}}\langle Te, e\rangle$$

for some orthonormal basis $\mathcal{E}$.

I devote the remainder of this section to demonstrating properties of $\mathcal{B}_1(\mathcal{H})$ that we will use in Section 4.

Remark 3.11. $\mathcal{B}_{00}(\mathcal{H}) \subseteq \mathcal{B}_1(\mathcal{H})$.

Theorem 3.12. *$\mathcal{B}_{00}(\mathcal{H})$ is dense in $\mathcal{B}_1(\mathcal{H})$ with respect to $\|\cdot\|_1$.*

Proof. Suppose $T \in \mathcal{B}_1(\mathcal{H})$. Then by Corollary 3.9 we have that $|T| \in \mathcal{K}(\mathcal{H})$. But $|T| \geq 0$. So, by the spectral theorem for compact normal operators, we have some orthonormal basis $\mathcal{E}$ of $\mathcal{H}$ that diagonalizes $|T|$; i.e. there exist $s_e \geq 0$ such that for all $e \in \mathcal{E}$ we have $|T|e = s_e e$.

Suppose $\varepsilon > 0$. Then since $T \in \mathcal{B}_1(\mathcal{H})$, there is some finite $\mathcal{E}_0 \subseteq \mathcal{E}$ such that

$$\sum_{e\in\mathcal{E}_0} s_e = \sum_{e\in\mathcal{E}_0}\langle |T|e, e\rangle > \|T\| - \varepsilon$$

and in particular

$$\sum_{e\in\mathcal{E}\setminus\mathcal{E}_0} s_e < \varepsilon$$

Define $F \in \mathcal{B}_{00}(\mathcal{H})$ by

$$Fe = \begin{cases} |T|e & e \in \mathcal{E}_0 \\ 0 & e \in \mathcal{E}\setminus\mathcal{E}_0 \end{cases}$$

Then

$$|T| - F = \sum_{e\in\mathcal{E}\setminus\mathcal{E}_0} s_e e\otimes e^* \geq 0$$

So

$$\Big||T| - F\Big| = |T| - F$$

and

$$\Big\||T| - F\Big\| = \sum_{e\in\mathcal{E}}\left\langle \Big||T| - F\Big|e, e\right\rangle = \sum_{e\in\mathcal{E}}\langle (|T| - F)e, e\rangle = \sum_{e\in\mathcal{E}\setminus\mathcal{E}_0}\langle |T|e, e\rangle = \sum_{e\in\mathcal{E}\setminus\mathcal{E}_0} s_e < \varepsilon$$

□ Theorem 3.12

Proposition 3.13. *Suppose $T \in \mathcal{B}_1(\mathcal{H})$. Then $\|T^*\|_1 = \|T\|_1$; in particular, $T^* \in \mathcal{B}_1(\mathcal{H})$.*

Proof. Let $T = W|T|$ be the polar decomposition of T. Then

$$(T^*)^*T^* = TT^* = W|T||T|W^* = W|T|W^*W|T|W^* = (W|T|W^*)^2$$

Also $W|T|W^* = \left(|T|^{\frac{1}{2}}W^*\right)^* \left(|T|^{\frac{1}{2}}W^*\right) \geq 0$. So, by uniqueness of square roots, we have that $|T^*| = W|T|W^*$. Now, let $\mathcal{E}_0$ be an orthonormal basis for $\ker(W^*)$; let $\mathcal{E}$ be an orthonormal basis for $\mathcal{H}$ extending $\mathcal{E}_0$. Let $\mathcal{F}_0 = \{\, W^*e : e \in \mathcal{E} \setminus \mathcal{E}_0 \,\}$. Then Fact 2.5 yields that W^* is a partial isometry, and thus $\mathcal{F}_0$ is orthonormal; let $\mathcal{F}$ be an orthonormal basis for $\mathcal{H}$ extending $\mathcal{F}_0$. Then

$$\begin{aligned}
\|T^*\|_1 &= \sum_{e\in\mathcal{E}} \langle |T^*|e, e\rangle \\
&= \sum_{e\in\mathcal{E}} \langle W|T|W^*e, e\rangle \\
&= \sum_{e\in\mathcal{E}} \langle |T|W^*e, W^*e\rangle \\
&= \sum_{f\in\mathcal{F}_0} \langle |T|f, f\rangle \\
&= \sum_{f\in\mathcal{F}} \langle |T|f, f\rangle \\
&= \|T\|_1
\end{aligned}$$

(since $f \in \mathcal{F} \setminus \mathcal{F}_0$ satisfies $f \in \operatorname{Ran}(W^*)^\perp = \ker(W) = \ker(|T|)$). ☐ Proposition 3.13

Lemma 3.14. *Suppose $A \in \mathcal{B}(\mathcal{H})$ and $T \in \mathcal{B}_1(\mathcal{H})$. Suppose $\mathcal{E}$ is an orthonormal basis for $\mathcal{H}$ diagonalizing $|T|$; that is, suppose there is $s_e \geq 0$ such that $|T|e = s_e e$ for all $e \in \mathcal{E}$. Then*

$$\sum_{e\in\mathcal{E}} |\langle ATe, e\rangle| \leq \|A\|\|T\|_1$$

Proof. Let $T = W|T|$ be the polar decomposition of T. Then

$$\begin{aligned}
\sum_{e\in\mathcal{E}} |\langle ATe, e\rangle| &= \sum_{e\in\mathcal{E}} \Big|\langle AW^*|T|e, e\rangle\Big| \\
&= \sum_{e\in\mathcal{E}} s_e |\langle AW^*e, e\rangle| \\
&\leq \sum_{e\in\mathcal{E}} s_e \|AW^*e\|\|e\| \\
&\leq \sum_{e\in\mathcal{E}} s_e \|A\| \\
&= \|A\| \sum_{e\in\mathcal{E}} s_e \\
&= \|A\| \sum_{e\in\mathcal{E}} \langle |T|e, e\rangle \\
&= \|A\|\|T\|_1
\end{aligned}$$

☐ Lemma 3.14

Theorem 3.15. *$\mathcal{B}_1(\mathcal{H})$ is a subspace of $\mathcal{B}(\mathcal{H})$, and $\|\cdot\|_1$ is a norm on $\mathcal{B}_1(\mathcal{H})$.*

(In fact, $(\mathcal{B}_1(\mathcal{H}), \|\cdot\|)$ is a Banach space; however, we won't see this until Corollary 4.6.)

A proof of the above theorem can be found in [2, Theorem 18.11]. (Though on the surface Conway appears to rely on properties of Hilbert-Schmidt operators, the only non-trivial use is to show that trace class operators are compact, which we verified in Corollary 3.9.) The only part that is not straightforward is verifying the triangle inequality; this follows from a somewhat tedious manipulation of sums.

The following remark then follows easily:

Remark 3.16. $\operatorname{tr}\colon \mathcal{B}_1(\mathcal{H}) \to \mathbb{F}$ is linear.

Theorem 3.17. *Suppose $A \in \mathcal{B}(\mathcal{H})$ and $T \in \mathcal{B}_1(\mathcal{H})$. Then $\|AT\|_1 \le \|A\|\|T\|_1$ and $\|TA\|_1 \le \|A\|\|T\|_1$; in particular, $\mathcal{B}_1(\mathcal{H})$ is a two-sided ideal of $\mathcal{B}(\mathcal{H})$.*

Proof. Suppose $A \in \mathcal{B}(\mathcal{H})$ and $T \in \mathcal{B}_1(\mathcal{H})$.

Claim 3.18. $\|AT\|_1 \le \|A\|\|T\|_1$.

Proof. Let $T = W|T|$ be the polar decomposition of T; let $AT = V|T|$ be the polar decomposition of AT. Then

TODO 2. *really?*

Corollary 3.9 yields that $T \in \mathcal{K}(\mathcal{H})$; thus $|T| = W^*T \in \mathcal{K}(\mathcal{H})$. But $|T| \ge 0$; so, by the spectral theorem for compact operators, we have an orthonormal basis $\mathcal{E}$ of $\mathcal{H}$ that diagonalizes $|T|$. But then Lemma 3.14 yields

$$\begin{aligned}
\|AT\|_1 &= \sum_{e\in\mathcal{E}} \langle |AT|e, e\rangle \\
&= \sum_{e\in\mathcal{E}} |\langle |AT|e, e\rangle| \\
&= \sum_{e\in\mathcal{E}} |\langle V^*ATe, e\rangle| \\
&\le \|V^*A\|\|T\|_1 \\
&\le \|A\|\|T\|_1
\end{aligned}$$

□ Claim 3.18

Claim 3.19. $\|TA\|_1 \le \|A\|\|T\|_1$.

Proof. Proposition 3.13 and Claim 3.18 yield

$$\|TA\|_1 = \|(A^*T^*)^*\|_1 = \|A^*T^*\|_1 \le \|A^*\|\|T^*\|_1 = \|A\|\|T\|_1$$

□ Claim 3.19

□ Theorem 3.17

Theorem 3.20. *Suppose $T \in \mathcal{B}_1(\mathcal{H})$ and $A \in \mathcal{B}(\mathcal{H})$. Then $|\operatorname{tr}(AT)| \le \|T\|_1\|A\|$.*

Proof. Corollary 3.9 yields that there is an orthonormal basis $\mathcal{E}$ for $\mathcal{H}$ diagonalizing $|T|$; Lemma 3.14 then yields that

$$\begin{aligned}
|\operatorname{tr}(AT)| &= \left|\sum_{e\in\mathcal{E}} \langle ATe, e\rangle\right| \\
&\le \sum_{e\in\mathcal{E}} |\langle ATe, e\rangle| \\
&\le \|A\|\|T\|_1
\end{aligned}$$

□ Theorem 3.20

Remark 3.21. It is easy to see that if $T \in \mathcal{B}_1(\mathcal{H})$, then $\operatorname{tr}(T) = \overline{\operatorname{tr}(T^*)}$.

Corollary 3.22. *Suppose $T \in \mathcal{B}_1(\mathcal{H})$ and $A \in \mathcal{B}(\mathcal{H})$. Then $|\operatorname{tr}(TA)| \le \|T\|_1\|A\|$.*

Proof. Note that

$$|\operatorname{tr}(TA)| = |\operatorname{tr}((A^*T^*)^*)| = |\operatorname{tr}(A^*T^*)| \le \|A^*\|\|T^*\|_1 = \|A\|\|T\|_1$$

(using Theorem 3.20 and Proposition 3.13). □ Corollary 3.22

I end this section by noting a neat generalization of one of the properties of the trace in finite dimensions that we will neither prove nor make use of:

Theorem 3.23. *Suppose $T \in \mathcal{B}_1(\mathcal{H})$ and $A \in \mathcal{B}(\mathcal{H})$. Then $\operatorname{tr}(AT) = \operatorname{tr}(TA)$.*

4 Duals

Besides being of independent interest, trace class operators arise naturally as the dual of the compact operators; a corollary of this will be that $(\mathcal{B}_1(\mathcal{H}), \|\cdot\|_1)$ is a Banach space. In this section, I demonstrate this; I further show that the bounded operators are the dual of the trace class operators.

The proofs of the theorems are almost entirely due to [2, Section 19]; the only modification beyond style and presentation that I made was to explicitly prove injectivity in Theorem 4.2.

The following easy technical lemma will prove useful.

Lemma 4.1. *Suppose $A \in \mathcal{B}(\mathcal{H})$, $g, h \in \mathcal{H}$. Then $\operatorname{tr}(A(g \otimes h^*)) = \langle Ag, h\rangle$.*

Proof. Recall that $g \otimes h^* \in \mathcal{B}_{00}(\mathcal{H}) \subseteq \mathcal{B}_1(\mathcal{H})$, so the trace is well-defined.

Case 1. Suppose $h = 0$. Then $\operatorname{tr}(A(g \otimes h^*)) = 0 = \langle Ag, h\rangle$.

Case 2. Suppose $h \neq 0$. Let $\mathcal{E}$ be an orthonormal basis extending $\frac{h}{\|h\|}$. Then

$$\begin{aligned}
\operatorname{tr}(A(g \otimes h^*)) &= \sum_{e \in \mathcal{E}} \langle A(g \otimes h^*)e, e\rangle \\
&= \sum_{e \in \mathcal{E}} \langle e, h\rangle \langle Ag, e\rangle \\
&= \left\langle \frac{h}{\|h\|}, h \right\rangle \left\langle Ag, \frac{h}{\|h\|} \right\rangle \\
&= \|h\| \left\langle Ag, \frac{h}{\|h\|} \right\rangle \\
&= \langle Ag, h\rangle
\end{aligned}$$

□ Lemma 4.1

We now proceed to our main theorems.

Theorem 4.2. *The dual of $\mathcal{K}(\mathcal{H})$ is $(\mathcal{B}_1(\mathcal{H}), \|\cdot\|_1)$.*

Proof. Let $\Psi \colon \mathcal{B}_1(\mathcal{H}) \to \mathcal{K}(\mathcal{H})^*$ be $\Psi_T(K) = \operatorname{tr}(TK)$. I claim that Ψ is an isometric isomorphism.

Claim 4.3. *Ψ_T is linear and Ψ is linear.*

Proof. To see that Ψ_T is linear, note that

$$\Psi_T(aK + L) = \operatorname{tr}(T(aK + L)) = \operatorname{tr}(aTK + TL) = a \operatorname{tr}(TK) + \operatorname{tr}(TL) = a\Psi_T(K) + \Psi_T(L)$$

by Remark 3.16. To see that Ψ is linear, note that

$$\Psi_{aS+T}(K) = \operatorname{tr}((aS + T)K) = \operatorname{tr}(aSK + TK) = a \operatorname{tr}(SK) + \operatorname{tr}(TK) = a\Psi_S(K) + \Psi_T(K)$$

again by Remark 3.16. □ Claim 4.3

Claim 4.4. *$\|\Psi_T\| \leq \|T\|_1$ for all $T \in \mathcal{B}_1(\mathcal{H})$.*

Proof. This is just Corollary 3.22. □ Claim 4.4

So Ψ is indeed a linear map $\mathcal{B}_1(\mathcal{H}) \to \mathcal{K}(\mathcal{H})^*$.

Claim 4.5. *Ψ is bijective and $\|\Psi_T\| \geq \|T\|_1$.*

Proof. Suppose $\alpha \in \mathcal{K}(\mathcal{H})^*$. We define a sesquilinear form on $\mathcal{H}$ by $[g,h] = \alpha(g \otimes h^*)$; it is easily seen that this is sesquilinear and that $|[g,h]| \leq \|\alpha\|\|g\|\|h\|$ for all $g, h \in \mathcal{H}$. Thus there is a unique $S \in \mathcal{H}$ such that $[g,h] = \langle Sg, h\rangle = \operatorname{tr}(S(g \otimes h^*))$ for all $g, h \in \mathcal{H}$ (where the last equality follows from Lemma 4.1).

Note that the uniqueness immediately implies that Ψ is injective, since if $T_1, T_2 \in \mathcal{B}_1(\mathcal{H})$ satisfy $\Psi_{T_1} = \Psi_{T_2} = \alpha$, then

$$\begin{aligned} \operatorname{tr}(T_1(g \otimes h^*)) &= \Psi_R(g \otimes h^*) \\ &= \alpha(g \otimes h^*) \\ &= [g,h] \end{aligned}$$

and likewise we have $\operatorname{tr}(T_2(g \otimes h^*)) = [g,h]$; uniqueness then yields that $T_1 = T_2$.

Now, let $S = W|S|$ be the polar decomposition of S. Suppose $\mathcal{E}$ is an orthonormal basis for $\mathcal{H}$; suppose $\mathcal{E}_0 \subseteq \mathcal{E}$ is finite. Then

$$\begin{aligned} \sum_{e \in \mathcal{E}_0} \langle |S|e, e\rangle &= \left|\sum_{e \in \mathcal{E}_0} \langle W^* S e, e\rangle\right| \\ &= \left|\sum_{e \in \mathcal{E}_0} \langle Se, We\rangle\right| \\ &= \left|\sum_{e \in \mathcal{E}_0} [e, We]\right| \\ &= \left|\sum_{e \in \mathcal{E}_0} \alpha(e \otimes (We)^*)\right| \\ &= \left|\alpha\left(\sum_{e \in \mathcal{E}_0} e \otimes (We)^*\right)\right| \\ &\leq \|\alpha\| \left\|\sum_{e \in \mathcal{E}_0} e \otimes e^*\right\| \|W\| \\ &\leq \|\alpha\| \end{aligned}$$

Thus

$$\|S\|_1 = \sum_{e \in \mathcal{E}} \langle |S|e, e\rangle \leq \|\alpha\| < \infty$$

and $S \in \mathcal{B}_1(\mathcal{H})$. Furthermore, as remarked above, we have $\Psi_S(g \otimes h^*) = [g,h] = \alpha(g \otimes h^*)$ for all $g, h \in \mathcal{H}$. But $\mathcal{B}_{00}(\mathcal{H})$ is spanned by elements of the form $g \otimes h^*$, and $\mathcal{B}_{00}(\mathcal{H})$ is $\|\cdot\|$-dense in $\mathcal{K}(\mathcal{H})$; furthermore, Ψ_S and α are linear functionals that are $\|\cdot\|$-continuous and agree on elements of the form $g \otimes h^*$. So Ψ_S and α agree on all of $\mathcal{K}(\mathcal{H})$, and $\Psi_S = \alpha$.

So Ψ is surjective. Furthermore, by injectivity, we have S is the unique preimage of α; this, combined with our earlier note that $\|S\|_1 \leq \|\alpha\|$, shows that $\|\Psi_T\| \geq \|T\|_1$ for all $T \in \mathcal{B}_1(\mathcal{H})$. (Note that we need the uniqueness of the preimage to conclude this for *all* $T \in \mathcal{B}_1(\mathcal{H})$; otherwise, all we have is that Φ_S has *some* preimage R with $\|\Psi_R\| \geq \|R\|$.) □ Claim 4.5

Putting the claims together, we see that Ψ is a bijective linear isometry. So Ψ is an isometric isomorphism. □ Theorem 4.2

Corollary 4.6. *$(\mathcal{B}_1, \|\cdot\|_1)$ is a Banach space.*

Theorem 4.7. *The dual of $(\mathcal{B}_1(\mathcal{H}), \|\cdot\|_1)$ is $\mathcal{B}(\mathcal{H})$.*

Proof. Let $\Phi\colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})^*$ be $\Phi_A(T) = \operatorname{tr}(AT)$. I claim that Φ is an isometric isomorphism.

That Φ is a linear map $\mathcal{B}(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})^*$ follows exactly as in Theorem 4.2.

Claim 4.8. $\|\Phi_A\| \leq \|A\|$.

Proof. This is just Theorem 3.20. □ Claim 4.8

Claim 4.9. $\|\Phi_A\| \geq \|A\|$.

Proof. Suppose $\varepsilon > 0$; we will show that $\|\Phi_A\| > \|A\| - \varepsilon$. Pick non-zero $g \in \mathcal{H}$ such that $\|Ag\| > (\|A\| - \varepsilon)\|g\|$. Then Lemma 4.1 yields

$$\begin{aligned}\Phi_A(g \otimes (Ag)^*) &= \mathrm{tr}(A(g \otimes (Ag)^*)) \\ &= \|Ag\|^2 \\ &> (\|A\| - \varepsilon)\|g\|\|Ag\| \\ &= (\|A\| - \varepsilon)\|g \otimes (Ag)^*\|\end{aligned}$$

So $\|\Phi_A\| > \|A\| - \varepsilon$ for all $\varepsilon > 0$. So $\|\Phi_A\| \geq \|A\|$. □ Claim 4.9

Claim 4.10. Φ *is surjective.*

Proof. This follows similarly to the proof of Theorem 4.2, albeit more simply. Suppose $\alpha \in \mathcal{B}_1(\mathcal{H})^*$. Consider the sesquilinear form on $\mathcal{H}$ given by $[g, h] = \alpha(g \otimes h^*)$. We again have that this is a bounded, sesquilinear form; we again get $A \in \mathcal{B}(\mathcal{H})$ such that $[g, h] = \langle Ag, h \rangle = \mathrm{tr}(A(g \otimes h^*)) = \Phi_A(g \otimes h^*)$. We again have that $\mathcal{B}_{00}(\mathcal{H})$ is spanned by elements of the form $g \otimes h^*$; Theorem 3.12 gives us that $\mathcal{B}_{00}(\mathcal{H})$ is $\|\cdot\|_1$-dense in $\mathcal{B}_1(\mathcal{H})$. Furthermore, we have that Φ_A and α are linear and $\|\cdot\|_1$-continuous, and they agree on elements of the form $g \otimes h^*$; so Φ_A and α agree on all of $\mathcal{B}_1(\mathcal{H})$, and $\Phi_A = \alpha$. So Φ is surjective. □ Claim 4.10

So Φ is a bijective linear isometry. So Φ is an isometric isomorphism. □ Theorem 4.7

References

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