Notes from NIP XIV-XVI

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Contents

1 Preliminaries

Theorem 1. *An o-minimal theory is NIP.*

Proof. Suppose $\varphi(x;y)$ is a partitioned formula, $A \subseteq \text{dom}(\mathcal{U})$ infinite. Pick some sequence $(a_i : i < \omega)$ in A such that $a_i \neq a_j$ when $i \neq j$. But every sequence in a totally ordered set has a monotone subsequence. So we have some monotone $(b_i : i < \omega)$; suppose for convenience that it is increasing. Then $b_0 < b_1 < \ldots$. Let $B = \{b_{2i} : i < \omega\}$. Then B is not the intersection of a definable set with A; if it were, the definable set would be a finite union of intervals and points, but each interval can contain at most one of the b_{2i}, a contradiction. So B is not definable, and thus is not defined by $\varphi(\mathcal{U};b) \cap A$ for any $b \in \text{dom}(\mathcal{U})$. Thus φ is NIP.

Thus all formulae with one parameter are NIP. Thus, by [Simon, Proposition 2.14], the theory is NIP. □

Consider a κ-saturated and strongly κ-homogeneous o-minimal L-structure $\mathcal{U}$, and a (not necessarily small) $A \subseteq \text{dom}(\mathcal{U})$. Unless otherwise stated, any types we consider are complete global 1-types.

Fact 2. *The complete global 1-types in $\mathcal{U}$ are just the realized types and the cuts.*

Proof. By o-minimality, if two types satisfy the same formulae with parameters in the order language, then they satisfy the same $L_{\mathcal{U}}$-formulae, since each formula $L_{\mathcal{U}}$-formula is equivalent to a formula with parameters in the language of order language. □

We call a cut infinite if it is either $\{x > a : a \in \text{dom}(\mathcal{U})\}$ or $\{x < a : a \in \text{dom}(\mathcal{U})\}$. We call a cut finite if it is not infinite. If there exists a supremum of the lower set or an infimum of the upper set of a cut p, then both must exist, and they will be equal; we call this the standard part of the cut (denoted $\text{st}(p)$). In this case, we have $d > \text{st}(p) \implies p \vdash x < d$ and $c < \text{st}(p) \implies p \vdash x > c$.

Example 3. Not all finite cuts have a standard part; consider $((-\infty, 0) \cup (0, \infty), <)$ as a model of DLO, and $p = \{x > a : a < 0\} \cup \{x < a : a > 0\}$.

Definition 4. Suppose p is a type. We say p is an **accumulation type** of A if and only if whenever $p \vdash c < x < d$, there is some $a \in A$ such that $c < a < d$.

Example 5. It is not the case that being an accumulation type of A is equivalent to its realization in an elementary extension being an accumulation point of A. Consider in $\mathbb{R}$ the type of an infinitesimal $s > 0$, with $A = \{n^{-1} : 0 < n < \omega\}$. Then this is an accumulation type of A, but in the hyperreals we have $(0, 2s) \cap A = \emptyset$ and $s \in (0, 2s)$.

Remark 6. Suppose p has a standard part and $p \vdash x > \mathrm{st}(p)$. Then p is an accumulation type of A if and only if $\mathrm{st}(p)$ is an accumulation point of $A \cap (\mathrm{st}(p), \infty)$ in the order topology.

Proof.

($\implies$) Suppose p is an accumulation type of A. Suppose $c < \mathrm{st}(p) < d$. Then $p \vdash \mathrm{st}(p) < x < d$, so we have some $a \in A$ such that $\mathrm{st}(p) < a < d$. But then $a \in A \cap (\mathrm{st}(p), \infty)$ and $c < a < d$. So $\mathrm{st}(p)$ is an accumulation point of $A \cap (\mathrm{st}(p), \infty)$.

($\impliedby$) Suppose $\mathrm{st}(p)$ is an accumulation point of $A \cap (\mathrm{st}(p), \infty)$. Suppose $p \vdash c < x < d$. Then $c \leq \mathrm{st}(p) < d$. Pick $c' < c$. Then $c' < \mathrm{st}(p) < d$, so by assumption there is some $a \in A \cap (\mathrm{st}(p), \infty)$ such that $c' < a < d$. Then $c \leq \mathrm{st}(p) < a < d$, and p is an accumulation type of A.

□

2 Finite Satisfiability

In an o-minimal structure, a realized type is finitely satisfiable in A if and only if its realization (unique since these are global types) is in A.

Proposition 7. *A finite cut p is finitely satisfiable in A if and only if p is an accumulation type of A.*

Proof.

($\implies$) Suppose p is not an accumulation type of A. Then we have c and d such that $p \vdash c < x < d$ but $(c, d) \cap A = \emptyset$. Then A contains no realizations of $c < x < d$, so p is not finitely satisfiable in A.

($\impliedby$) Suppose p is an accumulation type of A. Suppose $p \vdash \varphi(x)$, where φ is an L-formula with parameters. Then φ defines a finite union of intervals and points, so to show that φ has a realization in A it suffices to show that whenever $\psi(x)$ is $c < x < d$ defining an interval and $p \vdash \psi(x)$, then ψ is realized in A. (We don't need to consider formulae defining points because no formula defining a finite set of points is entailed by an omitted type.) But this is just the definition of an accumulation type. So φ is realized in A, and p is finitely satisfiable in A.

□

Corollary 8. *If a finite cut p has a standard part and $p \vdash x > \mathrm{st}(p)$, then it is finitely satisfiable in A if and only if $\mathrm{st}(p)$ is an accumulation point of $A \cap (\mathrm{st}(p), \infty)$ (in the order topology).*

For infinite cuts, the cut is finitely satisfiable in A if and only if A is unbounded in the appropriate direction. (To see this, note that any $\varphi(x)$ entailed by an infinite type must be unbounded in the appropriate direction in its realizations, and thus must have an unbounded interval in the appropriate direction by o-minimality. Thus any A unbounded in the appropriate direction will have a realization of φ.) For realized types, the type is finitely satisfiable in A if and only if its realization is in A.

3 Definability

Proposition 9. *A realized type is A-definable if and only if its realization is in* $\mathrm{dcl}(A)$.

Proof. Reverse direction easy. To get the forward direction, suppose a is a realization of an A-definable type. Consider $\varphi(x, y) = x < y$. Then $d\varphi(y)$ is an L_A-formula defining (a, ∞); thus $a \in \mathrm{dcl}(A)$ as the infimum of (a, ∞). □

Lemma 10. *Suppose p is a finite cut with a standard part, $p \vdash x > \mathrm{st}(p)$, and $\varphi(x, \bar{b})$ an L-formula with parameters. Then $p \vdash \varphi(x, \bar{b})$ if and only if*

$$\mathcal{U} \models \exists z(z > \mathrm{st}(p) \land \forall y(\mathrm{st}(p) < y < z \to \varphi(y, \bar{b})))$$

(i.e. there is some interval $(\mathrm{st}(p), z)$ such that $\varphi(x, \bar{b})$ holds everywhere on that interval.)

Proof.

($\Longrightarrow$) Suppose $p \vdash \varphi(x, \bar{b})$. By o-minimality, $\varphi(x, \bar{b})$ defines a finite union of intervals and points. But p is realized in some elementary extension, and $\varphi(x, \bar{b})$ will define the same union of intervals and points in the extension; thus the realization of p is equal to one of the points or is in one of the intervals. But p is omitted, and the points are in $\mathcal{U}$, so the realization must be in one of the intervals. Thus there exist $c, d \in \mathrm{dom}(\mathcal{U})$ such that $p \vdash c < x < d$ and $\mathcal{U} \models \forall y(c < y < d \to \varphi(y, \bar{b}))$. But then $c \leq \mathrm{st}(p) < d$, so

$$\mathcal{U} \models \forall y(\mathrm{st}(p) < y < d \to \varphi(y, \bar{b}))$$

So d is our required element, and:

$$\mathcal{U} \models \exists z(\forall y(\mathrm{st}(p) < y < z \to \varphi(y, \bar{b})))$$

($\Longleftarrow$) Suppose $p \nvdash \varphi(x, \bar{b})$. Then, as a complete type, $p \vdash \neg\varphi(x, \bar{b})$, so by the forward direction we have some $d > \mathrm{st}(p)$ such that whenever $y \in (\mathrm{st}(p), d)$ we have $\mathcal{U} \models \neg\varphi(y, \bar{b})$. But then whenever $z > \mathrm{st}(p)$, we may find some (indeed, any) $y \in (\mathrm{st}(p), \min(z, d)) \subseteq (\mathrm{st}(p), z)$ such that $\mathcal{U} \not\models \varphi(y, \bar{b})$. So

$$\mathcal{U} \not\models \exists z(\forall y(\mathrm{st}(p) < y < z \to \varphi(y, \bar{b})))$$

$\square$

Proposition 11. *A finite cut is p is A-definable if and only if $\mathrm{st}(p)$ exists and is in $\mathrm{dcl}(A)$.*

Proof.

($\Longrightarrow$) Let p be an A-definable cut. Consider $\varphi(x, y) = x < y$. Then $d\varphi(y)$ is an L_A-formula, and thus defines a finite union of intervals and points. Let s be the minimum of the points and the lower bounds of the intervals. (As p is a finite cut, $s > -\infty$.) It then follows that whenever $b < s$, we have $p \vdash x > b$, and whenever $c > s$, we have $p \vdash x < c$. So $s = \mathrm{st}(p)$. Furthermore, $d\varphi$ defines either (s, ∞) or $[s, \infty)$, so $s \in \mathrm{dcl}(A)$ as the infimum of $d\varphi^{\mathcal{U}}$.

($\Longleftarrow$) Let p be a cut whose standard part is in $\mathrm{dcl}(A)$. Suppose $p \vdash x > \mathrm{st}(p)$. Let $\varphi(x, \bar{y})$ be a formula without parameters. Then, by Lemma 10, we may let $d\varphi(\bar{y})$ be

$$\exists z(\forall w(\mathrm{st}(p) < w < z \to \varphi(w, \bar{y})))$$

As $\mathrm{st}(p) \in \mathrm{dcl}(A)$ and φ has no parameters, this is an L_A formula that holds if and only if $p \vdash \varphi(x, \bar{y})$. So p is A-definable. (The case where $p \vdash x < \mathrm{st}(p)$ is similar; one needs to check that the corresponding version of Lemma 10 holds.)

$\square$

Regarding the infinite types, $\varphi(x, \bar{y})$ holds in the infinite type if and only if $\mathcal{U} \models \exists z(\forall w(w > z \to \varphi(w, \bar{y})))$. But this has no parameters, so the infinite type is A-definable for every A; likewise with the negative infinite type.

4 Invariance

Definition 12. Let $A \subseteq \mathrm{dom}(\mathcal{U})$, $p \in S_x(\mathrm{dom}(\mathcal{U}))$. We say that p is **A-invariant** if for every formula $\varphi(x; y)$ and tuples $\overline{b}, \overline{b'} \in \mathrm{dom}(U)^{|\overline{b}|}$, if $\overline{b} \equiv_A \overline{b'}$, then we have

$$p \vdash \varphi(x, \overline{b}) \iff p \vdash \varphi(x, \overline{b'})$$

Note that this definition agrees with the primary definition given in Simon only in the case where A is small.

We can extend this definition to non-global types:

Definition 13. Let $A \subseteq B \subseteq \mathrm{dom}(\mathcal{U})$, $p \in S_x(B)$. We say that p is A-invariant if for every formula $\varphi(x; y)$ and tuples $\overline{b}, \overline{b'} \in B^{|b|}$, if $b \equiv_A b'$, then

$$p \vdash \varphi(x, \overline{b}) \iff p \vdash \varphi(x, \overline{b'})$$

We can then make sense of the assertion following remark 2.20 in Simon:

Remark 14. Suppose $p \in S(\mathcal{U})$ is A-invariant. Suppose $\mathcal{U} \preceq \mathcal{U}'$, and $\mathrm{dom}(\mathcal{U}) \subseteq V \subseteq \mathrm{dom}(\mathcal{U}')$. Then there is a unique extension of p to an element of $S(V)$ that is A-invariant.

Given a formula φ, o-minimality tells us that it is a finite union of intervals and points; however, such representations are not unique, and may not be nice. Lemma 15 gives us a canonical representation with some desirable properties.

Lemma 15. *Suppose φ is a formula with parameters. Then there is a representation of $\varphi^{\mathcal{U}}$ as a finite union of intervals and points where the intervals are disjoint and maximal as open interval subsets of $\varphi^{\mathcal{U}}$, and the points are contained in none of the intervals. Furthermore, this representation is unique.*

Proof.

Existence By o-minimality, there exists some representation of $\varphi^{\mathcal{U}}$ as a finite union of intervals and points; by well-ordering of ω, we may find such a representation using a minimum number of points and intervals. We claim that this is our desired representation.

Suppose the representation contains two intervals I and J such that $I \cap J \neq \emptyset$. Then $I \cup J \subseteq \varphi^{\mathcal{U}}$ is also an interval, so if we replace I and J by $I \cup J$, we obtain a representation using one fewer interval, contradicting minimality.

Suppose the representation contains a point c and an interval I such that $c \in I$. Then we may obtain a representation using one fewer point by simply omitting c, contradicting minimality.

Suppose the representation has an interval I and there is an open interval $J \subseteq \varphi^{\mathcal{U}}$ such that $I \subsetneq J$. Then $J \setminus I$ has an open interval subset, say K. K is infinite, so our representation doesn't cover it entirely with points. So there is some interval K' in our representation such that $K' \cap K \neq \emptyset$. Clearly $K' \neq I$. But $K' \cap J \supseteq K' \cap K \supsetneq \emptyset$, and K' and J are both open interval subsets of $\varphi^{\mathcal{U}}$. So $K' \cup J$ is an open interval subset of $\varphi^{\mathcal{U}}$. But then by replacing K' and I with $K' \cup J$, we obtain a representation using one fewer interval, contradicting minimality.

So our representation has the desired properties.

Uniqueness Suppose we have two such representations.

Suppose I is an interval in the first. I is infinite, so it cannot be covered entirely by points, so there is some interval J in the second representation such that $I \cap J \neq \emptyset$. Then $I \cup J$ is an open interval subset of $\varphi^{\mathcal{U}}$, so by maximality $I = I \cup J = J$. Similarly, any interval in the second representation shows up in the first representation.

Suppose c is a point in the first representation. Then it does not show up in any of the intervals in the first representation, and hence in the second representation, so it is a point in the second representation. Likewise, any point in the second representation is a point in the first representation.

So the representations are the same.

□

In talking about A-invariance, discussing types over A forces us to discuss L_A-formulae. Given an L_A-formula $\varphi(x)$, Lemma 15 gives us a canonical representation of $\varphi(x)$ as a quantifier-free formula in order language, but this representation may not be an L_A-formula. Lemma 16 tells us that while the parameters may not be in A, they will be in $\mathrm{dcl}(A)$.

Lemma 16. *Suppose φ is an L_A-formula. Write $\varphi^{\mathcal{U}}$ as a finite union of intervals and points as in Lemma 15. Then the finite endpoints and the individual points are all in $\mathrm{dcl}(A)$.*

Proof. Apply induction on the number of intervals and points in the representation. Base case trivial.

For the induction step, suppose $\varphi^{\mathcal{U}} \neq \emptyset$. By disjointness of the intervals and points, we may totally order the intervals and points in the natural way. Consider the rightmost interval or point.

Case 1. Suppose it is a point, say $c \in \mathrm{dom}(\mathcal{U})$. Then it is definable by:

$$\varphi(x) \wedge \forall y(y > x \rightarrow \neg\varphi(y))$$

and is therefore in $\mathrm{dcl}(A)$.

Consider then $\varphi(x) \wedge x \neq c$. This can be expressed as an L_A-formula, and its representation is clearly the representation of φ with c removed. So, by induction, the finite endpoints and individual points in the remaining objects in the representation of φ are all in $\mathrm{dcl}(A)$. So all the finite endpoints and individual points are in $\mathrm{dcl}(A)$.

Case 2. Suppose it is an interval. We consider the case where both endpoints are finite; the other cases are similar. Suppose then that it is (a, b) for $a, b \in \mathrm{dom}(\mathcal{U})$. Then b is in $\mathrm{dcl}(A)$ as the unique realization of

$$\forall y(y > x \rightarrow \neg\varphi(y)) \wedge \forall z(z < x \rightarrow \exists w(z < w \wedge \varphi(w)))$$

and a is in $\mathrm{dcl}(A)$ as the unique realization of

$$\forall y(x < y < b \rightarrow \varphi(y)) \wedge \forall z(z < x \rightarrow \exists w(z < w < b \wedge \neg\varphi(w)))$$

Consider then $\varphi(x) \wedge \neg(a < x < b)$. This can be expressed as an L_A-formula, and its representation is clearly the representation of φ with (a, b) removed. So, by induction, the finite endpoints and individual points in the remaining objects in the representation of φ are all in $\mathrm{dcl}(A)$. So all the finite endpoints and individual points are in $\mathrm{dcl}(A)$.

□

Corollary 17. *Suppose $b, c \in \mathrm{dom}(\mathcal{U})$, $b < c$, $b \not\equiv_A c$. Then $\mathrm{dcl}(A) \cap [b, c] \neq \emptyset$.*

Proof. Suppose $b \not\equiv_A c$. Take some L_A-formula $\varphi(x)$ such that $\mathcal{U} \models \varphi(b)$ but $\mathcal{U} \not\models \varphi(c)$. Then by Lemma 16, φ defines a finite union of intervals with endpoints in $\mathrm{dcl}(A)$ and points in $\mathrm{dcl}(A)$. Thus either b is one of the points, in which case $b \in \mathrm{dcl}(A)$, or there is some $a_1, a_2 \in \mathrm{dcl}(A)$ such that $b \in (a_1, a_2)$ and $c \notin (a_1, a_2)$. But then $b < a_2 < c$, so $a_2 \in [b, c]$. □

Proposition 18. *A finite cut p without a standard part is A-invariant if and only if p is an accumulation type of* $\mathrm{dcl}(A)$.

Proof.

($\Longrightarrow$) Say p is A-invariant. Suppose $p \vdash b < x < c$. p has no standard part, so we may find $d, e \in (b, c)$ such that $p \vdash d < x < e$. Then $d \not\equiv_A e$ by A-invariance, so by Corollary 17 there is some element of $\mathrm{dcl}(A)$ in $[d, e] \subseteq (b, c)$. So p is an accumulation type of $\mathrm{dcl}(A)$.

($\Longleftarrow$) Suppose p is an accumulation type of $\mathrm{dcl}(A)$. Then, by Proposition 7, p is finitely satisfiable in $\mathrm{dcl}(A)$, and thus $\mathrm{dcl}(A)$-invariant. Suppose $\overline{b} \equiv_A \overline{b'}$. Then $\overline{b} \equiv_{\mathrm{dcl}(A)} \overline{b'}$, so for any L-formula φ, we have $p \vdash \varphi(x, \overline{b}) \iff p \vdash \varphi(x, \overline{b'})$ by $\mathrm{dcl}(A)$-invariance. So p is A-invariant.

□

Proposition 19. *A finite cut p with a standard part is A-invariant if and only if p is an accumulation type of* $\mathrm{dcl}(A)$ *or* $\mathrm{st}(p) \in \mathrm{dcl}(A)$.

Proof.

($\Longrightarrow$) Say p with a standard part is A-invariant. Suppose $\mathrm{st}(p) \notin \mathrm{dcl}(A)$. Suppose $p \vdash x > \mathrm{st}(p)$; the other case is identical. Suppose $p \vdash b < x < c$. Then $b \leq \mathrm{st}(p)$. Also, by density we may find $d \in (\mathrm{st}(p), c)$ such that $p \vdash x < d$. Then $p \vdash \mathrm{st}(p) < x < d$. So $d \not\equiv_A \mathrm{st}(p)$ by A-invariance, so by Corollary 17, there is some element of $\mathrm{dcl}(A)$ in $[\mathrm{st}(p), d]$. But $\mathrm{st}(p) \notin \mathrm{dcl}(A)$, so there is some element of $\mathrm{dcl}(A)$ in $(\mathrm{st}(p), d] \subseteq (b, c)$. So p is an accumulation type of $\mathrm{dcl}(A)$.

($\Longleftarrow$) Suppose $\mathrm{st}(p) \in \mathrm{dcl}(A)$. Then, by Proposition 11, p is A-definable, and hence A-invariant.

Suppose p is an accumulation type of $\mathrm{dcl}(A)$. Then, by Proposition 7, p is finitely satisfiable in $\mathrm{dcl}(A)$, and thus $\mathrm{dcl}(A)$-invariant. Suppose $\bar{b} \equiv_A \overline{b'}$. Then $\bar{b} \equiv_{\mathrm{dcl}(A)} \overline{b'}$, so for any L-formula φ, we have $p \vdash \varphi(x, \bar{b}) \iff p \vdash \varphi(x, \overline{b'})$ by $\mathrm{dcl}(A)$-invariance. So p is A-invariant.

□

Corollary 20. *A finite cut p is A-invariant if and only if p is an accumulation type of* $\mathrm{dcl}(A)$ *or p has a standard part and* $\mathrm{st}(p) \in \mathrm{dcl}(A)$.

Corollary 21. *A finite cut is A-invariant if and only if it is A-definable or finitely satisfiable in* $\mathrm{dcl}(A)$.

Fact 22. *Suppose A is small. Then a type p realized by $c \in \mathrm{dom}(\mathcal{U})$ is A-invariant if and only if $c \in \mathrm{dcl}(A)$.*

Proof.

($\Longrightarrow$) Suppose $c \notin \mathrm{dcl}(A)$. A is small, so $A \cup \{c\}$ is small, and by saturation we may find some realization c' of

$$\{x > c\} \cup \{x < a : a \in A \cap (c, \infty)\}$$

Then $[c, c'] \cap A = \emptyset$, so by Corollary 17 we have that $c \equiv_A c'$. But $p \vdash x = c$, and $p \nvdash x = c'$. So p is not A-invariant.

($\Longleftarrow$) Suppose $c \in \mathrm{dcl}(A)$. By Proposition 9, p is A-definable, and thus A-invariant.

□

Example 23. The left-to-right direction of Fact 22 does not hold when A is not small. Consider $\mathrm{tp}(\pi/\mathbb{R})$ in $(\mathbb{R}, <)$. This is $\mathbb{Q}$-invariant, as $b \equiv_{\mathbb{Q}} b' \implies b = b'$, but $\pi \notin \mathrm{dcl}(\mathbb{Q})$.

Conjecture 24. A type p realized by $c \in \mathrm{dom}(\mathcal{U})$ is A-invariant if and only if $c \in \mathrm{dcl}(A)$ or

$$c \in \overline{\mathrm{dcl}(A) \cap (c, \infty)} \cap \overline{\mathrm{dcl}(A) \cap (-\infty, c)}$$

in the order topology. (Another way of stating the latter condition is that c is a limit point of $\mathrm{dcl}(A)$ both from above and below.)

Partial Proof.

($\Longrightarrow$) Suppose $c \notin \overline{\mathrm{dcl}(A) \cap (c, \infty)} \cap \overline{\mathrm{dcl}(A) \cap (-\infty, c)}$ and $c \notin \mathrm{dcl}(A)$; then there is some $b > c$ such that $\mathrm{dcl}(A) \cap [c, b] = \emptyset$ (the case where $b < c$ is identical). Then, by Corollary 17, $c \equiv_A b$. But $p \vdash x = c$, and $p \nvdash x = b$. So p is not A-invariant.

($\Longleftarrow$) Suppose $c \in \overline{\mathrm{dcl}(A) \cap [c, \infty)} \cap \overline{\mathrm{dcl}(A) \cap (-\infty, c]}$. We observe that if $c \in \mathrm{dcl}(A)$, then the result follows by Proposition 9; suppose then the $c \notin \mathrm{dcl}(A)$. Suppose $p \vdash \varphi(x, \bar{b})$ but $p \nvdash \varphi(x, \overline{b'})$; we aim to prove that $\bar{b} \not\equiv_A \overline{b'}$. Then $p \vdash \neg\varphi(x, \overline{b'})$. By o-minimality, $\varphi(x, \bar{b})$ and $\neg\varphi(x, \overline{b'})$ define a union of intervals and points. If either defines an interval around c, say $\varphi(x, \bar{b})$ does, then there is some $a_1, a_2 \in A$ in that interval such that $a_1 < c < a_2$; thus $\mathcal{U} \models \forall x(a_1 < x < a_2 \to \varphi(x, \bar{b}))$. But by assumption $\mathcal{U} \not\models \varphi(x, \overline{b'})$, so the same statement does not hold for $\overline{b'}$; thus $\bar{b} \not\equiv_A \overline{b'}$. Suppose then that in both $\varphi(x, \bar{b})$ and $\neg\varphi(x, \overline{b'})$, c is one of the finitely many points. Then $c \in \mathrm{dcl}(\bar{b})$ and $c \in \mathrm{dcl}(\overline{b'})$.

TODO 1. *Finish this. Probably want to use propositions 4.3 and 4.4 in Casanovas.*

□

As noted at the end of Section 3, the infinite types are A-definable for every $A \subseteq \mathrm{dom}(\mathcal{U})$, and thus A-invariant for every $A \subseteq \mathrm{dom}(\mathcal{U})$.

5 Product Types and Morley Sequences

Definition 25 (Product Type)**.** Let $p_x, q_y \in S(\mathcal{U})$ be two invariant types. We define the type $p_x \otimes q_y \in S_{xy}(\mathcal{U})$ as $\mathrm{tp}(ab/\mathcal{U})$ where $b \models q_y$ and $a \models p|\mathcal{U}b$.

Proposition 26. *Let T be an O-minimal theory, $\mathcal{U}$ an $|A|$-saturated model of T, and let p be the 1-type at infinity. Then, we claim that for every $c \in \mathrm{dcl}(\mathcal{U}y)$, we have $p_x \otimes p_y \vdash x > c$. That is, if $(a, b) \models p_x \otimes p_y$ then for all $c \in \mathrm{dcl}(\mathcal{U}b)$ we have $a > c$.*

Proof. Suppose to the contrary that there exists $(a, b) \models p \otimes p$ such there exists $c \in \mathrm{dcl}(\mathcal{U}b)$ such that $a \leq c$. By saturation, there exists $b' \in \mathcal{U}$ such that $b \equiv_A b'$ and so there exists an A-automorphism σ such that $\sigma(b') = b$. p is A-invariant and so by definition, $p \vdash \varphi(x; b) \iff \varphi(x; b')$. Let $\psi(z; b)$ be the formula defining c, then let $\varphi(x; b) := \exists y(x \leq y \wedge \psi(y; b))$. By hypothesis, $a \leq c$ so $p \vdash \varphi(x; b) \iff p \vdash \varphi(x; b')$.

Note that since $\psi(y; b)$ has a unique realization, so does $\psi(y; b')$ because take the formula $\gamma(z) := \exists! x\psi(x, z)$ and by elementary equivalence since $b \models \gamma(z)$ so does b'. Thus, $\psi(y; b')$ has a unique realization d, and by elementary equivalence of $\mathcal{U} \preceq \mathcal{V}$ we have that $d \in \mathcal{U}$ (Tarski-Vaught). Thus, since $p \vdash \varphi(x; b')$ we have $a \models \varphi(x; b')$ and so $a \leq d$ which is a contradiction because $a \models p_x$, the type at $+\infty$. □

Example 27 (2.22 in Notes). Let T be DLO and take $p = q$ to be the type at $+\infty$. Then $p_x \otimes q_y \vdash x > y$ since the dcl of a set in DLO is just itself.

Example 28 (RCOF). Let T be RCOF and let $p \in S_1(\mathcal{U})$, then for every polynomial $f \in \mathcal{U}[x]$ we have $p_x \otimes p_y \vdash a > f(y)$.

Proposition 29. *Let T be an O-minimal theory, $\mathcal{U}$ an $|A|$-saturated model of T and let p be an unrealized type with standard part $st(p) \in \mathrm{dcl}(A)$. Then, for all $c \in \mathrm{dcl}(\mathcal{U}y) \cap (st(p), \infty)$ we have $p_x \otimes p_y \vdash x < c$.*

Proof. This will follow in a similar way to the previous proof. Suppose to the contrary that there exists $(a, b) \models p_x \otimes p_y$ such that there exists $c \in \mathrm{dcl}(\mathcal{U}b)$ with $a \geq c > st(p)$. Again by saturation, there exists $b' \in \mathcal{U}$ such that $b \equiv_A b'$ and so there exists an A-automorphism σ with $\sigma(b) = \sigma(b')$. p is A-invariant, so $p \vdash \varphi(x; b) \iff \varphi(x; b')$. Let $\psi(x; b)$ define c, then let

$$\varphi(x; b) := \exists y(x \geq y \wedge \psi(y; b) \wedge y > st(p))$$

Then, $\psi(x; b')$ has a unique realization c' as well and c', by elementary equivalence, is in $\mathcal{U}$ as well. So since $a \models p$ we have $a \geq c' > st(p)$. This is a contradiction because $c' \in \mathcal{U}$ and since $a \models p$ we have that $a < x$ for all $x > st(p)$. □

Example 30. Let $p \in S_1(\mathcal{U})$ be a cut with standard part 0. If T is DLO, then, $p_x \otimes p_y \vdash x < y$ and if $(a, b) \models p_x \otimes p_y$ then $a, b > 0$ and $a < b$. If T is RCOF, then if $(a, b) \models p_x \otimes p_y$ then $a, b > 0$ and $a < f(b)$ where f is any polynomial in $\mathcal{U}[x]$.

Definition 31 (Morley Sequence)**.** Let p_x be an invariant type, we define by induction for $n \in \mathbb{N}^*$

$$p^{(1)}_{x_0} = p_{x_0}$$
$$p^{(n+1)}_{x_0,\ldots,x_n} = p_{x_n} \otimes p^{(n)}_{x_0,\ldots,x_{n-1}}$$

and define $p^{(\omega)}_{x_0,x_1,\ldots,} = \bigcup p^{(n)}$. For any $N \supset A$, a realization $(a_i : i < \omega)$ of $p^{(\omega)}|N$ is called a **Morley Sequence** of p over N.

Example 32. Let T be DLO and p be the type at $+\infty$ and consider $p^{(w)}_{x_0,\ldots,x_1}$, what do the $\emptyset$-Morley Sequences look like? By our characterization of products in DLO, it follows that if $\{a_i\}$ is an increasing sequence such that $a_0 > v$ for all $v \in U$ then $\{a_i\}$ is a Morley sequence over A.

If p is the type of an infinitessimal with standard part 0, then if a_i is a decreasing sequence of infinitessimals with standard part 0, then $\{a_i\}$ is a Morley sequence over A.

Example 33. Let T be RCOF, then if p is type at $+\infty$ then similar to the above example, a morley sequence consists of an increasing sequence such that $a_{i+1} > f(a_i)$ where f is any polynomial in $\mathcal{U} \cup \{a_0, \ldots, a_{i-1}\}[x]$. Similarly, if p is the type of an infintessimal with standard part 0, then if $\{a_i\}$ is a decreasing sequence of infinitessimals with standard part 0 such that $a_{i+1} < f(a_i)$ for each $f \in \mathcal{U} \cup \{a_0, \ldots, a_{i-1}\}[x]$, then $\{a_i\}$ is a Morely sequence of p over A.

Remark 34. This is tangentially related to the above but came up when we were thinking about invariant types. It is not the case that if $\mathcal{U}$ is a k-saturated model of a theory T, then if $\mathcal{V} \succeq \mathcal{U}$ then $\mathcal{V}$ is also κ-saturated. Let T be DLO and let $\mathcal{U}$ be an $\aleph_1$-saturated model of T. Let $\mathcal{V} = \mathcal{U} \sqcup \mathbb{Q}$ such that $\mathcal{U} < \mathbb{Q}$, i.e. every elements $u \in \mathcal{U}, q \in \mathbb{Q}$ we have $u < q$. $\mathcal{V}$ is a model of DLO and by model completeness $\mathcal{U} \preceq \mathcal{V}$. But $\mathcal{V}$ is not $\aleph_1$-saturated since $\{v > p : p \in \mathbb{Q}\}$ is not realized in $\mathcal{V}$.

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