

A categorical generalization of Hrushovski’s limit structure

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1 Introduction

In [4], Hrushovski refutes a conjecture of Zilber by constructing a strongly minimal structure with certain geometric properties. The first part of this construction consists of extracting a limit structure from a class of finite structures via an adaptation of Fraïssé’s amalgamation construction; we are interested in generalizations of the construction of this limit structure.

In [5], Wagner presents an axiomatization of the construction of Hrushovski’s limit structure. In [1], Droste and Göbel present a very abstract category theoretic construction that generalizes that of [5]. In this paper, we construct a limit structure in an intermediate level of abstraction: we generalize the construction of [5] using concepts from category theory, but we keep our focus on structures and embeddings, rather than going into the full generality of categories.

For a formal statement of our main result, see Theorem 4.8. Informally, suppose \mathcal{C} is a category whose objects are some class of structures and whose morphisms are some class of embeddings; further suppose that all the objects of \mathcal{C} are well-approximated by the finitely generated objects of \mathcal{C} (see Definition 3.8 for a precise definition). We can relativize familiar model-theoretic concepts such as universality, homogeneity, JEP, and AP to the category \mathcal{C} ; see Definition 3.2 and Definition 4.1. For example, an object X of \mathcal{C} is universal with respect to the finitely generated objects of \mathcal{C} if for every finitely generated object A of \mathcal{C} , there is a morphism $A \rightarrow X$. Our main result is that \mathcal{C} has an object that is universal and homogeneous with respect to the finitely generated objects of \mathcal{C} if and only if the finitely generated objects of \mathcal{C} have JEP and AP; furthermore, if such an object exists, it is unique up to an isomorphism in \mathcal{C} .

In Section 2, we give a brief overview of Wagner’s axiomatization of Hrushovski’s limit construction, as presented by Ferreira in [2, Section 4.1]. In Section 3, we present our ω -generated categories of L -structures and show that they cover the setting of Hrushovski amalgamations. In Section 4, we prove our generalization.

In this paper, we draw on both category theory and model theory. Some confusion may arise when dealing with a category of L -structures and L -embeddings such that the categorical and model-theoretic notions of isomorphism and automorphism do not coincide. To avoid confusion, whenever we are discussing the model theoretic notion, we will prefix it with the language in question; whenever we are discussing the categorical notion, we will prefix it the category in question. So “ L -automorphism” and “ L -isomorphism” refer to the model theory notions, while “ \mathcal{C} -automorphism” and “ \mathcal{C} -isomorphism” refer to the categorical notions.

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Given a language L , we denote by \mathcal{K}_L the category whose objects are all the L -structures and whose morphisms are all the L -embeddings. We can then justify our notation “ L -isomorphism” by viewing L as standing in for \mathcal{K}_L .

“Countable” means having cardinality at most \aleph_0 .

2 Amalgamation via Predimension

We present a summarized version of the results in [2, Section 4.1], which is itself a presentation of the results of [5]. While not all of the statements in this section appear in [2], the ones that don’t follow without much effort. One should note that the construction in [2], and thus the one presented here, is less general than the original presentation in [5].

For this section, it is convenient to allow structures with empty domain.

Definition 2.1. Let L be a countable relational language. Let \mathcal{S} be a class of finite L -structures closed under substructures and L -isomorphisms, and containing countably many structures (up to L -isomorphism). A **predimension** on \mathcal{S} is a real-valued function $\delta: \mathcal{S} \rightarrow [0, \infty)$ such that the following hold:

1. $\delta(\emptyset) = 0$.
2. Suppose $A, B \in \mathcal{S}$. Suppose $AB \in \mathcal{S}$ is an L -structure of domain $A \cup B$. Then $\delta(AB) + \delta(A \cap B) \leq \delta(A) + \delta(B)$.
3. If $A, B \in \mathcal{S}$ are L -isomorphic, then $\delta(A) = \delta(B)$.
4. There is no chain $A_0 \subseteq A_1 \subseteq \dots$ with $A_i \in \mathcal{S}$ and $\delta(A_i) > \delta(A_{i+1})$ for each $i < \omega$.

Given a predimension on \mathcal{S} , we define a binary relation \leq strengthening \subseteq .

Definition 2.2. Suppose δ is a predimension on \mathcal{S} . Suppose $A, B \in \mathcal{S}$. We say $A \leq B$ (A is **closed in** B) if $A \subseteq B$ and $\delta(A) \leq \delta(B')$ for all $A \subseteq B' \subseteq B$.

Note that if δ is a predimension on \mathcal{S} , then $\emptyset \leq A$ for all $A \in \mathcal{S}$; this is simply because $\delta(\emptyset) = 0$ is the minimum value of $\delta: \mathcal{S} \rightarrow [0, \infty)$.

Definition 2.3. Suppose δ is a predimension on \mathcal{S} . Suppose we have $A, B \in \mathcal{S}$ and an L -embedding $f: A \rightarrow B$. We say f is a **strong embedding** if $f(A) \leq B$.

It easily follows that L -isomorphisms are strong.

Example 2.4 ([2, Definition 5.1.1]). Let $L = \{R\}$ consist of a single ternary relation symbol. Given an L -structure A , let

$$R[A] = \{ (a, b, c) \in \text{dom}(A)^3 : A \models R(a, b, c) \}$$

Given a finite L -structure A , let $\delta(A) = |A| - |R[A]|$. Let \mathcal{S} be the class of finite L -structures A such that for all substructures $A' \subseteq A$, we have $\delta(A') \geq 0$. Then δ is a predimension on \mathcal{S} .

Definition 2.5. Suppose δ be a predimension on \mathcal{S} . We say (\mathcal{S}, \leq) has the **\leq -amalgamation property** if whenever $A, B, C \in \mathcal{S}$, $A \leq B$, and $A \leq C$, we have some $D \in \mathcal{S}$ and strong embeddings $f: B \rightarrow D$, $g: C \rightarrow D$ such that $f|_A = g|_A$.

Example 2.6. Let \mathcal{S} and δ be as in Example 2.4. Then (\mathcal{S}, \leq) has the \leq -amalgamation property.

Definition 2.7. Suppose D is an L -structure. We define $\text{age}(D)$, the **age** of D , to be the class of finitely generated L -structures A such that there is an L -embedding from A to D . (Note that in the case of a relational language, we can replace “finitely generated” with “finite”.)

Given a predimension δ on \mathcal{S} , it is convenient to talk about strong embeddings on certain infinite L -structures; to this end, we define an extension of \mathcal{S} and \leq . Let $\overline{\mathcal{S}}$ be the class of L -structures D such that $\text{age}(D) \subseteq \mathcal{S}$. We extend \leq to $\mathcal{S} \times \overline{\mathcal{S}}$ by letting $A \leq B$ if $A \subseteq B$ and whenever $A \subseteq C \subseteq_{\text{fin}} B$, we have $\delta(A) \leq \delta(C)$. We then extend this to $\overline{\mathcal{S}} \times \overline{\mathcal{S}}$ by letting $A \leq B$ if for all $C \subseteq_{\text{fin}} A$ such that $C \leq A$, we have $C \leq B$. (See [2] for a brief justification that these are, in fact, extensions.) We also use the analogous definition of a strong embedding between structures in $\overline{\mathcal{S}}$.

The following results can be derived from [2, Section 4.1].

Proposition 2.8. *Suppose δ is a predimension on \mathcal{S} . Then \leq satisfies the following properties as a relation on $\overline{\mathcal{S}}$:*

1. \leq is reflexive and transitive.
2. \leq is invariant under L -isomorphism. i.e. If $f: B \rightarrow B'$ is an L -embedding, then for any $A \subseteq B$, we have $A \leq B$ if and only if $f(A) \leq f(B)$.
3. For all $A \in \overline{\mathcal{S}}$, we have $\emptyset \leq A$.
4. Suppose $A, B, C \in \overline{\mathcal{S}}$ and $A \subseteq B \subseteq C$. Then if $A \leq C$, we have $A \leq B$.
5. Suppose $C \in \overline{\mathcal{S}}$, $A \subseteq_{\text{fin}} C$. Then there is a $B \in \mathcal{S}$ such that $A \subseteq B \leq C$ and for any $B' \in \mathcal{S}$ such that $A \subseteq B' \leq C$, we have $B \subseteq B'$. We denote this B by $\text{cl}_C(A)$.

Theorem 2.9 ([2, Theorems 4.1.12 and 4.1.13]). *Suppose δ is a predimension on \mathcal{S} and (\mathcal{S}, \leq) has the \leq -amalgamation property. Then there is a countable L -structure D such that the following holds:*

1. $\text{age}(D) \subseteq \mathcal{S}$
2. D is \leq -**homogeneous**; that is, any L -isomorphism between finite closed substructures of D extends to an L -automorphism of D .
3. D is \leq -**universal**; that is, given $A \in \mathcal{S}$, there is a strong embedding $f: A \rightarrow D$.

Furthermore, this structure is unique up to L -isomorphism. We call this structure the **generic model** of (\mathcal{S}, \leq) .

The generic model is constructed by inductively building a suitable chain of finite L -structures and taking their union. That this structure is \leq -universal follows from the construction of the chain; uniqueness and \leq -homogeneity follow from a back-and-forth argument.

Using Theorem 2.9 as a model, we wish to abstract away the use of the predimension; we will instead focus on the strong embeddings, and examine the existence and uniqueness of a universal homogeneous object.

3 ω -Generated Categories of L -structures

In the discussion of the previous section, the amalgamation was constructed with reference to the strong embeddings and their properties; little reference was made to the predimension itself, beyond defining the strong embeddings and proving some of their properties. A natural question is whether we can abstract away the predimension, and look only at the embeddings. This would be a statement about structures and the embeddings between them; category theory provides a natural framework for this question. In this section, we develop the setting in which we will work.

We begin by trying to phrase our question in a more categorical language. To that end, we define our primary object of study:

Definition 3.1. Let L be a language (not necessarily countable or relational). A **category of L -structures** is any subcategory of \mathcal{K}_L ; that is, some class of L -structures and some class of L -embeddings between them that form a category. If \mathcal{C} is a category of L -structures, we use the term **\mathcal{C} -embedding** to refer to a morphism of \mathcal{C} . For an L -structure A , we write $A \in \mathcal{C}$ to denote that $A \in \text{ob}(\mathcal{C})$. We use $\mathcal{C}_{\text{f.g.}}$ to denote the full subcategory of finitely generated (in the model theory sense) objects of \mathcal{C} ; that is,

$$\text{ob}(\mathcal{C}_{\text{f.g.}}) = \{ X \in \text{ob}(\mathcal{C}) : X \text{ finitely generated as an } L\text{-structure} \}$$

and given $A, B \in \text{ob}(\mathcal{C}_{\text{f.g.}})$, we have $\text{hom}_{\mathcal{C}_{\text{f.g.}}}(A, B) = \text{hom}_{\mathcal{C}}(A, B)$.

We would like to ask when a universal homogeneous (with respect to $\mathcal{C}_{\text{f.g.}}$) object exists; we thus need precise definitions of “universal” and “homogeneous”. For this, we use the following definitions from [1].

Definition 3.2. Suppose \mathcal{C} is any category, \mathcal{C}^* a full subcategory of \mathcal{C} , and $X \in \text{ob}(\mathcal{C})$. We say X is \mathcal{C}^* -**universal** if for every $A \in \text{ob}(\mathcal{C}^*)$, there is a morphism $f: A \rightarrow X$. We say X is \mathcal{C}^* -**homogeneous** if given $A \in \text{ob}(\mathcal{C}^*)$ and morphisms $f, g: A \rightarrow X$, there is an automorphism $\varphi: X \rightarrow X$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \downarrow g & \searrow \varphi & \\ X & & \end{array}$$

(Recall that in category theory, an isomorphism is a morphism with a two-sided inverse, and an automorphism is an isomorphism from an object to itself.)

If \mathcal{C} is a category of L -structures, then $D \in \mathcal{C}$ is \mathcal{C}^* -universal if and only if every element of \mathcal{C}^* has a \mathcal{C} -embedding into D ; also, D is \mathcal{C}^* -homogeneous if and only if every \mathcal{C} -isomorphism between objects of \mathcal{C}^* that \mathcal{C} -embed into D extends to a \mathcal{C} -automorphism of D .

We can then state our question as follows:

Question 3.3. Given a category \mathcal{C} of L -structures, when can we find a $\mathcal{C}_{f.g.}$ -universal, $\mathcal{C}_{f.g.}$ -homogeneous object of \mathcal{C} ? Furthermore, will such an object be unique up to \mathcal{C} -isomorphism, as it is in the predimension case?

Example 3.4. Suppose L is a countable relational language. Suppose \mathcal{S} is a class of finite L -structures closed under substructures and L -isomorphisms, and containing countably many structures up to L -isomorphism; suppose δ is a predimension on \mathcal{S} . Let \mathcal{C} be the category whose objects are the countable structures in \mathcal{S} and whose morphisms the strong embeddings.

We check that \mathcal{C} is indeed a category. Recall from Proposition 2.8 that \leq is reflexive, transitive, and isomorphism-invariant on $\overline{\mathcal{S}}$, and hence on the objects of \mathcal{C} . That the identity map on a given structure is strong follows from reflexivity of \leq ; that composition is associative is trivial. It remains to verify that the composition of strong embeddings is a strong embedding. Suppose we have strong embeddings $f: A \rightarrow B$ and $g: B \rightarrow C$. Then $f(A) \leq B$. By invariance of \leq under L -isomorphism, we then have that $g(f(A)) \leq g(B)$. But we also have that $g(B) \leq C$. So, by transitivity of \leq , we have $g(f(A)) \leq C$, and $g \circ f$ is strong. So \mathcal{C} is a category. By checking the definition, we see that \mathcal{C} is a category of L -structures.

It follows immediately from the definitions that an object of \mathcal{C} is $\mathcal{C}_{f.g.}$ -universal if and only if it is \leq -universal. Furthermore, an object of \mathcal{C} is $\mathcal{C}_{f.g.}$ -homogeneous if and only if it is \leq -homogeneous; this follows from the fact that all L -isomorphisms are strong embeddings. So Question 3.3 in this case is partially answered by Theorem 2.9.

Before tackling Question 3.3 in the general case, we need to translate another notion from model theory to category theory. Given a chain of substructures, we can take their union to be the structure whose universe is the union of the universes of the structures of the chain, whose constants are the constants of the structures of the chain (which are necessarily all equal), and whose functions and relations are the unions of the functions and relations of the structures of the chain (which necessarily extend each other); we call this the **model-theoretic union**. We would like a more general notion of union that doesn't rely on an identified subset embedding; to this end, we use the following notions from category theory.

Definition 3.5. Suppose \mathcal{C} is any category. Suppose we have objects $(X_i : i < \omega)$ and morphisms $f_{ij}: X_i \rightarrow X_j$ for $i < j < \omega$ such that for all $i < k < j < \omega$, the following diagram commutes:

$$\begin{array}{ccc} X_i & \xrightarrow{f_{ik}} & X_k \\ & \searrow f_{ij} & \downarrow f_{kj} \\ & & X_j \end{array} \tag{1}$$

Then the pair $\langle X_i, f_{ij} \rangle$ is an ω -**indexed directed system**. (All directed systems in this paper will be ω -indexed, so we will hereafter simply use the term "directed system".) It is convenient to allow f_{ii} to denote id_{X_i} ; observe that this respects (1).

Suppose $\langle X_i, f_{ij} \rangle$ is a directed system. A **direct limit** of $\langle X_i, f_{ij} \rangle$ is a pair $\langle X, f_{i\infty} \rangle$ of an object $X \in \text{ob}(\mathcal{C})$ and a family of morphisms $f_{i\infty}: X_i \rightarrow X$ such that the following holds:

1. For every $i < j < \omega$, the following diagram commutes:

$$\begin{array}{ccc} X_i & \xrightarrow{f_{ij}} & X_j \\ & \searrow f_{i\infty} & \downarrow f_{j\infty} \\ & & X \end{array}$$

2. Given any such pair $\langle Y_i, f'_{i\infty} \rangle$ satisfying Item 1, we have a unique morphism $u: X \rightarrow Y$ such that for all $i < \omega$, the following diagram commutes:

$$\begin{array}{ccc} X_i & \xrightarrow{f_{i\infty}} & X \\ & \searrow f'_{i\infty} & \downarrow u \\ & & Y \end{array}$$

Example 3.6. Let L be a language. Suppose $X_0 \subseteq X_1 \subseteq \dots$ is a chain of L -structures. Then, letting $f_{ij}: X_i \rightarrow X_j$ be the inclusion maps for $i < j$, it is easily seen that $f_{ij} = f_{kj} \circ f_{ik}$ for $i < k < j$; thus $\langle X_i, f_{ij} \rangle$ is a directed system in \mathcal{K}_L . Let X be the model-theoretic union of the chain, and $f_{i\infty}: X_i \rightarrow X$ the inclusion maps. Then $\langle X, f_{i\infty} \rangle$ is a direct limit of $\langle X_i, f_{ij} \rangle$ in \mathcal{K}_L .

Note that when a category \mathcal{C} of L -structures doesn't contain all L -embeddings as \mathcal{C} -embeddings, the correspondence between direct limit and model-theoretic unions no longer applies. For example, the following situations might occur:

- A direct limit exists but the model-theoretic union is not an object of the category.
- The model-theoretic union is an object of the category but no direct limit exists.
- A chain of substructures $(A_i : i < \omega)$ such that the model-theoretic union A is in \mathcal{C} and all the inclusion maps $f_{ij}: A_i \rightarrow A_j$ and $f_{i\infty}: A_i \rightarrow A$ are \mathcal{C} -embeddings, but $\langle A, f_{i\infty} \rangle$ is not a direct limit of $\langle A_i, f_{ij} \rangle$; furthermore, $\langle A_i, f_{ij} \rangle$ has a direct limit in \mathcal{C} that is not \mathcal{C} -isomorphic to A .

The following three facts follow from diagram chasing.

Fact 3.7.

1. Suppose \mathcal{C} is a category of L -structures. Suppose $A_0 \subseteq A_1 \subseteq \dots$ (that is, they are substructures in the model theory sense) with each $A_i \in \mathcal{C}$; let A be the model-theoretic union of the A_i . Suppose the inclusion maps $A_i \rightarrow A_j$ are in fact \mathcal{C} -embeddings; further suppose that $A \in \mathcal{C}$ and the inclusion maps $A_i \rightarrow A$ are \mathcal{C} -embeddings. Finally, suppose that $\langle X, f_{i\infty} \rangle$ is a direct limit of $\langle A_i, f_{ij} \rangle$ in \mathcal{C} . Then X and A are L -isomorphic.
2. Suppose \mathcal{C} is any category, $\langle A_i, f_{ij} \rangle$ a direct system in \mathcal{C} . Suppose $\langle X, f_{i\infty} \rangle$ and $\langle Y, f'_{i\infty} \rangle$ are direct limits of $\langle A_i, f_{ij} \rangle$ in \mathcal{C} . Then X and Y are \mathcal{C} -isomorphic by a unique \mathcal{C} -isomorphism.
3. Suppose \mathcal{C} is any category. Suppose $\langle X_i, f_{ij} \rangle$ is a directed system in \mathcal{C} with direct limit $\langle X, f_{i\infty} \rangle$. Suppose $\langle X_{n_i}, f_{n_i n_j} \rangle$ is a subsequence. Then $\langle X, f_{n_i \infty} \rangle$ is a direct limit of $\langle X_{n_i}, f_{n_i n_j} \rangle$.

We restrict our attention to categories of L -structures that are controlled by the finitely generated objects via direct limits. This is formalized in the following definition.

Definition 3.8. Suppose \mathcal{C} is a category of L -structures. Then \mathcal{C} is ω -generated if the following hold:

- G1** For every $A \in \mathcal{C}_{f.g.}$, $|A| \leq \aleph_0$.
- G2** $\mathcal{C}_{f.g.}$ has no more than \aleph_0 objects up to \mathcal{C} -isomorphism.
- G3** Every directed system in $\mathcal{C}_{f.g.}$ has a direct limit in \mathcal{C} .

G4 Given $X \in \mathcal{C}$, there is a directed system $\langle X_i, f_{ij} \rangle$ in $\mathcal{C}_{f.g.}$ and \mathcal{C} -embeddings $f_{i\infty}$ such that $\langle X, f_{i\infty} \rangle$ is a direct limit of $\langle X_i, f_{ij} \rangle$ in \mathcal{C} .

G5 Suppose $\langle X_i, f_{ij} \rangle$ is a direct system in $\mathcal{C}_{f.g.}$ with $\langle X, f_{i\infty} \rangle$ as a direct limit in \mathcal{C} . Suppose $A \in \mathcal{C}_{f.g.}$ and $g: A \rightarrow X$ is a \mathcal{C} -embedding. Then there is some $i_0 < \omega$ and \mathcal{C} -embedding $g': A \rightarrow X_{i_0}$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{g'} & X_{i_0} \\ & \searrow g & \downarrow f_{i_0\infty} \\ & & X \end{array}$$

In the setting of ω -generated categories of L -structures, we will be able to give an exact characterization of when a universal homogeneous object exists; we will see this in Section 4. We devote the remainder of this section to showing that the predimension situation of Section 2 gives rise to an ω -generated category of L -structures, and that the $\mathcal{C}_{f.g.}$ -universal, $\mathcal{C}_{f.g.}$ -homogeneous objects there are precisely the generic models.

Proposition 3.9. *Suppose L is a countable relational language. Suppose \mathcal{S} is a class of finite L -structures closed under substructures and L -isomorphisms, and containing countably many structures up to L -isomorphism; suppose δ is a predimension on \mathcal{S} . Let \mathcal{C} be the category whose objects are the countable structures in $\overline{\mathcal{S}}$ and whose morphisms are the strong embeddings. Then \mathcal{C} is an ω -generated category of L -structures.*

Proof. We showed in Example 3.4 that \mathcal{C} is a category of L -structures; it remains to show that \mathcal{C} is ω -generated.

One can easily check that the objects in $\mathcal{C}_{f.g.}$ are exactly the elements of \mathcal{S} .

The following lemma will be useful in proving that G1-G5 hold:

Lemma 3.10. *Suppose $B_0 \leq B_1 \leq \dots$ are in \mathcal{S} . Let B be their model-theoretic union. Then $B \in \overline{\mathcal{S}}$, B is countable, and $B_i \leq B$ for each $i < \omega$. Furthermore, let $g_{ij}: B_i \rightarrow B_j$ and $g_{i\infty}: B_i \rightarrow B$ be the inclusion maps (which are then strong embeddings); then $\langle B_i, g_{ij} \rangle$ is a directed system in \mathcal{C} of which $\langle B, g_{i\infty} \rangle$ is a direct limit.*

Proof of Lemma 3.10. B is the countable union of finite L -structures, and therefore countable. To show that $B \in \overline{\mathcal{S}}$, we check that $\text{age}(B) \subseteq \mathcal{S}$. Suppose C is a finite L -structure, $\chi: C \rightarrow B$ an L -embedding. Then $\chi(C) \subseteq_{\text{fin}} B$, and therefore $\chi(C)$ is a substructure of one of the $B_i \in \mathcal{S}$; since \mathcal{S} is closed under substructures and L -isomorphism, we then have that $C \in \mathcal{S}$. So $B \in \overline{\mathcal{S}}$.

We now check that $B_i \leq B$ for each $i < \omega$. Suppose $i < \omega$. Let $C = \text{cl}_B(B_i)$ (see Item 5 of Proposition 2.8). Then $C \subseteq_{\text{fin}} B$, so there is some $j < \omega$ such that $C \subseteq B_j$; clearly $j \geq i$. But then $B_i \subseteq C \subseteq B_j$ and $B_i \leq B_j$; so $B_i \leq C$ by Proposition 2.8. But by definition of cl_B , $C \leq B$. So $B_i \leq B$.

That $\langle B_i, g_{ij} \rangle$ is a directed system in \mathcal{C} is clear.

Since the g_{ij} and $g_{i\infty}$ are inclusion maps, it is also clear that the following diagram commutes for every $i < j < \omega$:

$$\begin{array}{ccc} B_i & \xrightarrow{g_{ij}} & B_j \\ & \searrow g_{i\infty} & \downarrow g_{j\infty} \\ & & B \end{array}$$

It then remains to check that if we have a countable $Y \in \overline{\mathcal{S}}$ and strong embeddings $\eta_i: B_i \rightarrow Y$ such that for all $i < j < \omega$, the following diagram commutes:

$$\begin{array}{ccc} B_i & \xrightarrow{g_{ij}} & B_j \\ & \searrow \eta_i & \downarrow \eta_j \\ & & Y \end{array}$$

then there is a unique strong embedding $u: X \rightarrow Y$ such that for all $i < \omega$, the following diagram commutes:

$$\begin{array}{ccc} B_i & \xrightarrow{g_{i\infty}} & B \\ & \searrow \eta_i & \downarrow u \\ & & Y \end{array}$$

Suppose we have such Y and $\eta_i: B_i \rightarrow Y$. Then, since the g_{ij} are inclusion maps, the η_i is a chain of L -embeddings. Let u be the union of the η_i (an L -embedding). Then clearly the following diagram commutes for each $i < \omega$:

$$\begin{array}{ccc} B_i & \xrightarrow{g_{i\infty}} & B \\ & \searrow \eta_i & \downarrow u \\ & & Y \end{array}$$

Note that we have yet to show that u is a strong embedding.

We claim that u is a strong embedding; we need to show that $u(B) \leq Y$. Suppose that $C \subseteq_{\text{fin}} u(B)$ and $C \leq u(B)$. But C is finite, and

$$u(B) = \bigcup_{i < \omega} u(B_i)$$

so there is some $i < \omega$ such that $C \subseteq u(B_i) \subseteq u(B)$. But then by Proposition 2.8, we have $C \leq u(B_i)$. But $u(B_i) = \eta_i(B_i) \leq Y$, since η_i is a strong embedding. So, by transitivity of \leq , we have $C \leq Y$. So u is indeed a strong embedding.

It remains to check that u is unique. Suppose v is a strong embedding such that the following diagram commutes for all $i < \omega$:

$$\begin{array}{ccc} B_i & \xrightarrow{g_{i\infty}} & B \\ & \searrow \eta_i & \downarrow v \\ & & Y \end{array}$$

Suppose $b \in B$. Pick some $i < \omega$ such that $b \in B_i$. Then $v(b) = \eta_i(b)$. So v is determined by the η_i , and is thus unique. \square Lemma 3.10

We now continue with the proof of Proposition 3.9.

- G1** By definition of \mathcal{C} , all of its objects are countable; thus all objects of $\mathcal{C}_{\text{f.g.}}$ are countable.
- G2** Recall that in the definition of a predimension, we required that \mathcal{S} , and thus $\mathcal{C}_{\text{f.g.}}$, have countably many L -isomorphism types; G2 then follows by recalling that L -isomorphisms are strong embeddings.
- G3** Suppose $\langle A_i, f_{ij} \rangle$ is a directed system in (\mathcal{S}, \leq) . \mathcal{S} is closed under L -isomorphism, so by inductively relabeling the domains of the A_i appropriately we can find a chain $B_0 \subseteq B_1 \subseteq \dots$ of structures in \mathcal{S} and L -isomorphisms $\varphi_i: A_i \rightarrow B_i$ such that for every $i < j < \omega$, the following diagram commutes:

$$\begin{array}{ccc} A_i & \xrightarrow{f_{ij}} & A_j \\ \downarrow \varphi_i & & \downarrow \varphi_j \\ B_i & \xrightarrow{g_{ij}} & B_j \end{array} \quad (2)$$

where $g_{ij}: B_i \rightarrow B_j$ is the inclusion map. In principle, we have yet to show that g_{ij} is a strong embedding. Given the B_i and the g_{ij} , however, we note that

$$\begin{aligned} g_{ij} \circ \varphi_i &= \varphi_j \circ f_{ij} \\ \implies g_{ij} &= \varphi_j \circ f_{ij} \circ \varphi_i^{-1} \end{aligned}$$

and thus that g_{ij} is indeed a strong embedding.

We now let B be the model-theoretic union of the B_i , and let $g_{i\infty}: B_i \rightarrow B$ be the inclusion maps.

Then, by Lemma 3.10, $B \in \mathcal{C}$, each $B_i \leq B$, and the $g_{i\infty}$ are strong embeddings; furthermore, $\langle B, g_{i\infty} \rangle$ is a direct limit of $\langle B_i, g_{ij} \rangle$. It is then a simple matter of diagram chasing to show that $\langle B, g_{i\infty} \circ \varphi_i \rangle$ is a direct limit of $\langle A, f_{ij} \rangle$ in \mathcal{C} .

G4 Suppose $X \in \mathcal{C}$. If X is finite, then $\langle X, \text{id}_X \rangle$ is a direct limit of $\langle X, \text{id}_X \rangle$, so G4 trivially holds. If $|X| = \aleph_0$, enumerate X as $(x_i : i < \omega)$.

Let $X_0 = \emptyset$. Given X_i , let $X_{i+1} = \text{cl}_X(X_i \cup \{x_i\}) \in \mathcal{S}$. Then clearly X is the model-theoretic union of the X_i . Let $f_{ij}: X_i \rightarrow X_j$ and $f_{i\infty}: X_i \rightarrow X$ be the inclusion maps. By definition of cl_X , each $f_{i\infty}$ is a strong embedding. For $i < j < \omega$, note that $X_i \subseteq X_j \subseteq X$ and $X_i \leq X$; thus, by Proposition 2.8, $X_i \leq X_j$, and each f_{ij} is a strong embedding. Then by Lemma 3.10, $\langle X_i, f_{ij} \rangle$ is a directed system in (\mathcal{S}, \leq) of which $\langle X, f_{i\infty} \rangle$ is a direct limit.

G5 Suppose $\langle X_i, f_{ij} \rangle$ is a directed system in (\mathcal{S}, \leq) of which $\langle X, f_{i\infty} \rangle$ is a direct limit.

Claim 3.11. $f_{0\infty}(X_0) \subseteq f_{1\infty}(X_1) \subseteq \dots$ and

$$X = \bigcup_{i < \omega} f_{i\infty}(X_i)$$

(the model-theoretic union).

Proof. Let $Y_i = f_{i\infty}(X_i)$. Then $Y_i \in \text{age}(X)$, so $Y_i \in \mathcal{S}$. That $Y_i \subseteq Y_{i+1}$ follows from the fact that $f_{i\infty} = f_{(i+1)\infty} \circ f_{i(i+1)}$. Let Y be the model-theoretic union

$$\bigcup_{i < \omega} Y_i$$

Each $Y_i \subseteq X$, so $Y \subseteq X$. Furthermore, $X \in \mathcal{C}$, so $|X| \leq \aleph_0$ and $\text{age}(X) \subseteq \mathcal{S}$, so $|Y| \leq \aleph_0$ and $\text{age}(Y) \subseteq \mathcal{S}$, so $Y \in \mathcal{C}$. We claim that $Y \leq X$. To see this, suppose $C \subseteq_{\text{fin}} Y$ and $C \leq Y$. Then $C \subseteq Y_i \subseteq Y$ for some $i < \omega$; then by Proposition 2.8, $C \leq Y_i$. But $Y_i \leq X$, since $f_{i\infty}$ is a \mathcal{C} -embedding; so $C \leq X$. So $Y \leq X$. Furthermore, noting that $Y_i \subseteq Y \subseteq X$ and $Y_i \leq X$, Proposition 2.8 yields that $Y_i \leq Y$ for each $i < \omega$.

Now, let $\psi_i: X_i \rightarrow Y$ be defined by $\psi_i(x) = f_{i\infty}(x)$. These are clearly L -embeddings; that they are strong embeddings follows from the fact that $\psi_i(X_i) = f_{i\infty}(X_i) = Y_i \leq Y$. Furthermore, by properties of the $f_{i\infty}$ we have that the following diagram commutes for each $i < j < \infty$:

$$\begin{array}{ccc} X_i & \xrightarrow{f_{ij}} & X_j \\ & \searrow \psi_i & \downarrow \psi_j \\ & & Y \end{array}$$

Then, since $\langle X, f_{i\infty} \rangle$ is a direct limit of $\langle X_i, f_{ij} \rangle$ there is a unique strong embedding $u: X \rightarrow Y$ such that the following diagram commutes for each $i < \omega$:

$$\begin{array}{ccc} X_i & \xrightarrow{f_{i\infty}} & X \\ & \searrow \psi_i & \downarrow u \\ & & Y \end{array}$$

Now, let $u': Y \rightarrow X$ be the inclusion map; then u' is a strong embedding since $Y \leq X$. By definition of ψ_i , the following diagram commutes for each $i < \omega$:

$$\begin{array}{ccc} X_i & \xrightarrow{f_{i\infty}} & X \\ & \searrow \psi_i & \uparrow u' \\ & & Y \end{array}$$

Then:

$$\begin{aligned} u' \circ u \circ f_{i\infty} &= u' \circ \psi_i \\ &= f_{i\infty} \end{aligned}$$

But there is supposed to be a unique strong embedding $v: X \rightarrow X$ such that $v \circ f_{i\infty} = f_{i\infty}$ for all $i < \omega$, and clearly the identity map is such a \mathcal{C} -embedding; so $u' \circ u = \text{id}_X$. So u' , the inclusion map from Y to X , is surjective; so $X = Y$. \square Claim 3.11

We now show G5. Suppose we have $A \in \mathcal{S}$ and a strong embedding $g: A \rightarrow X$. Then $g(A)$ is finite, so there is some $i < \omega$ such that $g(A) \subseteq f_{i\infty}(X_i) \subseteq X$. But $g(A) \leq X$, so by Proposition 2.8, $g(A) \leq f_{i\infty}(X_i)$. Let $g': A \rightarrow X_i$ be $g'(x) = f_{i\infty}^{-1}(g(x))$; then g' is an L -embedding. The invariance of \leq under L -isomorphism tells us that since $g(A) \leq f_{i\infty}(X_i)$, we have $g'(A) = f_{i\infty}^{-1}(g(A)) \leq X_i$, and g' is a strong embedding. Furthermore, the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{g'} & X_i \\ & \searrow g & \downarrow f_{i\infty} \\ & & X \end{array}$$

So G5 holds.

So \mathcal{C} is an ω -generated category of L -structures. \square Proposition 3.9

Proposition 3.12. *Suppose \mathcal{C} is as in Proposition 3.9. Then an L -structure X is a generic model of (\mathcal{S}, \leq) if and only if $X \in \mathcal{C}$ and X is $\mathcal{C}_{f.g.}$ -universal and $\mathcal{C}_{f.g.}$ -homogeneous in \mathcal{C} .*

Proof.

(\implies) Suppose X is a generic model of (\mathcal{S}, \leq) . By definition of the generic model, we have $|X| \leq \aleph_0$ and $\text{age}(X) \subseteq \mathcal{S}$, so $X \in \mathcal{C}$. We also have that X is \leq -universal and \leq -homogeneous; by Example 3.4, we then have that X is $\mathcal{C}_{f.g.}$ -universal and $\mathcal{C}_{f.g.}$ -homogeneous.

(\impliedby) Suppose $X \in \mathcal{C}$ is $\mathcal{C}_{f.g.}$ -universal and $\mathcal{C}_{f.g.}$ -homogeneous. Since $X \in \mathcal{C}$, it follows that $|X| \leq \aleph_0$ and $\text{age}(X) \subseteq \mathcal{S}$. Furthermore, by Example 3.4, we find that X is \leq -universal and \leq -homogeneous; so X is a generic model of (\mathcal{S}, \leq) .

\square Proposition 3.12

4 Amalgamation over ω -Generated Categories of L -structures

The results of this section, being of an intermediate level of abstraction, have similarities to both the full categorical generalization and the more concrete construction of Fraïssé. In particular, the proof of Lemma 4.6 is adapted from that of [1, Lemma 2.1]; the proof of the right-to-left direction of Theorem 4.8 draws inspiration from [3, Theorem 7.1.2].

We fix a language L (not necessarily countable or relational) and a category \mathcal{C} of L -structures.

Definition 4.1. We say \mathcal{C} has the **joint embedding property** (JEP) if whenever $A, B \in \mathcal{C}$, there is some $C \in \mathcal{C}$ and \mathcal{C} -embeddings $f: A \rightarrow C$, $g: B \rightarrow C$. We say \mathcal{C} has the **amalgamation property** (AP) if given $A, B, C \in \mathcal{C}$ and \mathcal{C} -embeddings $f: A \rightarrow B$, $g: A \rightarrow C$, there is some $D \in \mathcal{C}$ and \mathcal{C} -embeddings $f': B \rightarrow D$, $g': C \rightarrow D$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow g & & \downarrow f' \\ C & \xrightarrow{g'} & D \end{array}$$

Example 4.2. Suppose \mathcal{C} is as in Proposition 3.9. If (\mathcal{S}, \leq) has the \leq -amalgamation property, then $\mathcal{C}_{f.g.}$ has JEP and AP.

Proof.

JEP Suppose $A, B \in \mathcal{C}_{f.g.}$. Then $\emptyset \leq A, B$, so by \leq -amalgamation, we have some $C \in \mathcal{C}_{f.g.}$ and \mathcal{C} -embeddings $f: A \rightarrow C$ and $g: B \rightarrow C$ such that the following diagram commutes:

$$\begin{array}{ccc} \emptyset & \xrightarrow{\eta_0} & A \\ \downarrow \eta_1 & & \downarrow f \\ B & \xrightarrow{g} & C \end{array}$$

(where η_i are the inclusion maps). So $\mathcal{C}_{f.g.}$ has JEP.

AP Suppose we have $A, B, C \in \mathcal{C}_{f.g.}$ and \mathcal{C} -embeddings $f: A \rightarrow B$ and $g: A \rightarrow C$. With appropriate relabeling, we may find L -structures B', C' and L -isomorphisms $\varphi: B \rightarrow B'$, $\psi: C \rightarrow C'$ such that $A \subseteq B'$, $A \subseteq C'$, $\varphi \circ f = \eta_0$, and $\psi \circ g = \eta_1$ (where η_i are the inclusion maps). Since \mathcal{S} is closed under L -isomorphism, we then have that $B', C' \in \mathcal{C}_{f.g.}$. Then $\varphi \circ f$ and $\psi \circ g$ are \mathcal{C} -embeddings, as the composition of \mathcal{C} -embeddings, so $A \leq B'$ and $A \leq C'$. Then, by the \leq -amalgamation property, there is some $D \in \mathcal{C}_{f.g.}$ and \mathcal{C} -embeddings $f': B' \rightarrow D$ and $g': C' \rightarrow D$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\eta_0} & B' \\ \downarrow \eta_1 & & \downarrow f' \\ C' & \xrightarrow{g'} & D \end{array}$$

Substituting $\eta_0 = \varphi \circ f$ and $\eta_1 = \psi \circ g$, we find the following diagram commutes:

$$\begin{array}{ccccc} A & \xrightarrow{f} & & & B \\ \downarrow g & & & & \downarrow \varphi \\ & & & & B' \\ \downarrow \psi & & & & \downarrow f' \\ C & \xrightarrow{g} & C' & \xrightarrow{g'} & D \end{array}$$

So $\mathcal{C}_{f.g.}$ has the \leq -amalgamation property. □

Our eventual result (Theorem 4.8) will be that if \mathcal{C} is an ω -generated category of L -structures, then there is a \mathcal{C} -universal and $\mathcal{C}_{f.g.}$ -homogeneous object if and only if $\mathcal{C}_{f.g.}$ has JEP and AP; furthermore, such objects are unique up to \mathcal{C} -isomorphism. (One should note that we are now discussing \mathcal{C} -universality, as opposed to $\mathcal{C}_{f.g.}$ -universality. The reason for this is that in this context, the two are equivalent; see Lemma 4.5.)

Working up to Theorem 4.8, the following definition (also from [1], although they use the term “ \mathcal{C}^* -saturated”) will be useful.

Definition 4.3. Suppose \mathcal{C}^* is a full subcategory of \mathcal{C} ; suppose $X \in \mathcal{C}$. Then X satisfies the **extension property with respect to \mathcal{C}^*** (X has **\mathcal{C}^* -extension**) if whenever we have objects $A, B \in \mathcal{C}^*$ and \mathcal{C} -embeddings $f: A \rightarrow X$, $g: A \rightarrow B$, there is a \mathcal{C} -embedding $h: B \rightarrow X$ such that the following diagram commutes:

$$\begin{array}{ccc} B & \xrightarrow{h} & X \\ g \uparrow & \nearrow f & \\ A & & \end{array}$$

Lemma 4.4. *Suppose \mathcal{C}^* is a full subcategory of \mathcal{C} . Suppose $X \in \mathcal{C}$ is \mathcal{C}^* -universal and \mathcal{C}^* -homogeneous. Then X has \mathcal{C}^* -extension.*

Proof. Suppose $X \in \mathcal{C}$ is \mathcal{C}^* -universal and \mathcal{C}^* -homogeneous. Suppose $A, B \in \mathcal{C}^*$; suppose $f: A \rightarrow X$ and $g: A \rightarrow B$ are \mathcal{C} -embeddings. By \mathcal{C}^* -universality, we have a \mathcal{C} -embedding $h: B \rightarrow X$. Then f and $h \circ g$ are

both \mathcal{C} -embeddings from A to X ; so by \mathcal{C}^* -homogeneity, there is a \mathcal{C} -automorphism $\varphi: X \rightarrow X$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \downarrow g & & \uparrow \varphi \\ B & \xrightarrow{h} & X \end{array}$$

Then $\varphi \circ h$ is our desired map. So X has \mathcal{C}^* -extension. □ Lemma 4.4

Lemma 4.5. *Suppose \mathcal{C} is ω -generated. Suppose $X \in \mathcal{C}$ is $\mathcal{C}_{f.g.}$ -universal and has $\mathcal{C}_{f.g.}$ -extension. Then X is \mathcal{C} -universal.*

Proof. Suppose $Y \in \mathcal{C}$. By G4, there is a directed system $\langle Y_i, f_{ij} \rangle$ in $\mathcal{C}_{f.g.}$ and \mathcal{C} -embeddings $f_{i\infty}: Y_i \rightarrow Y$ such that $\langle Y, f_{i\infty} \rangle$ is a direct limit of $\langle Y_i, f_{ij} \rangle$. By $\mathcal{C}_{f.g.}$ -universality, pick some \mathcal{C} -embedding $h_0: Y_0 \rightarrow X$. By $\mathcal{C}_{f.g.}$ -extension, we may inductively choose $h_{i+1}: Y_{i+1} \rightarrow X$ such that the following diagram commutes:

$$\begin{array}{ccc} Y_i & \xrightarrow{f_{i(i+1)}} & Y_{i+1} \\ & \searrow h_i & \downarrow h_{i+1} \\ & & X \end{array}$$

It then easily follows that for each $i < j < \omega$, the following diagram commutes:

$$\begin{array}{ccc} Y_i & \xrightarrow{f_{ij}} & Y_j \\ & \searrow h_i & \downarrow h_j \\ & & X \end{array}$$

Then, by definition of direct limits, there is a unique \mathcal{C} -embedding $u: Y \rightarrow X$ such that for each $i < \omega$, the following diagram commutes:

$$\begin{array}{ccc} Y_i & \xrightarrow{f_{i\infty}} & Y \\ & \searrow h_i & \downarrow u \\ & & X \end{array}$$

In particular, $u: Y \rightarrow X$ is a \mathcal{C} -embedding. So X is \mathcal{C} -universal. □ Lemma 4.5

Lemma 4.6. *Suppose \mathcal{C} is an ω -generated category of L -structures. Suppose $X, Y \in \mathcal{C}$ both have $\mathcal{C}_{f.g.}$ -extension. Suppose $A \in \mathcal{C}_{f.g.}$ and we have \mathcal{C} -embeddings $\varphi^*: A \rightarrow X$ and $\psi^*: A \rightarrow Y$. Then there is a \mathcal{C} -isomorphism $u: X \rightarrow Y$ such that the following diagram commutes:*

$$\begin{array}{ccc} & & X \\ & \nearrow \varphi^* & \downarrow u \\ A & & Y \\ & \searrow \psi^* & \end{array}$$

Proof. By G4, choose directed systems $\langle X_i, f_{ij} \rangle, \langle Y_i, g_{ij} \rangle$ in $\mathcal{C}_{f.g.}$ and \mathcal{C} -embeddings $f_{i\infty}: X_i \rightarrow X, g_{i\infty}: Y_i \rightarrow Y$ such that $\langle X, f_{i\infty} \rangle$ is a direct limit of $\langle X_i, f_{ij} \rangle$ and $\langle Y, g_{i\infty} \rangle$ is a direct limit of $\langle Y_i, g_{ij} \rangle$. By G5, we have some $m_0, m_1 < \omega$ and \mathcal{C} -embeddings $\varphi: A \rightarrow X_{m_0}$ and $\psi: A \rightarrow Y_{m_1}$ such that the following diagrams commute:

$$\begin{array}{ccc} & & X \\ & \nearrow \varphi^* & \uparrow f_{m_0\infty} \\ A & \xrightarrow{\varphi} & X_{m_0} \end{array} \quad \begin{array}{ccc} & & Y \\ & \nearrow \psi^* & \uparrow g_{m_1\infty} \\ A & \xrightarrow{\psi} & Y_{m_1} \end{array}$$

Then, taking $n_0 = \max(m_0, m_1)$, and composing with $f_{m_0 n_0}$ and $g_{m_1 n_0}$ as appropriate, we may assume without loss of generality that $n_0 = m_0 = m_1$.

Claim 4.7. We can find subsequences $(X_{n_i} : i < \omega)$ and $(Y_{n_i} : i < \omega)$ and \mathcal{C} -embeddings $(h_i : i < \omega)$ such that for each $i < \omega$, $h_{2i} : X_{n_{2i}} \rightarrow Y_{n_{2i+1}}$ and $h_{2i+1} : Y_{n_{2i+1}} \rightarrow X_{n_{2i+2}}$, and with the further property that the following diagram commutes:

$$\begin{array}{ccc}
 & X_{n_0} & \\
 \varphi \nearrow & & \searrow h_0 \\
 A & & Y_{n_0} \xrightarrow{g_{n_0 n_1}} Y_{n_1} \\
 \psi \searrow & & \\
 & &
 \end{array} \tag{3}$$

and for each $i < \omega$, the following diagrams commute:

$$\begin{array}{ccc}
 X_{n_{2i}} & \xrightarrow{f_{n_{2i} n_{2i+2}}} & X_{n_{2i+2}} \\
 \searrow h_{2i} & & \nearrow h_{2i+1} \\
 & Y_{n_{2i+1}} &
 \end{array}
 \qquad
 \begin{array}{ccc}
 & X_{n_{2i+2}} & \\
 \nearrow h_{2i+1} & & \searrow h_{2i+2} \\
 Y_{n_{2i+1}} & \xrightarrow{g_{n_{2i+1} n_{2i+3}}} & Y_{n_{2i+3}}
 \end{array} \tag{4}$$

Proof. By construction of φ and ψ , n_0 is given. To choose n_1 and h_0 , use $\mathcal{C}_{f.g.}$ -extension of Y to find a \mathcal{C} -embedding $h_0^* : X_{n_0} \rightarrow Y$ such that the following diagram commutes:

$$\begin{array}{ccc}
 & X_{n_0} & \\
 \varphi \nearrow & & \searrow h_0^* \\
 A & & Y \\
 \psi \searrow & & \nearrow g_{n_0 \infty} \\
 & Y_{n_0} &
 \end{array}$$

By G5, we have some $n_1 < \omega$ such that the following diagram commutes:

$$\begin{array}{ccc}
 X_{n_0} & \xrightarrow{h_0} & Y_{n_1} \\
 \searrow h_0^* & & \downarrow g_{n_1 \infty} \\
 & & Y
 \end{array}$$

(By composing h_0 with appropriate g_{ij} , we may assume without loss of generality that $n_1 > n_0$.) Then the following diagram commutes:

$$\begin{array}{ccc}
 & X_{n_0} & \xrightarrow{h_0} & Y_{n_1} \\
 \varphi \nearrow & & & \downarrow g_{n_1 \infty} \\
 A & & & Y \\
 \psi \searrow & & & \uparrow g_{n_1 \infty} \\
 & Y_{n_0} & \xrightarrow{g_{n_0 n_1}} & Y_{n_1}
 \end{array}$$

So $g_{n_1 \infty} \circ g_{n_0 n_1} \circ \psi = g_{n_1 \infty} \circ h_0 \circ \varphi$. But $g_{n_1 \infty}$ is a \mathcal{C} -embedding, thus an L -embedding, thus injective, and thus left-cancellative; so the following diagram commutes:

$$\begin{array}{ccc}
 & X_{n_0} & \\
 \varphi \nearrow & & \searrow h_0 \\
 A & & Y_{n_0} \xrightarrow{g_{n_0 n_1}} Y_{n_1} \\
 \psi \searrow & & \\
 & &
 \end{array}$$

and we have (3).

We now give the construction of n_{2i+2} and h_{2i+1} given n_{2i+1} and h_{2i} ; the construction of n_{2i+3} and h_{2i+2} given n_{2i+2} and h_{2i+1} is identical. By $\mathcal{C}_{f.g.}$ -extension of X , we may find some \mathcal{C} -embedding $h_{2i+1}^*: Y_{n_{2i+1}} \rightarrow X$ such that the following diagram commutes:

$$\begin{array}{ccc} X_{n_{2i}} & & \\ & \searrow^{f_{n_{2i}\infty}} & \\ & & X \\ & \searrow^{h_{2i}} & \uparrow^{h_{2i+1}^*} \\ & & Y_{n_{2i+1}} \end{array}$$

By G5, we have some $n_{2i+2} < \omega$ and \mathcal{C} -embedding $h_{2i+1}: Y_{n_{2i+1}} \rightarrow X_{n_{2i+2}}$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \longleftarrow & X_{n_{2i+2}} \\ & \uparrow^{h_{2i+1}^*} & \nearrow^{h_{2i+1}} \\ Y_{n_{2i+1}} & & \end{array}$$

(Again, we may assume without loss of generality that $n_{2i+2} > n_{2i+1}$.) Then the following diagram commutes:

$$\begin{array}{ccc} X_{n_{2i}} & \xrightarrow{f_{n_{2i}n_{2i+2}}} & X_{n_{2i+2}} \\ & \searrow^{h_{2i}} & \\ & & X \\ & & \longleftarrow^{f_{n_{2i+2}\infty}} \\ & & X_{n_{2i+2}} \\ & & \nearrow^{h_{2i+1}} \\ & & Y_{n_{2i+1}} \end{array}$$

So $f_{n_{2i+2}\infty} \circ f_{n_{2i}n_{2i+2}} = f_{n_{2i+2}\infty} \circ h_{2i+1} \circ h_{2i}$. But $f_{n_{2i+2}\infty}$ is left-cancellative; so the following diagram commutes:

$$\begin{array}{ccc} X_{n_{2i}} & \xrightarrow{f_{n_{2i}n_{2i+2}}} & X_{n_{2i+2}} \\ & \searrow^{h_{2i}} & \nearrow^{h_{2i+1}} \\ & & Y_{n_{2i+1}} \end{array}$$

and we have (4). □ Claim 4.7

Having constructed our $(X_{n_i} : i < \omega)$, $(X_{n_i} : i < \omega)$, and $(h_i : i < \omega)$, we look to the construction of the desired h . By Fact 3.7, $\langle X, f_{n_{2i}\infty} \rangle$ is a direct limit of $\langle X_{n_{2i}}, f_{n_{2i}n_{2j}} \rangle$. Consider then the \mathcal{C} -embeddings $g_{n_{2i+1}\infty} \circ h_{2i}: X_{2i} \rightarrow Y$. From (4), it follows that for each $i < \omega$, the following diagram commutes:

$$\begin{array}{ccccc} X_{n_{2i}} & \xrightarrow{f_{n_{2i}n_{2i+2}}} & X_{n_{2i+2}} & & \\ & \searrow^{h_{2i}} & & \searrow^{h_{2i+2}} & \\ & & Y_{n_{2i+1}} & \xrightarrow{g_{n_{2i+1}n_{2i+3}}} & Y_{n_{2i+3}} \end{array}$$

It then follows that the following diagram commutes for all $i < j < \omega$:

$$\begin{array}{ccc}
 X_{n_{2i}} & \xrightarrow{f_{n_{2i}n_{2j}}} & X_{n_{2j}} \\
 \searrow h_{2i} & & \searrow h_{2j} \\
 & & Y_{n_{2j+1}} \\
 & & \swarrow g_{n_{2j+1}\infty} \\
 & & Y \\
 & \swarrow g_{n_{2i+1}\infty} & \\
 & & Y_{n_{2i+1}}
 \end{array}$$

So, by definition of direct limits, there is a unique \mathcal{C} -embedding $h: X \rightarrow Y$ such that for all $i < \omega$, the following diagram commutes:

$$\begin{array}{ccc}
 X_{n_{2i}} & \xrightarrow{f_{n_{2i}\infty}} & X \\
 \downarrow h_{2i} & & \downarrow h \\
 Y_{n_{2i+1}} & \xrightarrow{g_{n_{2i+1}\infty}} & Y
 \end{array} \quad (5)$$

By a similar argument, there is a unique \mathcal{C} -embedding $h': Y \rightarrow X$ such that for all $i < \omega$, the following diagram commutes:

$$\begin{array}{ccc}
 X_{n_{2i+2}} & \xrightarrow{f_{n_{2i+2}\infty}} & X \\
 \uparrow h_{2i+1} & & \uparrow h' \\
 Y_{n_{2i+1}} & \xrightarrow{g_{n_{2i+1}\infty}} & Y
 \end{array} \quad (6)$$

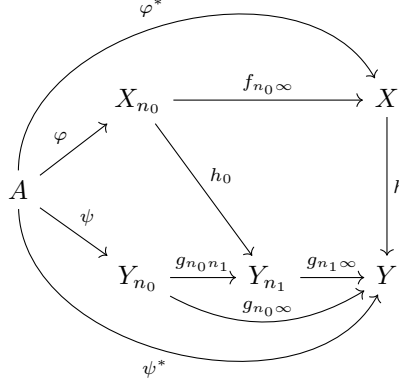
Then:

$$\begin{aligned}
 h' \circ h \circ f_{n_{2i}\infty} &= h' \circ g_{n_{2i+1}\infty} \circ h_{2i} && \text{(by (5))} \\
 &= f_{n_{2i+2}\infty} \circ h_{2i+1} \circ h_{2i} && \text{(by (6))} \\
 &= f_{n_{2i+2}\infty} \circ f_{n_{2i}n_{2i+2}} && \text{(by (4))} \\
 &= f_{n_{2i}\infty}
 \end{aligned}$$

But by the definition of a direct limit, the identity is the only \mathcal{C} -embedding $u: X \rightarrow X$ such that $u \circ f_{n_{2i}\infty} = f_{n_{2i}\infty}$ for each $i < \omega$. So $h' \circ h = \text{id}_X$. Similarly, $h \circ h' = \text{id}_Y$. So h is a \mathcal{C} -isomorphism. Furthermore:

$$\begin{aligned}
 h \circ \varphi^* &= h \circ f_{n_0\infty} \circ \varphi \\
 &= g_{n_1\infty} \circ h_0 \circ \varphi && \text{(by (5))} \\
 &= g_{n_1\infty} \circ g_{n_0n_1} \circ \psi && \text{(by (3))} \\
 &= g_{n_0\infty} \circ \psi \\
 &= \psi^*
 \end{aligned}$$

In diagram:



So h is our desired isomorphism. □ Lemma 4.6

Theorem 4.8. *Suppose \mathcal{C} is an ω -generated category of L -structures. Then \mathcal{C} contains a \mathcal{C} -universal $\mathcal{C}_{f.g.}$ -homogeneous object if and only if $\mathcal{C}_{f.g.}$ has JEP and AP. Furthermore, if such an object exists, it is unique up to \mathcal{C} -isomorphism.*

Proof.

(\implies) Suppose $X \in \mathcal{C}$ is \mathcal{C} -universal and $\mathcal{C}_{f.g.}$ -homogeneous. By G4, pick some directed system $\langle X_i, f_{ij} \rangle$ and \mathcal{C} -embeddings $f_{i\infty}: X_i \rightarrow X$ such that $\langle X, f_{i\infty} \rangle$ is a direct limit of $\langle X_i, f_{ij} \rangle$.

JEP Suppose $A, B \in \mathcal{C}_{f.g.}$. By \mathcal{C} -universality of X , there are \mathcal{C} -embeddings $\varphi: A \rightarrow X$ and $\psi: B \rightarrow X$. By G5, there is some $i_0, j_0 < \omega$ and \mathcal{C} -embeddings $\varphi': A \rightarrow X_{i_0}$, $\psi': B \rightarrow X_{j_0}$ such that the following diagrams commute:

$$\begin{array}{ccc} A & \xrightarrow{\varphi'} & X_{i_0} \\ & \searrow \varphi & \downarrow f_{i_0\infty} \\ & & X \end{array} \quad \begin{array}{ccc} B & \xrightarrow{\psi'} & X_{j_0} \\ & \searrow \psi & \downarrow f_{j_0\infty} \\ & & X \end{array}$$

Letting $k = \max(i_0, j_0)$, we have \mathcal{C} -embeddings $f_{i_0k} \circ \varphi': A \rightarrow X_k$ and $f_{j_0k} \circ \psi': B \rightarrow X_k$. But $X_k \in \mathcal{C}_{f.g.}$. So $\mathcal{C}_{f.g.}$ has JEP.

AP Suppose we have $A, B, C \in \mathcal{C}_{f.g.}$ and \mathcal{C} -embeddings $\varphi: A \rightarrow B$, $\psi: A \rightarrow C$. By \mathcal{C} -universality of X , there is some \mathcal{C} -embedding $\chi: A \rightarrow X$. By Lemma 4.4, X has $\mathcal{C}_{f.g.}$ -extension; so there are \mathcal{C} -embeddings $\chi_0: B \rightarrow X$ and $\chi_1: C \rightarrow X$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \downarrow \psi & \searrow \chi & \downarrow \chi_0 \\ C & \xrightarrow{\chi_1} & X \end{array}$$

By G5, there are $i_0, j_0 < \omega$ and \mathcal{C} -embeddings $\chi'_0: B \rightarrow X_{i_0}$ and $\chi'_1: C \rightarrow X_{j_0}$ such that the following diagrams commute:

$$\begin{array}{ccc} B & \xrightarrow{\chi'_0} & X_{i_0} \\ \downarrow \chi_0 & \swarrow & \downarrow f_{i_0\infty} \\ X & & X \end{array} \quad \begin{array}{ccc} C & \xrightarrow{\chi'_1} & X_{j_0} \\ \downarrow \chi_1 & \swarrow & \downarrow f_{j_0\infty} \\ X & & X \end{array}$$

Again, taking $k = \max(i_0, j_0)$ and composing by f_{i_0k} and f_{j_0k} as appropriate, we may assume without loss of generality that $i_0 = j_0 = k$. So the following diagram commutes:

$$\begin{array}{ccccc} A & \xrightarrow{\varphi} & B & \xrightarrow{x'_0} & X_k \\ \downarrow \psi & & & \swarrow f_{k\infty} & \\ C & & & & X \\ \downarrow x'_1 & \nearrow f_{k\infty} & & & \\ X_k & & & & \end{array}$$

But $f_{k\infty}$ is left-cancellative. So the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \downarrow \psi & & \downarrow x'_0 \\ C & \xrightarrow{x'_1} & X_k \end{array}$$

But $X_k \in \mathcal{C}_{f.g.}$. So $\mathcal{C}_{f.g.}$ has AP.

(\Leftarrow) Suppose $\mathcal{C}_{f.g.}$ has JEP and AP.

Claim 4.9. *There is a directed system $\langle X_i, f_{ij} \rangle$ in $\mathcal{C}_{f.g.}$ with the following property:*

Property 4.10. Suppose we have $A, B \in \mathcal{C}_{f.g.}$ and \mathcal{C} -embeddings $g: A \rightarrow B$ and $h: A \rightarrow X_i$ for some $i < \omega$. Then there is some $i \leq j < \omega$, some \mathcal{C} -embedding $\varphi: B \rightarrow X_j$ such that the following diagram commutes:

$$\begin{array}{ccc} B & \xrightarrow{\varphi} & X_j \\ g \uparrow & & f_{ij} \uparrow \\ A & \xrightarrow{h} & X_i \end{array}$$

Proof of Claim 4.9. By G2, $\mathcal{C}_{f.g.}$ has countably many isomorphism types. Let R be a set containing exactly one representative of each isomorphism type; then $|R| \leq \aleph_0$. We then claim that between each element of R , there are countably many \mathcal{C} -embeddings. To see this, suppose $A, B \in R$. A is finitely generated, so a \mathcal{C} -embedding $A \rightarrow B$ is determined by its action on the finitely many generators. By G1, B is countable, so there are countably many choices for each of the finitely many generators of A ; so there are countably many \mathcal{C} -embeddings $A \rightarrow B$.

For $X \in \mathcal{C}_{f.g.}$, let

$$S_X = \{ (A, B, g, h) : A, B \in R, g \in \text{hom}(A, B), h \in \text{hom}(A, X) \}$$

Then $|S_X| \leq \aleph_0$ for each $X \in \mathcal{C}_{f.g.}$.

We now construct $\langle X_i, f_{ij} \rangle$. Let $\pi: \omega^2 \rightarrow \omega$ be a bijection such that $\pi(i, j) \geq i$ for all $i, j < \omega$ (for example, the Cantor pairing function). Let X_0 be any structure in $\mathcal{C}_{f.g.}$. Let $i < \omega$, and suppose we have already constructed X_k for each $k \leq i$ and $f_{k\ell}$ for each $k < \ell \leq i$; we now construct X_{i+1} and $f_{k(i+1)}$ for each $k \leq i$. Enumerate S_{X_i} as $(A_{i,j}, B_{i,j}, g_{i,j}, h_{i,j} : j < r_i)$; we may assume $r_i \leq \omega$, since $|S_{X_i}| \leq \aleph_0$. Consider $(m, n) = \pi^{-1}(i)$; by the property of π , we have that $m \leq i$. Thus X_m is already defined, meaning that at some previous stage of the construction, we gave an enumeration of S_{X_m} as $(A_{m,j}, B_{m,j}, g_{m,j}, h_{m,j} : j < r_m)$ for some $r_m < \omega$. If $r_m \leq n$, we let $X_{i+1} = X_i$ and we let $f_{k(i+1)} = f_{ki}$ for $k < i + 1$. Otherwise, $n < r_m$, and the tuple $(A_{m,n}, B_{m,n}, g_{m,n}, h_{m,n})$ was defined in the enumeration of S_{X_m} . We then use AP to find $X_{i+1} \in \mathcal{C}_{f.g.}$ and \mathcal{C} -embeddings $\varphi: B_{m,n} \rightarrow X_{i+1}$ and $\psi: X_i \rightarrow X_{i+1}$ such that the following diagram commutes:

$$\begin{array}{ccccc} B_{m,n} & \xrightarrow{\varphi} & X_{i+1} & & \\ g_{m,n} \uparrow & & & & \psi \uparrow \\ A_{m,n} & \xrightarrow{h_{m,n}} & X_m & \xrightarrow{f_{mi}} & X_i \end{array}$$

We then let $f_{j(i+1)} = \psi \circ f_{ji}$ for $j < i + 1$. This completes our construction of $\langle X_i, f_{ij} \rangle$.

A simple induction shows that $\langle X_i, f_{ij} \rangle$ is indeed a directed system; it remains to show that it satisfies Property 4.10. Suppose we have $A, B \in \mathcal{C}_{f.g.}$ and \mathcal{C} -embeddings $g: A \rightarrow B$ and $h: A \rightarrow X_i$ for some $m < \omega$. R contains representatives of each isomorphism type, so there are elements of R isomorphic to A and B ; with some diagram chasing, we may assume that A and B are themselves elements of R . Thus $(A, B, g, h) \in S_{X_m}$. During the construction of X_{m+1} , we enumerated S_{X_m} as $(A_{m,j}, B_{m,j}, g_{m,j}, h_{m,j} : j < r_m)$; so we have some $n < r_m$ such that $(A, B, g, h) = (A_{m,n}, B_{m,n}, g_{m,n}, h_{m,n})$. Let $i = \pi(m, n)$. Then during the construction of X_{i+1} , we found \mathcal{C} -embeddings $\varphi: B \rightarrow X_{i+1}$ and $\psi: X_i \rightarrow X_{i+1}$ such that the following diagram commutes:

$$\begin{array}{ccc} B & \xrightarrow{\varphi} & X_{i+1} \\ g \uparrow & & \uparrow \psi \\ A & \xrightarrow{h} X_m \xrightarrow{f_{mi}} & X_i \end{array}$$

However, we defined $f_{m(i+1)} = \psi \circ f_{mi}$; so the following diagram commutes:

$$\begin{array}{ccc} B & \xrightarrow{\varphi} & X_{i+1} \\ g \uparrow & & \uparrow f_{m(i+1)} \\ A & \xrightarrow{h} & X_m \end{array}$$

So φ is our desired \mathcal{C} -embedding. □ Claim 4.9

Let $\langle X_i, f_{ij} \rangle$ be such a directed system in $\mathcal{C}_{f.g.}$. By G3, there is a direct limit of $\langle X_i, f_{ij} \rangle$ in \mathcal{C} , say $\langle X, f_{i\infty} \rangle$. We claim that X is $\mathcal{C}_{f.g.}$ -universal and has $\mathcal{C}_{f.g.}$ -extension.

$\mathcal{C}_{f.g.}$ -universal Suppose $A \in \mathcal{C}_{f.g.}$. By JEP, there is some $B \in \mathcal{C}_{f.g.}$ and \mathcal{C} -embeddings $\mu: A \rightarrow B$ and $\nu: X_0 \rightarrow B$. Applying Property 4.10 to $(X_0, B, \nu, \text{id}_{X_0})$, we find that there is some $i < \omega$ and \mathcal{C} -embedding $\varphi: B \rightarrow X_i$ such that the following diagram commutes:

$$\begin{array}{ccc} B & \xrightarrow{\varphi} & X_j \\ \nu \uparrow & & \uparrow f_{0j} \\ X_0 & \xrightarrow{\text{id}_{X_0}} & X_0 \end{array}$$

Then $f_{j\infty} \circ \varphi \circ \mu: A \rightarrow X$. So X is $\mathcal{C}_{f.g.}$ -universal.

$\mathcal{C}_{f.g.}$ -extension Suppose we have $A, B \in \mathcal{C}_{f.g.}$ and \mathcal{C} -embeddings $g: A \rightarrow X$, $h: A \rightarrow B$. By G5, there is some $i < \omega$ and \mathcal{C} -embedding $g': A \rightarrow X_i$ such that the following diagram commutes:

$$\begin{array}{ccc} & & X \\ & \nearrow g & \uparrow f_{i\infty} \\ A & \xrightarrow{g'} & X_i \end{array} \tag{7}$$

Then, applying Property 4.10 to (A, B, h, g') , we find that there is some $i \leq j < \omega$ and \mathcal{C} -embedding $\varphi: B \rightarrow X_j$ such that the following diagram commutes:

$$\begin{array}{ccc} B & \xrightarrow{\varphi} & X_j \\ h \uparrow & & \uparrow f_{ij} \\ A & \xrightarrow{g'} & X_i \end{array} \tag{8}$$

Observe that by (7) and by definition of direct limits, the following diagram commutes:

$$\begin{array}{ccc}
 & & X_j \\
 & \swarrow & \uparrow f_{ij} \\
 & X & \\
 \nearrow g & & \\
 A & \xrightarrow{g'} & X_i
 \end{array}$$

Then, applying (8), we get that the following diagram commutes:

$$\begin{array}{ccc}
 B & \xrightarrow{\varphi} & X_j \\
 \uparrow h & & \swarrow f_{j\infty} \\
 & X & \\
 \nearrow g & & \\
 A & &
 \end{array}$$

So $f_{(i+1)\infty} \circ \varphi$ is our desired \mathcal{C} -embedding. So X has $\mathcal{C}_{f.g.}$ -extension.

That X is \mathcal{C} -universal then follows from Lemma 4.5. To get that X is $\mathcal{C}_{f.g.}$ -homogeneous, suppose we have $A \in \mathcal{C}_{f.g.}$ and \mathcal{C} -embeddings $\varphi, \psi: A \rightarrow X$. Then, since X has $\mathcal{C}_{f.g.}$ -extension, Lemma 4.6 applies, and there is a \mathcal{C} -automorphism $u: X \rightarrow X$ such that the following diagram commutes:

$$\begin{array}{ccc}
 & X & \\
 \nearrow \varphi & & \downarrow u \\
 A & & X \\
 \searrow \psi & &
 \end{array}$$

So X is $\mathcal{C}_{f.g.}$ -homogeneous.

To show uniqueness, suppose $X, Y \in \mathcal{C}$ are both \mathcal{C} -universal and $\mathcal{C}_{f.g.}$ -homogeneous. Clearly X and Y are $\mathcal{C}_{f.g.}$ -universal. Then by Lemma 4.4, X and Y have $\mathcal{C}_{f.g.}$ -extension. Pick any $A \in \mathcal{C}_{f.g.}$. By $\mathcal{C}_{f.g.}$ -universality, we have \mathcal{C} -embeddings $\varphi: A \rightarrow X$ and $\psi: A \rightarrow Y$. Then, by Lemma 4.6, there is a \mathcal{C} -isomorphism $u: X \rightarrow Y$ such that the following diagram commutes:

$$\begin{array}{ccc}
 & X & \\
 \nearrow \varphi & & \downarrow u \\
 A & & Y \\
 \searrow \psi & &
 \end{array}$$

In particular, X and Y are \mathcal{C} -isomorphic. So such objects are unique up to \mathcal{C} -isomorphism.

□ Theorem 4.8

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