A brief introduction to monads (Now with Universal Algebra™!)

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1 Introduction

When first encountering the definition of adjoint functors, one might remark that being adjoint is a weakening of being mutually inverse. This naturally leads to questions about the composition of a pair of adjoint functors: do such compositions have any properties of note? Can we recover the original adjunction from the composition? The former question leads to the notion of a monad; the goal of this paper (Theorem 4.3) is to give a sense in which the Eilenberg-Moore algebras are the best we can do in regard to the latter question. The particular sense in which these are the best is formalized by way of universal algebra.

Section 2 gives an overview of monads and Eilenberg-Moore algebras. Section 3 gives a minimal and very fast introduction to the universal algebra needed to state our main result. Section 4 is devoted to stating and proving Theorem 4.3.

Our notation in universal algebra occasionally diverges from established usage in the interest of readability to the uninitiated. To maintain a decent length, we will necessarily gloss over some relevant universal algebra. However, it should be noted that nothing involved is at all deep; the curious reader is encouraged to seek further edification in [1, Chapter II]. [2, Section VI] is a good reference for monads.

We make use of the horizontal composition of functors and natural transformations as defined in [2, Section II.5]. Briefly, given a natural transformation $\eta\colon G \to G'$ between functors $G, G'\colon \mathcal{B} \to \mathcal{C}$ and given functors $H\colon \mathcal{A} \to \mathcal{B}$ and $F\colon \mathcal{C} \to \mathcal{D}$, there is a natural transformation $F\eta H\colon FGH \to FG'H$ given by $(F\eta H)_A = F(\eta_{HA})\colon FGHA \to FG'HA$ for $A \in \mathrm{Ob}(\mathcal{A})$.

2 Monads

Consider a pair of adjoint functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ (with $F \leftrightarrows G$); a useful example to keep in mind throughout this section is the free-forgetful adjunction between **Set** and **Grp**. In general we have no right to expect that F and G be mutually inverse, or even that their composition be naturally isomorphic to the identity functor. Looking at the definition of adjunctions, however, there seems to be a sense in which F and G "do opposite things". Indeed, an equivalent definition of being adjoint requires the existence of natural transformations $\eta\colon \mathrm{id}_{\mathcal{C}} \to GF$ and $\varepsilon\colon FG \to \mathrm{id}_{\mathcal{D}}$ such that

$$(\varepsilon F) \circ (F\eta) = \mathrm{id}_F$$
$$(G\varepsilon) \circ (\eta G) = \mathrm{id}_G$$

(See for example [2, Theorem IV.1.2].) Writing adjunction in this form, we see that being adjoint is a weakening of being an equivalence of categories; the compositions aren't naturally isomorphic to the identity functors, but there is some kind of cancellation taking place. A natural question to ask is "what *can* we say about the composition GF?" (One might also ask about FG; this leads to the study of *comonads*, which we will not consider.)

Our answer is Theorem 2.2; to get there, however, we need a technical lemma.

Lemma 2.1. *Suppose $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ have $F \leftrightarrows G$; let $\alpha\colon \mathrm{Hom}_{\mathcal{D}}(F-,-) \to \mathrm{Hom}_{\mathcal{C}}(-,G-)$ be the natural isomorphism yielding $F \leftrightarrows G$. Then for any $A, B \in \mathrm{Ob}(\mathcal{C})$ and any $\varphi\colon A \to B$, we have*

$$\alpha_{B,FB}(\mathrm{id}_{FB}) \circ \varphi = GF\varphi \circ \alpha_{A,FA}(\mathrm{id}_{FA})$$

Likewise, for any $M, N \in \mathrm{Ob}(\mathcal{D})$ and any $\psi\colon M \to N$, we have

$$\psi \circ \alpha^{-1}_{GM,M}(\mathrm{id}_{GM}) = \alpha^{-1}_{GN,N}(\mathrm{id}_{GN}) \circ FG\psi$$

Proof. Note that by naturality of α we get

$$\alpha_{B,FB}(\mathrm{id}_{FB}) \circ \varphi = \alpha_{A,FB}(\mathrm{id}_{FB} \circ F\varphi) = \alpha_{A,FB}(F\varphi) = \alpha_{A,FB}(F\varphi \circ \mathrm{id}_{FA}) = GF\varphi \circ \alpha_{A,FA}(\mathrm{id}_{FA})$$

Likewise, by naturality of α^{-1} we get

$$\psi \circ \alpha^{-1}_{GM,M}(\mathrm{id}_{GM}) = \alpha^{-1}_{GM,N}(G\psi \circ \mathrm{id}_{GM}) = \alpha^{-1}_{GM,N}(G\psi) = \alpha^{-1}_{GM,N}(\mathrm{id}_{GN} \circ G\psi) = \alpha^{-1}_{GN,N}(\mathrm{id}_{GN}) \circ FG\psi$$

as desired. □ Lemma 2.1

We are now ready to answer the question "what can we say about GF?"

Theorem 2.2. *Suppose $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ have $F \leftrightarrows G$; let $T = G \circ F\colon \mathcal{C} \to \mathcal{C}$. Then we have natural transformations $\eta\colon \mathrm{id}_{\mathcal{C}} \to T$ and $\mu\colon T^2 \to T$ such that the following diagrams commute:*

$$\begin{array}{ccc} T^3 & \xrightarrow{T\mu} & T^2 \\ \downarrow{\scriptstyle \mu T} & & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow{\mu} & T \end{array} \qquad \begin{array}{ccc} T & \xrightarrow{T\eta} & T^2 \\ \downarrow{\scriptstyle \eta T} & \searrow^{\mathrm{id}_T} & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow{\mu} & T \end{array}$$

(Here $T^2 = T \circ T$, which is sensible because T is an endofunctor.)

Proof. Let $\alpha\colon \mathrm{Hom}_{\mathcal{D}}(F-,-) \to \mathrm{Hom}_{\mathcal{C}}(-,G-)$ be the natural isomorphism yielding the adjunction $F \leftrightarrows G$. For $A \in \mathrm{Ob}(\mathcal{C})$ we note that $\alpha_{A,FA}\colon \mathrm{Hom}_{\mathcal{D}}(FA,FA) \to \mathrm{Hom}_{\mathcal{C}}(A,GFA)$; we may then set $\eta_A = \alpha_{A,FA}(\mathrm{id}_{FA})\colon A \to GFA = TA$.

Claim 2.3. *η as defined above is a natural transformation.*

Proof. Suppose $A, B \in \mathrm{Ob}(\mathcal{C})$; suppose $f\colon A \to B$. We wish to check that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle \eta_A} & & \downarrow{\scriptstyle \eta_B} \\ GFA & \xrightarrow{GFf} & GFB \end{array}$$

But by Lemma 2.1 we get that

$$\eta_B \circ f = \alpha_{B,FB}(\mathrm{id}_{FB}) \circ f = GFf \circ \alpha_{A,FA}(\mathrm{id}_{FA}) = GFf \circ \eta_A$$

as desired. □ Claim 2.3

Along similar lines, we note that for $A \in \mathrm{Ob}(\mathcal{C})$ we have $\alpha^{-1}_{GFA,FA}\colon \mathrm{Hom}_{\mathcal{C}}(GFA,GFA) \to \mathrm{Hom}_{\mathcal{D}}(FGFA,FA)$; we may then set $\mu_A = G(\alpha^{-1}_{GFA,FA}(\mathrm{id}_{GFA})) \in \mathrm{Hom}_{\mathcal{C}}(GFGFA,GFA) = \mathrm{Hom}_{\mathcal{C}}(T^2A,TA)$.

Claim 2.4. *μ as defined above is a natural transformation.*

Proof. Suppose $A, B \in \mathrm{Ob}(\mathcal{C})$; suppose $f \colon A \to B$. We wish to check that the following diagram commutes:

$$\begin{array}{ccc} GFGFA & \xrightarrow{GFGFf} & GFGFB \\ \downarrow{\scriptstyle \mu_A} & & \downarrow{\scriptstyle \mu_B} \\ GFA & \xrightarrow{GFf} & GFB \end{array}$$

But by Lemma 2.1 we get that

$$Ff \circ \alpha^{-1}_{GFA,FA}(\mathrm{id}_{GFA}) = \alpha^{-1}_{GFB,FB}(\mathrm{id}_{GFB}) \circ FGFf$$

Hence, applying G, we get that

$$\begin{aligned} GFf \circ \mu_A &= GFf \circ G(\alpha^{-1}_{GFA,FA}(\mathrm{id}_{GFA})) && \text{(by definition of } \mu) \\ &= G(Ff \circ \alpha^{-1}_{GFA,FA}(\mathrm{id}_{GFA})) && \text{(since } G \text{ is a functor)} \\ &= G(\alpha^{-1}_{GFB,FB}(\mathrm{id}_{GFB}) \circ FGFf) && \text{(by equation above)} \\ &= G\alpha^{-1}_{GFB,FB}(\mathrm{id}_{GFB}) \circ GFGFf && \text{(since } G \text{ is a functor)} \\ &= \mu_B \circ GFGFf && \text{(by definition of } \mu) \end{aligned}$$

as desired. ☐ Claim 2.4

It remains to check that the following diagrams commute:

$$\begin{array}{ccc} T^3 & \xrightarrow{T\mu} & T^2 \\ \downarrow{\scriptstyle \mu T} & & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow{\mu} & T \end{array} \qquad \begin{array}{ccc} T & \xrightarrow{T\eta} & T^2 \\ \downarrow{\scriptstyle \eta T} & \searrow{\scriptstyle \mathrm{id}_T} & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow{\mu} & T \end{array}$$

Suppose $A \in \mathrm{Ob}(\mathcal{C})$; we check that the following diagrams commute:

$$\begin{array}{ccc} T^3A & \xrightarrow{T(\mu_A)} & T^2A \\ \downarrow{\scriptstyle \mu_{TA}} & & \downarrow{\scriptstyle \mu_A} \\ T^2A & \xrightarrow{\mu_A} & TA \end{array} \qquad \begin{array}{ccc} TA & \xrightarrow{T(\eta_A)} & T^2A \\ \downarrow{\scriptstyle \eta_{TA}} & \searrow{\scriptstyle \mathrm{id}_{TA}} & \downarrow{\scriptstyle \mu_A} \\ T^2A & \xrightarrow{\mu_A} & TA \end{array}$$

For the first, we note that

$$\begin{aligned} \mu_A \circ \mu_{TA} &= G(\alpha^{-1}_{GFA,FA}(\mathrm{id}_{GFA})) \circ G(\alpha^{-1}_{GFGFA,FGFA}(\mathrm{id}_{GFGFA})) && \text{(by definition of } \mu) \\ &= G(\alpha^{-1}_{GFA,FA}(\mathrm{id}_{GFA}) \circ \alpha^{-1}_{GFGFA,FGFA}(\mathrm{id}_{GFGFA})) && \text{(since } G \text{ is a functor)} \\ &= G(\alpha^{-1}_{GFA,FA}(\mathrm{id}_{GFA}) \circ FG(\alpha^{-1}_{GFA,FA}(\mathrm{id}_{GFA}))) && \text{(by Lemma 2.1)} \\ &= G(\alpha^{-1}_{GFA,FA}(\mathrm{id}_{GFA})) \circ GFG(\alpha^{-1}_{GFA,FA}(\mathrm{id}_{GFA})) && \text{(since } G \text{ is a functor)} \\ &= \mu_A \circ T(\mu_A) && \text{(by definition of } \mu) \end{aligned}$$

For the second, we note that

$$\begin{aligned} \mu_A \circ T(\eta_A) &= G(\alpha^{-1}_{GFA,FA}(\mathrm{id}_{GFA})) \circ GF(\alpha_{A,FA}(\mathrm{id}_{FA})) && \text{(by definitions of } \mu \text{ and } \eta) \\ &= G(\alpha^{-1}_{GFA,FA}(\mathrm{id}_{GFA}) \circ F(\alpha_{A,FA}(\mathrm{id}_{FA}))) && \text{(since } G \text{ is a functor)} \\ &= G(\alpha^{-1}_{A,FA}(\mathrm{id}_{GFA} \circ \alpha_{A,FA}(\mathrm{id}_{FA}))) && \text{(by naturality of } \alpha^{-1}) \\ &= G(\alpha^{-1}_{A,FA}(\alpha_{A,FA}(\mathrm{id}_{FA}))) \\ &= G(\mathrm{id}_{FA}) \\ &= \mathrm{id}_{TA} \end{aligned}$$

and

$$\begin{aligned}
\mu_A \circ \eta_{TA} &= G(\alpha^{-1}_{GFA,FA}(\mathrm{id}_{GFA})) \circ \alpha_{GFA,FGFA}(\mathrm{id}_{FGFA}) && \text{(by definitions of } \mu \text{ and } \eta\text{)} \\
&= \alpha_{GFA,FA}(\alpha^{-1}_{GFA,FA}(\mathrm{id}_{GFA}) \circ \mathrm{id}_{FGFA}) && \text{(by naturality of } \alpha\text{)} \\
&= \alpha_{GFA,FA}(\alpha^{-1}_{GFA,FA}(\mathrm{id}_{GFA})) \\
&= \mathrm{id}_{TA}
\end{aligned}$$

So the desired diagrams commute. □ Theorem 2.2

This motivates the following definition:

Definition 2.5. A *monad* in a category $\mathcal{C}$ is a triple (T, η, μ) where

- $T : \mathcal{C} \to \mathcal{C}$ is an endofunctor
- $\eta : \mathrm{id}_{\mathcal{C}} \to T$ is a natural transformation, called the *unit*
- $\mu : T^2 \to T$ is a natural transformation, called the *multiplication*

such that the following diagrams commute:

$$\begin{array}{ccc} T^3 & \xrightarrow{T\mu} & T^2 \\ \downarrow{\scriptstyle \mu T} & & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow{\mu} & T \end{array} \qquad \begin{array}{ccc} T & \xrightarrow{T\eta} & T^2 \\ \downarrow{\scriptstyle \eta T} & \searrow^{\mathrm{id}_T} & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow{\mu} & T \end{array}$$

When η and μ are clear from context, we will use T to refer to the monad.

The terms "unit" and "multiplication" arise from the observation that the monad axioms resemble the monoid axioms. Indeed, a monoid can be defined as a triple (X, e, m) where $X \in \mathrm{Ob}(\mathbf{Set})$, $e \in \mathrm{Hom}_{\mathbf{Set}}(1, X)$ (where $1 = \{ 0 \}$ is the identity of $\times$), and $m \in \mathrm{Hom}_{\mathbf{Set}}(X^2, X)$ such that the following diagrams commute:

$$\begin{array}{ccc} X^3 & \xrightarrow{\mathrm{id}_X \times m} & X^2 \\ \downarrow{\scriptstyle m \times \mathrm{id}_X} & & \downarrow{\scriptstyle m} \\ X^2 & \xrightarrow{m} & X \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\mathrm{id}_X \times e} & X^2 \\ {\scriptstyle e \times \mathrm{id}_X}\downarrow & \searrow^{\mathrm{id}_X} & \downarrow{\scriptstyle m} \\ X^2 & \xrightarrow{m} & X \end{array}$$

(where in the second diagram we regard X as $X \times \{ 0 \}$ or $\{ 0 \} \times X$ as necessary). If we allow an arbitrary category $\mathcal{C}$ with well-behaved products to replace **Set** in the above, we get the notion of a *monoid-like object*. In particular, if we use the category of endofunctors $\mathcal{C} \to \mathcal{C}$ and squint enough that we overlook the difference between X^2 as a product object and T^2 the composition of two endofunctors, we notice that we have recovered exactly the definition of a monad; hence the facetious explanation "a monad is just a monoid in the category of endofunctors" given by Haskell enthusiasts when they feel like being unhelpful. (To do away with the squinting entirely, one needs the notion of a *monoidal category*.)

Theorem 2.2 then says that any pair of adjoint functors gives rise to a monad in a canonical way; so monads are generalizations of compositions of adjoint functors. In fact, it turns out that all monads arise in this way.

Definition 2.6. Suppose (T, η, μ) is a monad in $\mathcal{C}$. We define $\mathcal{C}_T$, the *Kleisli category of* T, as follows:

- For each $A \in \mathrm{Ob}(\mathcal{C})$ we define a new object $A_T \in \mathrm{Ob}(\mathcal{C}_T)$.
- For each $f \in \mathrm{Hom}_{\mathcal{C}}(A, TB)$ we define a new morphism $f_T \in \mathrm{Hom}_{\mathcal{C}_T}(A_T, B_T)$.
- Given $f \in \mathrm{Hom}_{\mathcal{C}}(A, TB)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(B, TC)$, we define $g_T \circ f_T = (\mu_C \circ Tg \circ f)_T \in \mathrm{Hom}_{\mathcal{C}_T}(A, C)$.

One can without too much trouble construct adjoint functors $F\colon \mathcal{C} \to \mathcal{C}_T$ and $G\colon \mathcal{C}_T \to \mathcal{C}$ such that T arises from the adjunction $F \leftrightarrows G$; in particular, our F will act on objects by $FA = A_T$ and our G will act on objects by $GA_T = TA$. (The details can be found in [2, Theorem VI.5.1].) Hence we indeed get that every monad arises from an adjunction.

Veteran Haskell coders will recognize $A \to TB$ as a relabelling of the type signature `a -> m b` that haunts their dreams; indeed, the Kleisli category applies very well to computer science. From an algebraic perspective, however, it is less ideal: for most adjunctions that one encounters in the wild, the Kleisli category of the associated monad is a proper subcategory of the original one. For example, in the free-forgetful adjunction between **Set** and **Grp**, the Kleisli category of the associated monad turns out to be the full subcategory of free groups, rather than **Grp**; see [2, Exercise VI.5.2].

A better way to recover an adjunction from a monad turns out to be the following:

Definition 2.7. Suppose (T, η, μ) is a monad in $\mathcal{C}$. An *Eilenberg-Moore algebra over* T is a pair (A, h) where $A \in \mathrm{Ob}(\mathcal{C})$ and $h\colon TA \to A$ such that the following diagrams commute:

$$\begin{array}{ccc} T^2A & \xrightarrow{Th} & TA \\ \downarrow{\scriptstyle \mu_A} & & \downarrow{\scriptstyle h} \\ TA & \xrightarrow{h} & A \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\eta_A} & TA \\ & {\scriptstyle \mathrm{id}_A}\searrow & \downarrow{\scriptstyle h} \\ & & A \end{array}$$

A morphism of Eilenberg-Moore algebras $(A, h) \to (A', h')$ is some $f\colon A \to A'$ such that the following diagram commutes:

$$\begin{array}{ccc} TA & \xrightarrow{Tf} & TA' \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h'} \\ A & \xrightarrow{f} & A' \end{array}$$

This defines a category $\mathcal{C}^T$, the *Eilenberg-Moore category of* T.

Here too we can find adjoint functors $F\colon \mathcal{C} \to \mathcal{C}^T$ and $G\colon \mathcal{C}^T \to \mathcal{C}$ such that T arises from the adjunction $F \leftrightarrows G$; in particular, our F will act on objects by $FA = (TA, \mu_A)$, and our G will act on objects by $G(A, h) = A$. One remarks that this adunction bears some resemblance to a free-forgetful adjunction. (The details of this construction can be found in [2, Theorem VI.2.1].)

At first glance, the definition of an Eilenberg-Moore algebra is quite opaque; it's not at all clear how to justify my claim that the Eilenberg-Moore category is "probably" the category the original adjunction came from. Why should an Eilenberg-Moore algebra over the free group monad have any correspondence to a group? Phrasing my assertion in a suitable level of generality and proving it will be the content of the next two sections.

3 A whirlwind tour of universal algebra

To give formal meaning to my assertion above, we will need the language of universal algebra; this section is devoted to covering the necessary definitions and theorems. Throughout this section, we use monoids as a concrete example to which we can apply the concepts and theorems; another good example the reader may wish to keep in mind would be rings. As noted in the introduction, [1, Section II.5] is an excellent reference for elementary universal algebra.

Definition 3.1. A *signature* or *type* is a collection $\mathscr{F}$ of symbols, each with an associated arity (which is allowed to be 0). An *algebra* $\mathbf{A}$ of type $\mathscr{F}$ is a non-empty set A together with a function $f^{\mathbf{A}}\colon A^n \to A$ for each n-ary $f \in \mathscr{F}$; we then write $\mathbf{A} = (A, (f^{\mathbf{A}} : f \in \mathscr{F}))$. We call A the *underlying set* of $\mathbf{A}$; we call the $f^{\mathbf{A}}$ the *fundamental operations* of $\mathbf{A}$.

We will use boldface to denote algebras and functions that output an algebra; we will use roman to denote sets and functions that output a set. When a binary relation symbol is conveniently expressed as an infix operator, we will do so.

Example 3.2. Let $\mathscr{M} = \{ \cdot, 1 \}$ where $\cdot$ is a binary symbol and 1 is a nullary symbol; we call this the *signature of monoids*. An algebra of type $\mathscr{M}$ then consists of a set together with a binary operation and an identified constant; for example:

- $(\mathbb{N}, +^{\mathbb{N}}, 0^{\mathbb{N}})$ (where $+^{\mathbb{N}}$ and $0^{\mathbb{N}}$ are the normal addition and 0 of $\mathbb{N}$).
- $(\mathbb{N}, -^{\mathbb{N}}, 0^{\mathbb{N}})$ (where $-^{\mathbb{N}}$ is binary subtraction in $\mathbb{N}$); note that there is no requirement that an algebra of type $\mathscr{M}$ actually be a monoid.

Definition 3.3. Suppose $\mathbf{A}$ and $\mathbf{B}$ are algebras of type $\mathscr{F}$. A *homomorphism* $\mathbf{A} \to \mathbf{B}$ is a map $\varphi\colon A \to B$ such that given any $f \in \mathscr{F}$ of arity n and any $a_1, \ldots, a_n \in A$ we have

$$\varphi(f^{\mathbf{A}}(a_1, \ldots, a_n)) = f^{\mathbf{B}}(\varphi(a_1), \ldots, \varphi(a_n))$$

One checks that the algebras of type $\mathscr{F}$ together with the homomorphisms form a category.

We now give an appropriate generalization of subobjects and Cartesian products.

Definition 3.4. Suppose $\mathbf{A}$ is an algebra of type $\mathscr{F}$. A *subalgebra* of $\mathbf{A}$ is an algebra $\mathbf{B}$ of type $\mathscr{F}$ such that $B \subseteq A$ and for all n-ary $f \in \mathscr{F}$ we have $f^{\mathbf{B}} = f^{\mathbf{A}} \restriction B^n$.

Definition 3.5. Suppose $(\mathbf{A}_i : i \in I)$ are algebras of type $\mathscr{F}$. We define their *direct product* $\prod_{i\in I} \mathbf{A}_i$ to have domain set $\prod_{i \in I} A_i$ and given n-ary $f \in \mathscr{F}$ we define

$$f^{\prod_{i\in I} \mathbf{A}_i}((a_{1i} : i \in I), \ldots, (a_{ni} : i \in I)) = (f^{\mathbf{A}_i}(a_{1i}, \ldots, a_{n-i}) : i \in I)$$

This gives us a notion of a "nice" class of algebras:

Definition 3.6. A *variety* is a non-empty class of algebras of a fixed type that is closed under direct products, subalgebras, and homomorphic images.

Example 3.7. Let $\mathscr{M} = \{ \cdot, 1 \}$ be the signature of monoids; let V be the class of algebras of type $\mathscr{M}$ that are actually monoids (i.e. the $\mathbf{A}$ of type $\mathscr{M}$ such that $\cdot$ is associative and 1 is a multiplicative identity.) One can easily verify that V is closed under direct products, subalgebras, and homomorphic images, and is thus a variety. On the other hand, let V' be the class of algebras of type $\mathscr{M}$ whose elements are groups; then V' is not a variety, as it is not closed under subalgebras. (For example, we have that $(\mathbb{N}, +^{\mathbb{N}}, 0^{\mathbb{N}}) \notin V'$ is a subalgebra of $(\mathbb{Z}, +^{\mathbb{Z}}, 0^{\mathbb{Z}}) \in V'$.)

While not directly helpful in proving our main result, the following characterization of varieties is quite nice:

Fact 3.8 ([1, Theorem II.11.9])**.** *Suppose V is a class of algebras of type $\mathscr{F}$. Then V is a variety if and only if there some set Σ of "identities" such that the elements of V are exactly the algebras of type $\mathscr{F}$.*

To avoid bogging the reader down with definitions that are not relevant to our main result, I forgo giving a proper definition of "identities" in favour of an example:

Example 3.9. Let $\mathscr{M}$, V, and V' be as in the previous example. Another way to see that V is a variety is to note that V is exactly the class of algebras of type $\mathscr{M}$ satisfying the following identities:

$$(x \cdot y) \cdot z \approx x \cdot (y \cdot z)$$
$$x \cdot 1 \approx x$$
$$1 \cdot x \approx x$$

One wonder whether adding the identities $x \cdot x^{-1} \approx 1$ and $x^{-1} \cdot x \approx 1$ would yield a proof that V' is a variety; it does not, because ${}^{-1}$ is not a symbol in our signature. However, if we expand our signature to $\mathscr{G} = \{ \cdot, 1, {}^{-1} \}$, then the class of algebras of type $\mathscr{G}$ that form a group does form a variety.

Definition 3.10. Given a class K of algebras of type $\mathscr{F}$, the *category associated to* K is the full subcategory of the category of algebras of type $\mathscr{F}$ whose objects are the elements of K.

Before we consider the free monad, we need to know that free objects exist. Remarkably, it turns out that as long as V is a variety, we can guarantee the existence of free objects. For this we will need the notion of a *term*:

Definition 3.11. Suppose $\mathscr{F}$ is a signature containing a 0-ary symbol; suppose X is a set (which we will think of as a set of variables). We define the set of *terms* to be the smallest set $T_{\mathscr{F}}(X)$ satisfying:

- $X \subseteq T_{\mathscr{F}}(X)$.
- Given n-ary $f \in \mathscr{F}$ and $t_1, \ldots, t_n \in T_{\mathscr{F}}(X)$, we have that the tuple $(f, t_1, \ldots, t_n) \in T_{\mathscr{F}}(X)$; roughly speaking, we think of this as saying that the "string" $f(t_1, \ldots, t_n)$ is in $T_{\mathscr{F}}(X)$.

We make this into an algebra $\mathbf{T}(X)$ of type $\mathscr{F}$ as follows: given n-ary $f \in \mathscr{F}$ and $t_1, \ldots, t_n \in T_{\mathscr{F}}(X)$, we let $f^{\mathbf{T}(X)}(t_1, \ldots, t_n) = (f, t_1, \ldots, t_n)$. Given $t \in T_{\mathscr{F}}(X)$ and $x_1, \ldots, x_n \in X$, we write $t(x_1, \ldots, x_n)$ to mean that the variables in t are from the $x_1, \ldots, x_n$.

The requirement that $\mathscr{F}$ contain a 0-ary symbol allows us to consider $T_{\mathscr{F}}(\emptyset)$; one could dispense with it at the cost of requiring that X be non-empty. When we are being informal, we will write $f(t_1, \ldots, t_n)$ instead of $(f, t_1, \ldots, t_n)$; we will also write binary operators that are conveniently expressed as infix operators in infix notation.

Example 3.12. Let $\mathscr{M} = \{\, \cdot, 1 \,\}$ as before. Then x, $1 \cdot 1$, and $x \cdot (y \cdot z)$ are elements of $T_{\mathscr{M}}(\{\, x, y, z \,\})$; they would formally be written x, $(\cdot, 1, 1)$, and $(\cdot, x, (\cdot, y, z))$, respectively. If $t = x \cdot y \in T_{\mathscr{M}}(\{\, x, y, z \,\})$, we might refer to t by $t(x, y)$ to assert that x and y are the only variables appearing in t. (It would also be correct to refer to t by $t(x, y, z)$; we make no requirement that *all* of the variables show up in t.)

We think of terms as functions in the following way:

Definition 3.13. Suppose A is an algebra of type $\mathscr{F}$. We recursively define $t^{\mathbf{A}} \colon A^n \to A$ for $t(x_1, \ldots, x_n) \in T_{\mathscr{F}}(X)$:

- $x_i^{\mathbf{A}} \colon A^n \to A$ is projection onto the i^{th} coordinate.
- Suppose $t = (f, t_1, \ldots, t_k)$ for some k-ary $f \in \mathscr{F}$ and $t_1, \ldots, t_k \in T_{\mathscr{F}}(X)$. Given $a_1, \ldots, a_n \in A$, we set
$$t^{\mathbf{A}}(a_1, \ldots, a_n) = f^{\mathbf{A}}(t_1^{\mathbf{A}}(a_1, \ldots, a_n), \ldots, t_k^{\mathbf{A}}(a_1, \ldots, a_n))$$

Note that $t^{\mathbf{A}}$ depends on the presentation $t = t(x_1, \ldots, x_n)$, which is not unique; whenever we use $t^{\mathbf{A}}$, we will be careful to specify a presentation in advance.

Example 3.14. Let $\mathscr{M} = \{\, \cdot, 1 \,\}$ as before; let $t = t(x, y) = x \cdot (y \cdot y) \in T_{\mathscr{M}}(\{\, x, y \,\})$. Let $\mathbf{A} = (\mathbb{N}, +^{\mathbb{N}}, 0^{\mathbb{N}})$. Then $t^{\mathbf{A}}$ is the map $\mathbb{N}^2 \to \mathbb{N}$ given by $(m, n) \mapsto m + 2n$.

Remark 3.15. For $t(x_1, \ldots, x_n) \in T(X)$ we have $t^{\mathbf{T}(X)}(x_1, \ldots, x_n) = t$.

When working in a variety V, we may have non-trivial relations between the terms. For example, let V be the variety of monoids; let $t_1(x) = x$ and $t_2(x) = x \cdot 1$. Then for any $\mathbf{A} \in V$ we have $t_1^{\mathbf{A}} = t_2^{\mathbf{A}}$. This motivates the following definition:

Definition 3.16. Suppose $\mathscr{F}$ is a signature containing a 0-ary symbol and K is a class of algebras of type $\mathscr{F}$; suppose X is a set. We define an equivalence relation on $T_{\mathscr{F}}(X)$ by $t_1(x_1, \ldots, x_n) \sim_K t_2(x_1, \ldots, x_n)$ if and only if for all $\mathbf{A} \in K$ we have $t_1^{\mathbf{A}} = t_2^{\mathbf{A}}$. (Note that while this superficially only applies when t_1 and t_2 have the same free variables, we can always add variables to the presentations of t_1 and t_2 until they have the same presentation; one also checks that $\sim_K$ is independent of the common presentation chosen.)

We then define $F_K(X) = T_{\mathscr{F}}(X)/{\sim_K}$. To avoid clutter, we use $\overline{t}$ to denote the equivalence class of t in $F_K(X)$. We make this into an algebra $\mathbf{F}_K(X)$ of type $\mathscr{F}$ as follows: given n-ary $f \in \mathscr{F}$ and $\overline{t_1}, \ldots, \overline{t_n} \in F_K(X)$, we define $f^{\mathbf{F}_K(X)}(\overline{t_1}, \ldots, \overline{t_n}) = \overline{f^{\mathbf{T}(X)}(t_1, \ldots, t_n)}$. (One checks that this is well-defined.)

Fact 3.17 ([1, Theorem II.10.10])**.** *Given any $\mathbf{A} \in K$ and any set map $\varphi \colon X \to A$ there is a unique homomorphism $\widehat{\varphi} \colon \mathbf{F}_K(X) \to \mathbf{A}$ such that the following diagram commutes:*

$$\begin{array}{ccc} X & \longrightarrow & F_K(X) \\ & \searrow^{\varphi} & \downarrow \widehat{\varphi} \\ & & A \end{array}$$

Fact 3.18 ([1, Theorem II.10.12])**.** *If V is a variety, then $\mathbf{F}_V(X) \in V$.*

So these $\mathbf{F}_V(X)$ play the role of free objects. In fact, we can make $\mathbf{F}_V$ into a functor from **Set** to the category associated to V as follows: given a set map $\varphi\colon X \to Y$, we let $\pi\colon Y \to F_V(Y)$ be the map $y \mapsto \overline{y}$; we then set $\mathbf{F}_V(\varphi) = \widehat{\pi \circ \varphi}\colon \mathbf{F}_V(X) \to \mathbf{F}_V(Y)$ (the unique homomorphism $\mathbf{F}_V(X) \to \mathbf{F}_V(Y)$ extending $\pi \circ \varphi$).

4 Eilenberg-Moore algebras over F_V

Throughout this section, we work in a fixed variety V over a signature $\mathscr{F}$ containing a 0-ary symbol. (The latter is a technical requirement to avoid having to deal with $\emptyset \in \mathrm{Ob}(\mathbf{Set})$ as a special case.) We let $\mathcal{V}$ be the category associated to V.

From the previous section, we get that:

Proposition 4.1. *$\mathbf{F}_V$ is left adjoint to the forgetful functor.*

In keeping with the conventions of the previous section, we use F_V to denote the associated monad (which is just the composition of the forgetful functor and $\mathbf{F}_V$).

The following proposition results from working through definitions and the details of the construction in Theorem 2.2.

Proposition 4.2.

1. *Suppose $\varphi\colon X \to Y$ and $t(x_1, \dots, x_n) \in T_{\mathscr{F}}(X)$. Then $F_V(\varphi)(\overline{t}) = t^{\mathbf{F}_V(Y)}(\overline{\varphi(x_1)}, \dots, \overline{\varphi(x_n)})$.*
2. *The unit η of F_V is given by $\eta_X\colon X \to F_V(X)$ is $x \mapsto \overline{x}$.*
3. *The multiplication μ of F_V is given as follows. Suppose $t(\overline{t_1}, \dots, \overline{t_n}) \in T(F_V(X))$. (Recall that the "variables" in $T(F_V(X))$ are elements of $F_V(X)$, and thus take the form $\overline{t_i}$ for some $t_i \in T(X)$.) Then $\mu_X(\overline{t}) = t^{\mathbf{F}_V(X)}(\overline{t_1}, \dots, \overline{t_n})$.*

Theorem 4.3. *$\mathcal{V}$ and $\mathbf{Set}^{F_V}$ are isomorphic categories.*

Proof. We first define a functor $\Phi\colon \mathcal{V} \to \mathbf{Set}^{F_V}$. Given $\mathbf{A} \in V$, the universal property of free objects yields that $\mathrm{id}_A\colon A \to A$ extends to a unique homomorphism $h_{\mathbf{A}}\colon \mathbf{F}_V(A) \to \mathbf{A}$; i.e. such that the following diagram commutes:

$$\begin{array}{ccc} A & \longrightarrow & F_V(A) \\ & \searrow^{\mathrm{id}_A} & \downarrow h_{\mathbf{A}} \\ & & A \end{array}$$

Claim 4.4. *$(A, h_{\mathbf{A}})$ is an Eilenberg-Moore algebra over F_V.*

Proof. We must check that the following diagrams commute:

$$\begin{array}{ccc} F_V(F_V(A)) & \xrightarrow{F_V(h_{\mathbf{A}})} & F_V(A) \\ \downarrow \mu_A & & \downarrow h_{\mathbf{A}} \\ F_V(A) & \xrightarrow{h_{\mathbf{A}}} & A \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\eta_A} & F_V(A) \\ & \searrow^{\mathrm{id}_A} & \downarrow h_{\mathbf{A}} \\ & & A \end{array}$$

For the first, suppose we are given $\overline{t} \in F_V(F_V(A))$ where $t(\overline{t_1}, \dots, \overline{t_n}) \in T_{\mathscr{F}}(F_V(A))$. Then

$$\begin{aligned} h_{\mathbf{A}}(F_V(h_{\mathbf{A}})(\overline{t})) &= h_{\mathbf{A}}(t^{\mathbf{F}_V(A)}(\overline{h_{\mathbf{A}}(\overline{t_1})}, \dots, \overline{h_{\mathbf{A}}(\overline{t_n})})) && \text{(by Proposition 4.2)} \\ &= t^{\mathbf{A}}(h_{\mathbf{A}}(\overline{h_{\mathbf{A}}(\overline{t_1})}), \dots, h_{\mathbf{A}}(\overline{h_{\mathbf{A}}(\overline{t_n})})) && \text{(since } h_{\mathbf{A}} \text{ is a homomorphism)} \\ &= t^{\mathbf{A}}(h_{\mathbf{A}}(\overline{t_1}), \dots, h_{\mathbf{A}}(\overline{t_n})) && \text{(by definition of } h_{\mathbf{A}}) \\ &= h_{\mathbf{A}}(t^{\mathbf{F}_V(A)}(\overline{t_1}, \dots, \overline{t_n})) && \text{(since } h_{\mathbf{A}} \text{ is a homomorphism)} \\ &= h_{\mathbf{A}}(\mu_A(\overline{t})) && \text{(by Proposition 4.2)} \end{aligned}$$

For the second, note that Proposition 4.2 and the definition of $h_{\mathbf{A}}$ yield that for $a \in A$ we have $h_{\mathbf{A}}(\eta_A(a)) = h_{\mathbf{A}}(\overline{a}) = a$. ☐ Claim 4.4

We may then set $\Phi(\mathbf{A}) = (A, h_{\mathbf{A}})$.

Claim 4.5. *Given $\mathbf{A}, \mathbf{B} \in V$ and a homomorphism $\varphi\colon \mathbf{A} \to \mathbf{B}$, we have that $\varphi\colon \Phi(\mathbf{A}) \to \Phi(\mathbf{B})$ is a morphism of Eilenberg-Moore algebras over F_V.*

Proof. We must check that the following diagram commutes:

$$\begin{array}{ccc} F_V(A) & \xrightarrow{F_V(\varphi)} & F_V(B) \\ \downarrow{\scriptstyle h_{\mathbf{A}}} & & \downarrow{\scriptstyle h_{\mathbf{B}}} \\ A & \xrightarrow{\quad\varphi\quad} & B \end{array}$$

But if $\bar{t} \in F_V(A)$ where $t(a_1, \dots, a_n) \in T_{\mathscr{F}}(A)$, then

$$\begin{aligned} h_{\mathbf{B}}(F_V(\varphi)(\bar{t})) &= h_{\mathbf{B}}(t^{\mathbf{F}_V(B)}(\overline{\varphi(a_1)}, \dots, \overline{\varphi(a_n)})) && \text{(by Proposition 4.2)} \\ &= t^{\mathbf{B}}(h_{\mathbf{B}}(\overline{\varphi(a_1)}), \dots, h_{\mathbf{B}}(\overline{\varphi(a_n)})) && \text{(since } h_{\mathbf{B}} \text{ is a homomorphism)} \\ &= t^{\mathbf{B}}(\varphi(a_1), \dots, \varphi(a_n)) && \text{(by definition of } h_{\mathbf{B}}) \\ &= t^{\mathbf{B}}(\varphi(h_{\mathbf{A}}(\overline{a_1})), \dots, \varphi(h_{\mathbf{A}}(\overline{a_n}))) && \text{(by definition of } h_{\mathbf{A}}) \\ &= \varphi(t^{\mathbf{A}}(h_{\mathbf{A}}(\overline{a_1}), \dots, h_{\mathbf{A}}(\overline{a_n}))) && \text{(since } \varphi \text{ is a homomorphism)} \\ &= \varphi(h_{\mathbf{A}}(t^{\mathbf{F}_V(A)}(\overline{a_1}, \dots, \overline{a_n}))) && \text{(since } h_{\mathbf{A}} \text{ is a homomorphism)} \\ &= \varphi(h_{\mathbf{A}}(\bar{t})) && \text{(by Proposition 4.2)} \end{aligned}$$

as desired. ☐ Claim 4.5

Given $\mathbf{A}, \mathbf{B} \in V$ and a homomorphism $\varphi\colon \mathbf{A} \to \mathbf{B}$, we may then set $\Phi(\varphi) = \varphi\colon \Phi(\mathbf{A}) \to \Phi(\mathbf{B})$; it is then immediate that Φ preserves composition and identity morphisms, and is thus a functor.

We now define a functor $\Psi\colon \mathbf{Set}^{F_V} \to \mathcal{V}$. Suppose $(A, h) \in \mathrm{Ob}(\mathbf{Set}^{F_V})$; then $h\colon F_V(A) \to A$. Note that given n-ary $f \in \mathscr{F}$ and given $a_1, \dots, a_n \in A$, we have that $(f, a_1, \dots, a_n) \in T_{\mathscr{F}}(A)$; thus $\overline{(f, a_1, \dots, a_n)} \in F_V(A)$, and $h(\overline{(f, a_1, \dots, a_n)}) \in A$. We can thus define an algebra $\mathbf{A}_h$ of type $\mathscr{F}$ to have underlying set A and fundamental operations $f^{\mathbf{A}_h}(a_1, \dots, a_n) = h(\overline{(f, a_1, \dots, a_n)})$.

Claim 4.6. *h is a homomorphism $\mathbf{F}_V(A) \to \mathbf{A}_h$.*

Proof. Suppose we have n-ary $f \in \mathscr{F}$; suppose $\overline{t_1}, \dots, \overline{t_n} \in F_V(A)$. Let $t = (f, \overline{t_1}, \dots, \overline{t_n}) \in T_{\mathscr{F}}(F_V(A))$. Since (A, h) form an Eilenberg-Moore algebra over F_V, we have that the following diagram commutes:

$$\begin{array}{ccc} F_V(F_V(A)) & \xrightarrow{\mu_A} & F_V(A) \\ \downarrow{\scriptstyle F_V(h)} & & \downarrow{\scriptstyle h} \\ F_V(A) & \xrightarrow{\quad h\quad} & A \end{array}$$

Thus

$$\begin{aligned} h(f^{\mathbf{F}_V(A)}(\overline{t_1}, \dots, \overline{t_n})) &= h(\mu_A(\overline{(f, \overline{t_1}, \dots, \overline{t_n})})) && \text{(by Proposition 4.2)} \\ &= h(\mu_A(\bar{t})) && \text{(by definition of } t) \\ &= h(F_V(h)(\bar{t})) && \text{(by the above commuting diagram)} \\ &= h(t^{\mathbf{F}_V(A)}(\overline{h(\overline{t_1})}, \dots, \overline{h(\overline{t_n})}) && \text{(by Proposition 4.2)} \\ &= h(\overline{(f, h(\overline{t_1}), \dots, h(\overline{t_n}))}) && \text{(by definition of } t) \\ &= f^{\mathbf{A}_h}(h(\overline{t_1}), \dots, h(\overline{t_n})) && \text{(by definition of } f^{\mathbf{A}_h}) \end{aligned}$$

So h is a homomorphism. ☐ Claim 4.6

Since (A, h) is an Eilenberg-Moore algebra over F_V, we get that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & F_V(A) \\ & \underset{\mathrm{id}_A}{\searrow} & \downarrow h \\ & & A \end{array}$$

In particular, since id_A is surjective, we get that h is as well. So $\mathbf{A}_h$ is a homomorphic image of $\mathbf{F}_V(A) \in V$; so $\mathbf{A}_h \in V$, since V is a variety. We may thus set $\Psi((A, h)) = \mathbf{A}_h$.

Claim 4.7. *Given $(A, h), (A', h') \in \mathbf{Set}^{F_V}$ and a morphism $\psi\colon (A, h) \to (A', h')$, we have that $\psi\colon \Psi((A, h)) \to \Psi((A', h'))$ is a homomorphism.*

Proof. Suppose we have n-ary $f \in \mathscr{F}$; suppose $a_1, \ldots, a_n \in A$. Since ψ is a morphism of Eilenberg-Moore algebras over F_V, we have that the following diagram commutes:

$$\begin{array}{ccc} F_V(A) & \xrightarrow{F_V(\psi)} & F_V(A') \\ \downarrow h & & \downarrow h' \\ A & \xrightarrow{\quad\psi\quad} & A' \end{array}$$

Thus

$$\begin{aligned} \psi(f^{\mathbf{A}_h}(a_1, \ldots, a_n)) &= \psi(h(\overline{(f, a_1, \ldots, a_n)})) && \text{(by definition of } f^{\mathbf{A}_h}) \\ &= h'(F_V(\psi)(\overline{(f, a_1, \ldots, a_n)})) && \text{(by the above commuting diagram)} \\ &= h'(\overline{(f, \psi(a_1), \ldots, \psi(a_n))}) && \text{(by Proposition 4.2)} \\ &= f^{\mathbf{A}_{h'}}(\psi(a_1), \ldots, \psi(a_n)) && \text{(by definition of } f^{\mathbf{A}_{h'}}) \end{aligned}$$

and $\psi\colon \Psi((A, h)) \to \Psi((A', h'))$ is a homomorphism. □ Claim 4.7

Given $(A, h), (A', h') \in \mathbf{Set}^{F_V}$ and a morphism $\psi\colon (A, h) \to (A', h')$, we may then set $\Psi(\psi) = \psi\colon \Psi((A, h)) \to \Psi((A', h'))$; it is again immediate that Ψ preserves compositions and identity morphisms, and is thus a functor.

Claim 4.8. *Φ and Ψ are mutually inverse.*

Proof. We first note that both Φ and Ψ preserve the underlying set: the underlying set of $\Phi(\mathbf{A})$ is A, and the underlying set of $\Psi((A, h))$ is A. We further observe that both Φ and Ψ preserve the underlying functions of morphisms; since morphisms in $\mathbf{Set}^{F_V}$ and $\mathcal{V}$ are functions satisfying additional properties, it follows that to show that Φ and Ψ are mutually inverse it suffices to check that they are mutually inverse on objects.

We now check that $\Phi \circ \Psi = \mathrm{id}_{\mathbf{Set}^{F_V}}$. Suppose $(A, h) \in \mathrm{Ob}(\mathbf{Set}^{F_V})$. Since Φ and Ψ preserve underlying sets, we have that $\Phi(\Psi((A, h))) = (A, h')$ for some map $h'\colon F_V(A) \to A$; it remains to check that $h = h'$. Recall by definition of Φ that h' is the unique homomorphism $\mathbf{F}_V(A) \to \Psi((A, h)) = \mathbf{A}_h$ extending the identity map $A \to A$. But Claim 4.6 tells us that $h\colon \mathbf{F}_V(A) \to \mathbf{A}_h$ is a homomorphism. Furthermore, since (A, h) is an Eilenberg-Moore algebra over F_V we get that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & F_V(A) \\ & \underset{\mathrm{id}_A}{\searrow} & \downarrow h \\ & & A \end{array}$$

Since Proposition 4.2 tells us that η_A is just canonical map $A \to F_V(A)$, we get that h extends id_A. So h is a homomorphism $\mathbf{F}_V(A) \to \mathbf{A}_h$ extending the identity map $A \to A$. But h' was the unique such homomorphism $\mathbf{F}_V(A) \to \mathbf{A}_h$; so $h = h'$. So $\Phi(\Psi((A, h))) = (A, h)$; so $\Phi \circ \Psi = \mathrm{id}_{\mathbf{Set}^{F_V}}$.

We now check that $\Psi \circ \Phi = \mathrm{id}_{\mathcal{V}}$. Suppose $\mathbf{A} \in V$. Since Φ and Ψ preserve underlying sets, we have that $\Psi(\Phi(\mathbf{A})) = \mathbf{A}'$ where $\mathbf{A}$ and $\mathbf{A}'$ both have A as their underlying set; it remains to check that they have the same fundamental operations. Suppose then that we have an n-ary $f \in \mathscr{F}$; suppose $a_1, \dots, a_n \in A$. Then by definitions of Φ and Ψ we get that

$$\begin{aligned} f^{\mathbf{A}'}(a_1, \dots, a_n) &= h_{\mathbf{A}}(\overline{(f, a_1, \dots, a_n)}) && \text{(by definition of } f^{\mathbf{A}'}\text{; i.e. by definition of } \Psi) \\ &= h_{\mathbf{A}}(f^{\mathbf{F}_V(A)}(\overline{a_1}, \dots, \overline{a_n})) && \text{(by definition of } f^{\mathbf{F}_V(A)}) \\ &= f^{\mathbf{A}}(h_{\mathbf{A}}(\overline{a_1}), \dots, h_{\mathbf{A}}(\overline{a_n})) && \text{(since } h_{\mathbf{A}} \text{ is a homomorphism)} \\ &= f^{\mathbf{A}}(a_1, \dots, a_n) && \text{(since } h_{\mathbf{A}} \text{ extends } \mathrm{id}_A) \end{aligned}$$

So $f^{\mathbf{A}'} = f^{\mathbf{A}}$; so $\Psi(\Phi(\mathbf{A})) = \mathbf{A}$, and $\Psi \circ \Phi = \mathrm{id}_{\mathcal{V}}$. $\square$ Claim 4.8

So Φ is an isomorphism of categories; so $\mathcal{V}$ and $\mathbf{Set}^{F_V}$ are isomorphic categories. $\square$ Theorem 4.3

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