Course notes for PMATH 863

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Lectures by Stephen New

Contents

1 Introduction

Lectures by Stephen New, office MC 5419, office hours 2:30-3:20 MWF (also after about 5:30 MWF if you tell him ahead of time).

Course outline found on his website.

Collaboration encouraged but acknowledge help (aside from him and books). (Write your own assignment though.) Assignments will be challenging, exam easier. (Foreknowledge of topics will be given for the exam.)

Warm thanks to Andrej Vukovic for the notes for the lecture I missed.

A somewhat vague introduction (formality later):

Definition 1.1. A *Lie grape* is both a C^∞ manifold and a grape G with smooth grape operations. (i.e. multiplication $m\colon G \times G \to G$ and inversion $v\colon G \to G$ are smooth).

Example 1.2.

- $\mathbb{R}^n$ under $+$

- $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ under component-wise multiplication
- $M_n(\mathbb{R})$ under $+$
- $\mathrm{GL}_n(\mathbb{R})$ under matrix multiplication

Definition 1.3. A *Lie algebra* is a vector space $\mathfrak{g}$ with an operation $[\cdot,\cdot]\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ which is alternating, bilinear, and satisfies the Jacobi identity. i.e. for $X, Y, Z \in \mathfrak{g}$ we have

- $[X, Y] = -[Y, X]$ (or equivalently, in the presence of bilinearity, $[X, X] = 0$).
- $[X, Y + Z] = [X, Y] + [X, Z]$ and $[X, cY] = c[X, Y]$, etc.
- $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$.

Example 1.4.

- $M_n(\mathbb{R})$ with $[X, Y] = XY - YX$.
- The set of smooth vector fields on a manifold M with $[X, Y] = XY - YX$ as differential operators.
- When G is a Lie grape the set of left-invariant vector fields on G is a Lie algebra, which we identify with $\mathfrak{g} = T_eG$; we call this the *Lie algebra of* G.

There is a map called the *exponential map* from $\mathfrak{g} = T_eG$ to G given (roughly) by taking a tangent vector $X \in T_eG$, using it to induce a left-invariant vector field X on all of G, finding the integral curve α of X with $\alpha(0) = e$ and $\alpha'(t) = X(\alpha(t))$, and setting $\exp(X) = \alpha(1)$.

One can show that $\exp\colon \mathfrak{g} \to G$ is a local diffeomorphism. For the classical matrix Lie grapes

$$\begin{aligned}
\mathrm{GL}(n,\mathbb{R}) &= \{\, A \in M_n(\mathbb{R}) : \det(A) \neq 0 \,\} \\
\mathrm{SL}(n,\mathbb{R}) &= \{\, A \in \mathrm{GL}_n(\mathbb{R}) : \det(A) = 1 \,\} \\
\mathrm{O}(n,\mathbb{R}) &= \{\, A \in \mathrm{GL}_n(\mathbb{R}) : A^T A = I \,\} \\
\mathrm{SO}(n,\mathbb{R}) &= \{\, A \in \mathrm{O}(n,\mathbb{R}) : \det(A) = 1 \,\} \\
\mathrm{U}(n) &= \{\, A \in \mathrm{GL}(n,\mathbb{C}) : A^* A = I \,\}
\end{aligned}$$

etc. we can identify the Lie algebra $\mathfrak{g}$ with a matrix algebra

$$\begin{aligned}
\mathfrak{gl}(n,\mathbb{R}) &= M_n(\mathbb{R}) \\
\mathfrak{sl}(n,\mathbb{R}) &= \{\, A \in M_n(\mathbb{R}) : \mathrm{tr}(A) = 0 \,\}
\end{aligned}$$

etc., and then $\exp\colon \mathfrak{g} \to G$ is given by

$$\exp(A) = e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$$

Definition 1.5. A *representation* of a grape is a grape homomorphism $\rho\colon G \to \mathrm{Perm}(X)$ for some set X, or $\rho\colon G \to \mathrm{GL}(n,\mathbb{R})$ or $\rho\colon G \to \mathrm{GL}(V)$ for some vector space V.

This gives an action of G on X or $\mathbb{R}^n$ or V: for $a \in G$ and $x \in X$ or $\mathbb{R}^n$ or V we write $a \cdot x = \rho(a)(x)$; this gives a G-module structure on V.

Given a representation $\rho\colon G \to \mathrm{GL}(V)$ we get $\rho = \mathrm{d}\rho\colon T_eG \to T_e\,\mathrm{GL}(V)$; i.e. $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$.

A representation $\rho\colon G \to \mathrm{GL}(V)$ induces a character $\chi\colon G \to \mathbb{R}$ given by $\chi(a) = \mathrm{tr}(\rho(a))$; one can show that for a Lie grape a representation is determined by its character. In the finite-dimensional case, one can always decompose a representation into irreducible subrepresentations.

1.1 Stuff we probably won't get to

A compact Lie grape G has a maximal torus T (i.e. a torus subgrape of maximal dimension) that is unique up to conjugation; its dimension is called the rank of the grape.

Example 1.6. SU(3) has maximal torus $T = \{ \operatorname{diag}(\exp(i2\pi t_1), \exp(i2\pi t_2), \exp(i2\pi t_3)) : \sum t_i = 0 \}$, which has Lie algebra

$$\mathfrak{t} = \{ \operatorname{diag}(t_1, t_2, t_3) : t_i \in \mathbb{R} \}$$

with $\exp(t_1, t_2, t_3) = (\exp(i2\pi t_1), \exp(i2\pi t_2), \exp(i2\pi t_3))$. A given representation $\rho\colon G \to \mathrm{GL}(n, \mathbb{R})$ reduces

TODO 1. *Is this $\mathbb{R}$ or $\mathbb{C}$?*

to give $\rho\colon T \to \mathrm{GL}(n, \mathbb{C})$. The irreducible representations of T are known, and are all 1-dimensional. The irreducible representations are classified by *weights* in $\mathfrak{t}$. The set of weights $\Omega \subseteq \mathfrak{t}$ is an integral lattice in $\mathfrak{t}$ related to the kernel of the exponential map. For SU(3) we have $\Omega = \ker(\exp) = \{ \operatorname{diag}(k_1, k_2, k_3) : \text{ each } k_i \in \mathbb{Z}, \sum k_i = 0 \}$.

For $u_1 = \operatorname{diag}(1, -1, 0)$ and $u_2 = \operatorname{diag}(0, 1, -1)$ the "angle" is given by

$$\Theta(u_1, u_2) = \cos^{-1} \frac{u_1 \cdot u_2}{|u_1||u_2|} = \frac{2\pi}{3}$$

The integral span of these (ignoring the diag) gives a lattice of equilateral triangles. The weights of the adjoint representation are called roots: for $a \in G$ we define $c_a\colon G \to G$ by $c_a(x) = axa^{-1}$; the gives a map $\mathrm{d}c_a\colon \mathfrak{g} \to \mathfrak{g}$, which gives the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$ (with $\mathrm{Ad}(a) = \mathrm{d}c_a$).

In SU(3) the weights of Ad are $\pm u_1, \pm u_2, \pm(u_1 + u_2)$.

2 Manifolds

Definition 2.1. Suppose M is a topological space.

- We say M is *second-countable* if there is a countable basis for the topology on M.
- We say M is *Hausdorff* if for all $p, q \in M$ there are disjoint open sets $U, V \subseteq M$ with $p \in U$ and $q \in V$.
- We say M is *locally homeomorphic to* $\mathbb{R}^n$ if for all $p \in M$ there is an open $U \subseteq M$ containing p, open $V \subseteq \mathbb{R}^n$, and a homeomorphism $\varphi\colon U \to V$.

  Such φ are called *(local coordinate) charts* on M at p. A set of charts whose domains cover M is called an *atlas* on M.

Remark 2.2. Note that when $\varphi\colon U \subseteq M \to \varphi(U) \subseteq \mathbb{R}^n$ and $\psi\colon V \subseteq M \to \psi(V) \subseteq \mathbb{R}^n$ are charts at p (so $p \in U \cap V$) then $\psi \circ \varphi^{-1}\colon \varphi(U \cap V) \to \psi(U \cap V)$ is a homeomorphism between two open sets in $\mathbb{R}^n$; such a map $\psi \circ \varphi^{-1}$ is called a *change in coordinates* map or a *transition map*.

Definition 2.3. An *n-dimensional topological manifold* is a topological space M that is separable, Hausdorff, and locally homeomorphic to $\mathbb{R}^n$.

Definition 2.4. An *n-dimensional smooth (or $\mathcal{C}^\infty$) manifold* is an n-dimensional topological manifold which has an atlas whose transition maps are $\mathcal{C}^\infty$.

Example 2.5. Some $\mathcal{C}^\infty$ manifolds:

- $\mathbb{R}^n$ (with one chart, the identity map)
- $\mathbb{S}^n = \{ x \in \mathbb{R}^{n+1} : |x| = 1 \}$ (using, for example, the $2n + 2$ charts

  $$\varphi_k\colon \{ (x_1, \dots, x_{n+1}) \in \mathbb{S}^n : x_k > 0 \} \to B = \{ y \in \mathbb{R}^n : |y| < 1 \}$$

  given by $\varphi_k(x_1, \dots, x_{n+1}) = (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_{n+1})$ and

  $$\psi_k\colon \{ (x_1, \dots, x_{n+1}) \in \mathbb{S}^{n+1} : x_k < 0 \} \to B$$

  given by the same formula).

- $\mathbb{P}^n = (\mathbb{R}^{n+1} \setminus \{0\})/(\mathbb{R} \setminus \{0\}) = \{ [x] : x \in \mathbb{R}^{n+1} \setminus \{0\} \}$ where $[x] = \{ tx : 0 \neq t \in \mathbb{R} \}$ using the $n+1$ charts $\varphi_k \colon U_k \to \mathbb{R}^n$ where $U_k = \{ [x_1, \ldots, x_{n+1}] : x_k \neq 0 \}$ and

$$\varphi([x_1, \ldots, x_{n+1}]) = \left(\frac{x_1}{x_k}, \ldots, \frac{x_{k-1}}{x_k}, \frac{x_{k+1}}{x_k}, \ldots, \frac{x_{n+1}}{x_k} \right)$$

- Every open subset of a manifold is also a manifold (of the same dimension). If N, M are manifolds then so is $N \times M$. In particular, we get

$$\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$$

Remark 2.6. If M is both an n-dimensional and an m-dimensional manifold then $m = n$. (One can see this by looking at the Jacobians of the transition maps.) If M is n-dimensional we write $\dim(M) = n$.

Definition 2.7. Suppose N and M are $\mathcal{C}^\infty$ manifolds with $\dim(N) = n$ and $\dim(M) = m$. Suppose $f \colon N \to M$. We say that f is *smooth* or $\mathcal{C}^\infty$ at $p \in M$ when there is some (and hence for all) charts $\varphi \colon U \subseteq N \to \varphi(N) \subseteq \mathbb{R}^n$ and $\psi \colon V \subseteq M \to \psi(V) \subseteq \mathbb{R}^m$ with $p \in U$ and $f(p) \in V$ such that $\psi \circ f \circ \varphi^{-1}$ is smooth ($\mathcal{C}^\infty$) at $\varphi(p)$.

In this case we define the *rank* of f at p to be equal to the rank of $D(\psi f \varphi^{-1})(\varphi(p))$. We sometimes denote the matrix $D(\psi f \varphi^{-1})(\varphi(p))$ by $Df(p)$.

There are a few different sensible notions of submanifold.

Definition 2.8. Let M be a smooth manifold. A *regular submanifold* of M is a subset $N \subseteq M$ which is a manifold such that for all $p \in N$ there are charts $\varphi \colon U \subseteq N \to \varphi(U) \subseteq \mathbb{R}^n$ at p on N and $\psi \colon V \subseteq M \to \psi(V) \subseteq \mathbb{R}^m$ at p on M such that

- $U = V \cap N$
- $\varphi(p) = 0$ and $\psi(p) = 0$ (if you want)
- For $x = (x_1, \ldots, x_n) \in \varphi(U) \subseteq \mathbb{R}^n$ we have $\psi\varphi^{-1}(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0, \ldots, 0)$.

Definition 2.9. Suppose N and M are $\mathcal{C}^\infty$-manifolds with $\dim(N) = n$, $\dim(M) = m$, and $n \leq m$. A function $f \colon N \to M$ is called an *immersion* when f is smooth and has maximal rank everywhere (i.e. $\operatorname{rank} Df(p) = n$ for all $p \in N$). An *immersed submanifold* of M is the image $f(N)$ of some injective immersion; we use the topology and charts induced by the map f (so that $f \colon N \to f(N)$ is a diffeomorphism).

Note that immersions need not be injective. Note also that the topology on an immersed submanifold $N \subseteq M$ need not be the subspace topology inherited from M.

Definition 2.10. Suppose N and M are $\mathcal{C}^\infty$ manifolds with $\dim(N) = n$ and $\dim(M) = m$; suppose $f \colon N \to M$. We say that f is an *embedding* (or a *regular immersion*) when f is an injective immersion and the topology on $f(N)$ induced by the map f agrees with the subspace topology on $f(N)$. The image $f(N)$ of such an embedding $f \colon N \to M$ is called an *embedded submanifold* of M.

Example 2.11. Consider the map $f \colon (-\pi, \pi) \to \mathbb{R}^2$ given by $f(t) = (\sin(t), \sin(2t))$ (image looks like an infinity sign). The image is an immersed submanifold but not an embedded submanifold because of the behaviour around the origin.

Example 2.12. Consider $f \colon \mathbb{R} \to \mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ given by $f(t) = (\exp(iat), \exp(ibt))$ with $a, b \in \mathbb{R}$. If $a \neq 0$ and $\frac{b}{a} \notin \mathbb{Q}$ then the image of f is dense in $\mathbb{T}^2$; this is an immersed manifold but not an embedded manifold.

Theorem 2.13. *Suppose $f \colon N \to M$ is an injective immersion of smooth manifolds; suppose that N is compact. Then f is an embedding.*

Proof. Consider $f(N)$ with the subspace topology. Suppose $K \subseteq N$ is closed (and hence compact, since N is compact). Since K is compact and $f\colon N \to f(N) \subseteq M$ is continuous, we get that $f(K)$ is compact. Since $f(K)$ is compact and M is Hausdorff, we get that $f(K)$ is closed. So f sends closed sets to closed sets; so, since $f\colon N \to f(N)$ is bijective, we get that f is open, and thus a homeomorphism $N \to f(N)$. □ Theorem 2.13

Theorem 2.14 (Rank theorem)**.** *Suppose N, M are $\mathcal{C}^\infty$ manifolds with $\dim(N) = n$ and $\dim(M) = m$. Suppose $f\colon N \to M$ is a smooth map of constant rank r around p (i.e. $\operatorname{rank}(Df(p)) = r$ for all x in some neighbourhood of p). Then there exist a chart $\varphi\colon U \subseteq N \to \varphi(U) \subseteq \mathbb{R}^n$ at p on N and a chart $\psi\colon V \subseteq M \to \psi(V) \subseteq \mathbb{R}^m$ at $f(p)$ on M such that $\varphi(p) = 0$ and $\psi(p) = 0$ (if you want) and for all $x = (x_1, \ldots, x_n) \in \varphi(U)$ we have $\psi\varphi^{-1}(x_1, \ldots, x_n) = (x_1, \ldots, x_r, 0, \ldots, 0)$. (In particular, if the rank is globally constant, then for all $q \in f(N)$ we have $K = f^{-1}(q) = \{\, x \in N : f(x) = q \,\}$ is a closed regular embedded submanifold of M.)*

Corollary 2.15. *Every injective immersion $f\colon N \to M$ of smooth manifolds is locally an embedding.*

Corollary 2.16. *If $f\colon N \to M$ is an embedding of smooth manifolds then $f(N)$ is a regular submanifold.*

Remark 2.17. One can define variations on the definition of a manifold. For example, an n-dimensional *complex* or $\mathbb{C}$-*manifold* is a $2n$-dimensional topological manifold with charts such that the transition maps are all holomorphic. One could also define $\mathcal{C}^k$ or analytic manifolds.

TODO 2. *Section?*

3 Lie grapes and Lie algebras

Definition 3.1. A *Lie grape* is a set G which is both a $\mathcal{C}^\infty$ manifold and a grape such that the grape operations multiplication $m\colon G \to G$ and inversion $v\colon G \to G$ are smooth.

Example 3.2.

- $\mathbb{R}^n$ under addition
- $M_n(\mathbb{R})$ under addition
- $\mathbb{R}^*$ or $\mathbb{C}^*$ or $\mathbb{S}^1$ under multiplication
- $\mathbb{T}^n$ under (component-wise) multiplication
- $\mathrm{GL}(n, \mathbb{R}) = \{\, A \in M_n(\mathbb{R}) : A \text{ is invertible} \,\}$ is a Lie grape under multiplication. Indeed, $M_n(\mathbb{R})$ is diffeomorphic to (and can be identified with) $\mathbb{R}^{n^2}$ using the map $F\colon M_n(\mathbb{R}) \to \mathbb{R}^{n^2}$ given by

  $$F(u_1, \ldots, u_n) = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$$

  where each $u_k \in \mathbb{R}^n$. The determinant map $\varphi\colon M_n(\mathbb{R}) \to \mathbb{R}$ given by $\varphi(X) = \det(X)$ is a polynomial in the entries of X (and so is $\mathcal{C}^\infty$). So $\mathrm{GL}(n, \mathbb{R}) = \varphi^{-1}(\mathbb{R} \setminus \{\, 0 \,\})$ is open in $M_n(\mathbb{R})$, and is thus endowed with a smooth structure.

  Note as well that the map $m(X, Y) = XY$ is polynomial in the entries of X and Y, and is thus smooth; also

  $$v(x) = \frac{1}{\det(X)} \operatorname{Adj}(X)$$

  is a quotient of a polynomial by a non-zero polynomial in the entries of X, and is thus smooth.

Definition 3.3. Suppose H and G are Lie grapes. A map $f\colon H \to G$ is called a *Lie grape homomorphism* when f is a smooth grape homomorphism. (*Isomorphisms* and *isomorphic* are defined accordingly.)

Example 3.4.

- The map $F\colon M_n(\mathbb{R}) \to \mathbb{R}^{n^2}$ above given by
$$F(u_1, \dots, u_n) = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$$
is an isomorphism of Lie grapes.
- The determinant map $\varphi\colon \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}^2$ is a Lie grape homomorphism.
- For $a \in \mathbb{R}^n$ we can define $\varphi\colon \mathbb{R} \to \mathbb{T}^n$ given by $\varphi(t) = (\exp(ia_1 t), \dots, \exp(ia_n t))$ and $\psi\colon \mathbb{R} \to \mathbb{T}^n$ given by $\psi(t_1, \dots, t_n) = (\exp(ia_1 t_1), \dots, \exp(ia_n t_n))$ are Lie grape homomorphisms.

Definition 3.5. Suppose G is a Lie grape. An *(immersed) Lie subgrape* of G is the image $\varphi(H)$ of a Lie grape homomorphism $\varphi\colon H \to G$ which is an immersion. An *embedded* (or *regular immersed*) Lie subgrape of G is the image $\varphi(H)$ of some Lie grape homomorphism $\varphi\colon H \to G$ which is an embedding.

Theorem 3.6. *Suppose H and G are Lie grapes; suppose $\varphi\colon H \to G$ is a homomorphism of Lie grapes. Then φ has constant rank.*

Proof. For $a \in H$ we let $\ell_a\colon H \to H$ be left-multiplication by a (so $\ell_a(x) = ax$ for $x \in H$). For all $a, x \in H$ we have $\varphi(ax) = \varphi(a)\varphi(x)$; i.e. $\varphi(\ell_a(x)) = \ell_{\varphi(a)}(\varphi(x))$. So, implicitly fixing charts, we apply chain rule to the above to get that
$$D\varphi(ax) \cdot D\ell_a(x) = D\ell_{\varphi(a)}(\varphi(x)) \cdot D\varphi(x)$$
Since ℓ_a and $\ell_{\varphi(a)}$ are diffeomorphisms (with inverses $\ell_{a^{-1}}$ and $\ell_{(\varphi(a))^{-1}}$, respectively), the matrices $D\ell_a(x)$ and $D\ell_{\varphi(a)}(\varphi(x))$ are invertible. So
$$\mathrm{rank}(D\varphi(ax)) = \mathrm{rank}(D\varphi(x))$$
for all $a, x \in H$. In particular, taking $a = x^{-1}$ gives $\mathrm{rank}(D\varphi(x)) = \mathrm{rank}(D\varphi(e))$ for all $x \in H$.
☐ Theorem 3.6

Example 3.7. Show that the grapes

$$\begin{aligned}
\mathrm{SL}(n, \mathbb{R}) &= \{\, A \in \mathrm{GL}(n, \mathbb{R}) : \det(A) = 1 \,\} && \text{(special linear grape)} \\
\mathrm{O}(n, \mathbb{R}) &= \{\, A \in \mathrm{GL}(n, \mathbb{R}) : A^T A = I \,\} && \text{(orthogonal grape)} \\
\mathrm{SO}(n, \mathbb{R}) &= \{\, A \in \mathrm{O}(n, \mathbb{R}) : \det(A) = 1 \,\} && \text{(special orthogonal grape)} \\
\mathrm{GL}(n, \mathbb{C}) &= \{\, A \in M_n(\mathbb{C}) : A \text{ is invertible} \,\} && \text{(general linear grape over } \mathbb{C}\text{)} \\
\mathrm{U}(n) &= \{\, A \in \mathrm{GL}(n, \mathbb{C}) : A^* A = I \,\} && \text{(unitary grape)} \\
\mathrm{SU}(n) &= \{\, A \in \mathrm{U}(N) : \det(A) = 1 \,\} && \text{(special unitary grape)} \\
\mathrm{GL}(n, \mathbb{H}) &= \{\, A \in M_n(\mathbb{H}) : A \text{ is invertible} \,\} && \text{(general linear grape over the quaternions)} \\
\mathrm{Sp}(n) &= \{\, A \in \mathrm{GL}(n, \mathbb{H}) : A^* A = I \,\} && \text{(symplectic grape)}
\end{aligned}$$

are regular Lie subgrapes of $\mathrm{GL}(m, \mathbb{R})$ for some m. (Here $A^* = (\overline{A})^T$ and $\mathbb{H} = \{\, a+bi+cj+dk : a, b, c, d \in \mathbb{R} \,\}$ with $i^2 = j^2 = k^2 = -1$ with $i^2 = j^2 = k^2 = ijk = -1$ and $ij = k, jk = i, ki = jk$.)

We do some sample computations:

($\mathrm{SL}(n, \mathbb{R})$) The determinant map $\varphi\colon \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}^*$ is a Lie grape homomorphism and $\mathrm{SL}(n, \mathbb{R}) = \ker(\varphi)$. So $\mathrm{SL}(n, \mathbb{R})$ is a closed, regular Lie subgrape.

Exercise 3.8 (Possibly worthwhile). Compute the Jacobian of φ and show directly that the rank is 1.

($\mathrm{O}(n, \mathbb{R})$) Consider the map $\varphi\colon \mathrm{GL}(n, \mathbb{R})$ given by $\varphi(X) = X^T X$.

Claim 3.9. *φ has constant rank.*

Proof. For $A \in \mathrm{GL}(n, \mathbb{R})$ we let $L_A, R_A \colon \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ be left- and right-multiplication by A, respectively. Then for $A, X \in \mathrm{GL}(n, \mathbb{R})$ we have

$$\varphi(R_A(x)) = \varphi(XA) = A^T X^T X A = L_{A^T}(R_A(\varphi(x)))$$

So by the chain rule (again implicitly fixing charts) we have

$$D\varphi(XA) \cdot DR_A(X) = DL_{A^T}(X^T X A) \cdot DR_A(\varphi(X)) \cdot D\varphi(X)$$

Then since R_A and L_{A^T} are diffeomorphisms we get that $\operatorname{rank}(D\varphi(XA)) = \operatorname{rank}(D\varphi(X))$ for all X. In particular for $A = X^{-1}$ we get $\operatorname{rank}(D\varphi(X)) = \operatorname{rank}(D\varphi(I))$. □ Claim 3.9

Hence $\mathrm{O}(n, \mathbb{R})$ is a closed regular Lie subgrape of $\mathrm{GL}(n, \mathbb{R})$ because $\mathrm{O}(n, \mathbb{R}) = \varphi^{-1}(I)$.

($\mathrm{SO}(n, \mathbb{R})$) It's the kernel of the determinant map.

Exercise 3.10. Check the rest.

Remark 3.11. We can also define *complex Lie grapes*. Some examples include $\mathrm{GL}(n, \mathbb{C}), \mathrm{SL}(n, \mathbb{C}), \mathrm{O}(n, \mathbb{C}), \mathrm{SO}(n, \mathbb{C}) = \{ A \in M_n(\mathbb{C}) : A^T A = I, \det(A) = 1 \}, \mathrm{Sp}(2n, \mathbb{C})$.

Fact 3.12. *The only connected compact complex Lie grapes are complex tori.*

Exercise 3.13. Which of the real Lie grapes exhibited above are compact?

Definition 3.14. Suppose M is a $\mathcal{C}^\infty$ manifold of dimension n and $p \in M$. A *tangent vector* on M at p is a set of ordered pairs (φ, u) with one pair for each chart φ at p and each $u \in \mathbb{R}^n$ obtained from the following procedure: pick a smooth curve $\alpha \colon (-\varepsilon, \varepsilon) \to M$ with $\alpha(0) = p$, and define $\alpha'(0)$ to be the set of pairs (φ, u) where given a chart φ at p we let $u = \beta'(0)$ where $\beta(t) = \varphi(\alpha(t))$. The space of tangent vectors on M at p is denoted by $T_p M$.

Remark 3.15. When ψ is another chart and (ψ, v) is another pair induced by α, we have $v = \gamma'(0)$ where

$$\gamma(t) = \psi(\alpha(t)) = \psi(\varphi^{-1}(\beta(t))) = (\psi\varphi^{-1})(\beta(t))$$

so $\gamma'(t) = D(\psi\varphi^{-1})(\beta(t)) \cdot \beta'(t)$, and $v = \gamma'(0) = D(\psi\varphi^{-1})(\varphi(p))u$. Thus u and v are related by

$$\begin{aligned} v &= D(\psi\varphi^{-1})(\varphi(p)) \cdot u \\ v_k &= \sum_{i=1}^n \frac{\partial(\psi\varphi^{-1})_k}{\partial x_i} u_i \end{aligned}$$

Definition 3.16. Suppose M is a $\mathcal{C}^\infty$ manifold and $p \in M$. A *derivation* on M at p is a linear map $D \colon \mathcal{C}_p^\infty(M, \mathbb{R}) \to \mathbb{R}$ such that $D(fg) = D(f) \cdot g + f \cdot D(g)$ for $f, g \in \mathcal{C}_p^\infty(M, \mathbb{R})$ where $\mathcal{C}_p^\infty(M, \mathbb{R})$ is the space of locally smooth functions on M at p; i.e. smooth functions $g \colon U \subseteq M \to \mathbb{R}$ where U is open in M with $p \in U$, and two such functions $g \colon U \subseteq M \to \mathbb{R}$ and $h \colon V \subseteq M \to \mathbb{R}$ are considered equivalent when they agree in some open $W \subseteq U \cap V$ with $p \in W$.

Remark 3.17. A tangent vector $X \in T_p M$ acts as a derivation on M at p as follows: choose a locally smooth curve $\alpha \colon (-\varepsilon, \varepsilon) \to M$ with $\alpha(0) = p$ and $\alpha'(0) = X$. Then we define $X(g) = h'(0)$ where $h(t) = g(\alpha(t))$.

Note that if X is given locally in the chart φ at p by $u \in \mathbb{R}^n$ then $u = \beta'(0)$ where $\beta(t) = \varphi(\alpha(t))$; so $h(t) = g(\alpha(t)) = (g\varphi^{-1})(\beta(t))$, and $h'(t) = D(g\varphi^{-1})(\beta(t)) \cdot \beta'(t)$. So $X(g) = h'(0) = D(g\varphi^{-1})(\varphi(p)) \cdot u$.

If we write $g\varphi^{-1}$ simply as g and $x = \varphi(p)$ then

$$X(g) = D(g\varphi^{-1})(\varphi(p)) \cdot u = \sum_{i=1}^n \frac{\partial(g\varphi^{-1})}{\partial x_i}(x) u_i$$

i.e.

$$X(g) = \sum_{i=1}^{n} u_i \frac{\partial g}{\partial x_i}$$

Because of this formula, it is customary to write the standard basis vectors in $\mathbb{R}^n$ (with $u \in \mathbb{R}^n$) as $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$; so

$$u = \sum_{i=1}^{n} u_i \frac{\partial}{\partial x_i}$$

Definition 3.18. Suppose $f\colon N \to M$ is a smooth map of smooth manifolds. Then f induces a linear map $f_*\colon T_pN \to T_{f(p)}M$ for each $p \in N$. (The map f_* is also denoted $\mathrm{d}f$ or Df.) Indeed, given $X \in T_pN$ we choose $\alpha\colon (-\varepsilon, \varepsilon) \to N$ with $\alpha(0) = p$ and $\alpha'(0) = X$, and then define $f_*X = \beta'(0)$ where $\beta(t) = f(\alpha(t))$.

TODO 3. *Roman d?*

Remark 3.19. If X is given locally in φ by $u \in \mathbb{R}^n$ then $u = \gamma'(0)$ where $\gamma(t) = \varphi(\alpha(t))$. Then $\beta(t) = f(\alpha(t)) = (f\varphi^{-1})(\gamma(t))$, and $\beta'(t) = D(f\varphi^{-1})(\gamma(t)) \cdot \gamma'(t)$; so

$$f_*X = \beta'(0) = D(f\varphi^{-1})(\varphi(p)) \cdot u$$

Theorem 3.20.

1. *Suppose $f\colon M \to N$ and $g\colon N \to L$ are smooth maps of smooth manifolds. Then $(g \circ f)_* = g_* \circ f_*\colon T_pM \to T_{g(f(p))}L$.*
2. *Suppose $f\colon N \to M$ is smooth; suppose $g\colon U \subseteq M \to \mathbb{R}$ where $U \subseteq M$ is open with $f(p) \in U$. (Or suppose $g\colon M \to \mathbb{R}$ is smooth.) Then for $X \in T_pN$ we have $(f_*X)(g) = X(g \circ f)$.*

Definition 3.21. Suppose M is a smooth manifold. A *vector field on M* is a map $X\colon M \to \bigcup_{p \in M} T_pM$ such that $X(p) \in T_pM$ for all $p \in M$. We sometimes write X_p to denote $X(p)$. A vector field X on M is given locally (in a chart $\varphi\colon U \subseteq M \to \varphi(U) \subseteq \mathbb{R}^n$) by a vector $u = u(x) \in \mathbb{R}^n$ at each point $x \in \varphi(U)$. We say that X is *continuous (or smooth, or $\mathcal{C}^k$))* when for some (hence for every) chart φ the resulting function $u(x)$ is continuous (or smooth, or $\mathcal{C}^k$). The space of all smooth vector fields on M is denoted $\Gamma(M, TM)$.

Remark 3.22. When $f\colon N \to M$ is a smooth map of smooth manifolds and $X \in \Gamma(N, TN)$, we don't necessarily have a well-defined vector field on M: if f is not injective we might have $p \neq q$ in N wth $f(p) = f(q)$ but $f_*X_p \neq f_*X_q$ in $T_{f(p)}M = T_{f(q)}M$. If f is surjective then f_*X is well-defined as a vector field on $f(N) \subseteq M$. If $f\colon N \to M$ is a diffeomorphism then f_* gives a well-defined map $\Gamma(N, TN) \to \Gamma(M, TM)$. If $f\colon N \to M$ is an injective immersion then f is a smooth diffeomorphism as a map $f\colon N \to f(N)$ (where the latter is endowed with the topology and smooth structure induced from N via f).

Theorem 3.23. *Suppose M is a smooth manifold; suppose $X, Y \in \Gamma(M, TM)$. Then there exists a (unique) smooth vector field Z on M such that $Z(g) = X(Y(g)) - Y(X(g))$ for all smooth maps $g\colon M \to \mathbb{R}$.*

Proof. Suppose X, Y are given locally in a chart $\varphi\colon U \to \varphi(U)$ by vectors $u, v \in \mathbb{R}^n$. Write $x = \varphi(p)$ and $g\varphi^{-1}$ as g. Then

$$\begin{aligned}
X(Y(g)) - Y(X(g)) &= \sum_{i=1}^{n} u_i \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n} v_j \frac{\partial g}{\partial x_j}\right) - \sum_{i=1}^{n} v_i \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n} u_j \frac{\partial g}{\partial x_j}\right) \\
&= \sum_{i,j} u_i \left(\frac{\partial v_j}{\partial x_i}\frac{\partial g}{\partial x_j} + v_j \frac{\partial^2 g}{\partial x_i \partial x_j}\right) - \sum_{i,j} v_i \left(\frac{\partial u_j}{\partial x_i}\frac{\partial g}{\partial x_j} + u_j \frac{\partial^2 g}{\partial x_i \partial x_j}\right) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \left(\frac{\partial v_j}{\partial x_i} u_i - \frac{\partial u_j}{\partial x_i} v_i\right) \frac{\partial g}{\partial x_j}
\end{aligned}$$

Thus $X(Y(g)) - Y(X(g)) = Z(g)$ where Z is the smooth vector field given locally in the chart φ by

$$w = \sum_{j=1}^{n} \sum_{i=1}^{n} \left(\frac{\partial v_j}{\partial x_i} u_i - \frac{\partial u_j}{\partial x_i} v_i \right) \frac{\partial}{\partial x_j}$$

(where again $\frac{\partial}{\partial x_j}$ is the j^{th} standard basis vector). i.e.

$$w = Dv \cdot u - Du \cdot v$$

$\square$ Theorem 3.23

Exercise 3.24. Check that if you change coordinates that w satisfies the rule.

Fact 3.25. *A tangent vector is determined by its action as a derivation. Hence a smooth vector field is determined by its action on locally smooth functions. Using smooth bump functions this shows that a smooth vector field is determined by its action on global smooth functions.*

Definition 3.26. The vector field Z in the above theorem is called the *Lie bracket* of X and Y and is denoted $[X, Y]$.

Theorem 3.27. *Suppose $f\colon N \to M$ is a smooth map of smooth manifolds. Suppose $X, Y \in \Gamma(N, TN)$; suppose $U, V \in \Gamma(M, TM)$ satisfy*

$$f_* X_p = U_{f(p)}$$
$$f_* Y_p = V_{f(p)}$$

for all $p \in N$. Then $(f_*[X, Y])_p = ([U, V])_p$ for all $p \in N$.

Exercise 3.28. Prove this. Hint: if $f, g\colon \mathbb{R} \to \mathbb{R}$ and $h = g \circ f$ then $h' = (g' \circ f) \cdot f'$.

Proof. For any smooth map $g\colon M \to \mathbb{R}$ and for all $p \in N$ we have

$$U(g)(f(p)) = U_{f(p)}(g) = (f_* X_p)(g) = X_p(g \circ f) = X(g \circ f)(p)$$

(The third equality was an exercise.)

TODO 4. *ref?*

Hence for all smooth $g\colon M \to \mathbb{R}$ we have $X(g \circ f) = U(g) \circ f$. Thus for all smooth $g\colon M \to \mathbb{R}$ and all $p \in N$ we have

$$\begin{aligned} f_*[X, Y]_p(g) &= [X, Y]_p(g \circ f) \\ &= X_p(Y(g \circ f)) - Y_p(X(g \circ f)) \\ &= X_p(V(g) \circ f) - Y_p(U(g) \circ f) \\ &= U_{f(p)}(V(g)) - V_{f(p)}(U(g)) \text{ (by our earlier equalities, with } g \text{ replaced by } V(g) \text{ and } U(g)) \\ &= [U, V]_{f(p)}(g) \end{aligned}$$

So $f_*[X, Y]_p(g) = [U, V]_{f(p)}(g)$ for all smooth $g\colon M \to \mathbb{R}$. So $f_*[X, Y]_p = [U, V]_{f(p)}$. $\square$ Theorem 3.27

Definition 3.29. A *Lie algebra* is a vector space V over a field F equipped with a binary operation $[\cdot, \cdot]$ called the *Lie bracket* satisfying:

(Skew-symmetry) $[a, b] = -[b, a]$ for all $a, b \in V$

(Bilinearity) $[ta, b] = t[a, b]$ and $[a + b, c] = [a, c] + [b, c]$ for all $a, b, c \in V$ and all $t \in F$

(Jacobian identity) $[[a, b], c] + [[b, c], a] + [[c, a], b] = 0$ for all $a, b, c \in V$.

We define a Lie algebra homomorphism and a Lie algebra isomorphism in the expected way.

Remark 3.30. Bilinearity of the Lie bracket in the second parameter follows from bilinearity of the first and skew-symmetry.

Example 3.31.

- $M_n(F)$ is a Lie algebra under $[A,B] = AB - BA$. Indeed,

$$\begin{aligned}[[A,B],C] &= [A,B]C - C[A,B] \\ &= (AB-BA)C - C(AB-BA) \\ &= ABC - BAC - CAB + CBA \\ [[B,C],A] &= BCA - CBA - ABC + ACB \\ [[C,A],B] &= CAB - ACB - BCA + BAC\end{aligned}$$

- When M is a smooth manifold, the space of smooth vector fields $\Gamma(M,TM)$ is a Lie algebra with Lie bracket given by $[X,Y](g) = X(Y(g)) - Y(X(g))$ for smooth $g\colon M \to \mathbb{R}$.

Definition 3.32. Suppose G is a Lie grape. For $a \in G$ let $\ell_a\colon G \to G$ denote left multiplication: $\ell_a(x) = a \cdot x$.

Note that $\ell_a\colon G \to G$ is a diffeomorphism (with inverse $\ell_{a^{-1}}$) and so $\mathrm{d}\ell_a = (\ell_a)_*$ defines a linear map from $\Gamma(G,TG) \to \Gamma(G,TG)$.

Definition 3.33. For a smooth vector field X on G, we say X is *left-invariant* when $\mathrm{d}\ell_a X = X$ for all $a \in G$; i.e. $\mathrm{d}\ell_a X_b = X_{ab}$ for all $a,b \in G$.

Remark 3.34. If X is a left-invariant vector field on G with $X(e) = X_e = A \in T_eG$ then we must have $X(p) = X_p = \mathrm{d}\ell_p A$ for all $p \in G$. On the other hand, given $A \in T_eG$, if we define $X(p) = X_p = \mathrm{d}\ell_p A$ for $p \in G$ then X is a left-invariant vector field; indeed

TODO 5. *smoothness?*

$$\mathrm{d}\ell_a X_b = \mathrm{d}\ell_a(\mathrm{d}\ell_b A) = (\mathrm{d}\ell_a \circ \mathrm{d}\ell_b)A \underbrace{=}_{(*)} \mathrm{d}(\ell_a \circ \ell_b)A = \mathrm{d}\ell_{ab}A = X_{ab}$$

where (*) was an exercise.

TODO 6. *ref?*

Note also that when X and Y are left-invariant vector fields on G the Lie bracket $[X,Y]$ is also left-invariant. Indeed

$$\mathrm{d}\ell_a[X,Y] = [\mathrm{d}\ell_a X, \mathrm{d}\ell_a Y] = [X,Y]$$

by the previous theorem.

Definition 3.35. For a Lie grape G we define the *Lie algebra* of G to be the vector space $\mathfrak{g} = T_eG$ with Lie bracket defined by $[A,B] = [X,Y]_e$ where X and Y are the (unique) left-invariant vecor fields on G with $X_e = A$ and $Y_e = B$.

Remark 3.36. Using standard identifications from differential geometry, when G is a (real) Lie subgrape of $\mathrm{GL}(n,\mathbb{F})$ where $\mathbb{F} \in \{\mathbb{R},\mathbb{C},\mathbb{H}\}$, then for any $p \in G$ we can identify T_pG with the set of $\alpha'(0) \in M_n(F)$ where α is a locally smooth map $\alpha\colon (-\varepsilon,\varepsilon) \to G$ with $\alpha(0) = p$.

We briefly describe the identifications. When U is an open set in $\mathbb{R}^n$ we get that U is a smooth manifold with atlas consisting of one chart φ where $\varphi\colon U \to U$ is the identity map. So a vector $X = \alpha'(0) \in T_pU$ is given by the one vector $u = \beta'(0) \in \mathbb{R}^n$ where $\beta(t) = \varphi(\alpha(t)) = \alpha(t)$; so $u = \alpha'(0) \in \mathbb{R}^n$. So we identify $T_pU = \mathbb{R}^n$.

Also when M is an (immersed) submanifold of $\mathbb{R}^m$ and $f\colon M \to \mathbb{R}^n$ is the inclusion given by $f(p) = p$, we identify $T_pM = f_*(T_pM) \subseteq \mathbb{R}^m$. Indeed, for $X = \alpha'(0)$ where $\alpha\colon (-\varepsilon,\varepsilon) \to M$ we have $f_*X = \beta'(0)$ where $\beta(t) = f(\alpha(t)) = \alpha(t)$. So $f_*X = \alpha'(0) \in \mathbb{R}^m$.

Finally we identify $M_n(\mathbb{R})$ with $\mathbb{R}^{n^2}$ (and $M_n(\mathbb{C})$ with $\mathbb{R}^{2n^2}$ and $M_n(\mathbb{H})$ with $\mathbb{R}^{4n^2}$).

So if $\alpha(t) = (A_{ij}(t))_{ij}$ then $\alpha'(t) = (A'_{ij}(t))_{ij}$.

Theorem 3.37. *Suppose G is a Lie subgrape of $\mathrm{GL}(n,\mathbb{F})$ for $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ so that*

$$\mathfrak{g} = T_I G = \{\, \alpha'(0) \in M_n(\mathbb{F}) : \alpha \text{ is a locally smooth map } (-\varepsilon,\varepsilon) \to G, \alpha(0) = I \,\} \subseteq M_n(\mathbb{F})$$

Then

1. *For $A \in \mathfrak{g} = T_I G$ the (unique) left-invariant vector field U on G with $U(I) = A$ is given by $U(P) = U_P = PA$ (matrix multiplication).*
2. *For $A, B \in \mathfrak{g} = T_I G \subseteq M_n(\mathbb{F})$ the Lie bracket of A and B is given by the commutator $[A,B] = AB - BA$ (matrix multiplication).*

Proof.

Case 1. Suppose $G = \mathrm{GL}(n,\mathbb{R}) \subseteq M_n(\mathbb{R})$, so $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{R}) = M_n(\mathbb{R})$.

1. Suppose $A \in \mathfrak{g} = M_n(\mathbb{R})$; let U be the left-invariant vector field on $G = \mathrm{GL}(n,\mathbb{R})$ with $U_I = A$. Then for all $P \in G$ we have $U(P) = U_P = DL_P A$ where L_P is left-multiplication by P and $DL_P = \mathrm{d}L_P = (L_P)_*$. (Note that DL_P when written as a matrix is $n^2 \times n^2$.)
   We have for $X \in M_n(\mathbb{R})$ that $L_P X = PX$; so
   $$(L_P)_{k,\ell}(x) = (L_P X)_{k,\ell} = (PX)_{k,\ell} = \sum_m P_{k,m} X_{m,\ell}$$
   So
   $$(DL_P)_{k\ell,ij}(x) = \frac{\partial (L_P)_{k\ell}}{\partial X_{ij}}(x) = \delta_{j,\ell} P_{k,i}$$
   So
   $$(U_P)_{k,\ell} = (DL_P A)_{k,\ell} = \sum_{i,j} (DL_P)_{k\ell,ij} A_{ij} = \sum_{i,j} \delta_{j,\ell} P_{ki} A_{ij} = \sum_i P_{ki} A_{i\ell} = (PA)_{k\ell}$$
   Thus $U(P) = U_P = PA$.
2. Next, given $A, B \in \mathfrak{g} = T_I G$ we must calculate $[A,B] = [U,V]_I$ where $U(x) = U_X = XA$ and $V(X) = V_X = XB$.
   We have $[U,V] = DV \cdot U - DU \cdot V$. So
   $$\begin{aligned}[U,V]_{k\ell} &= (DV \cdot U - DU \cdot V)_{k\ell} \\ &= (DV \cdot U)_{k\ell} - (DU \cdot V)_{k\ell} \\ &= \sum_{ij} (DV)_{k\ell,ij} U_{ij} - (DU)_{k\ell,ij} V_{ij}\end{aligned}$$
   We have $U(X) = XA$; so
   $$U_{k\ell}(X) = \sum_m X_{km} A_{m\ell}$$
   and
   $$\frac{\partial U_{k,\ell}}{\partial X_{i,j}}(x) = \delta_{ik} A_{j\ell}$$
   So
   $$[U,V]_{k\ell} = \sum_{ij} \delta_{ik} B_{j\ell} U_{ij} - \sum_{ij} \delta_{ik} A_{j\ell} V_{ij} = \sum_j (B_{j\ell} U_{kj} - A_{j\ell} V_{kj}) = (U(x)B - V(x)A)_{k\ell}$$
   Thus
   $$[U,V](x) = U(x) \cdot B - V(x) \cdot A$$
   for all x. Hence $[A,B] = [U,V]_I = U(I) \cdot B - V(I) \cdot A = AB - BA$.

Case 2. Suppose G is a Lie subgrape of $\mathrm{GL}(n,\mathbb{R})$. We identify $\mathfrak{g} = T_I G$ with $F_* T_I G \subseteq \mathrm{GL}(n,\mathbb{R}) = M_n(\mathbb{R})$ where $F\colon G \to \mathrm{GL}(n,\mathbb{R})$ is the inclusion map $F(P) = P$. For $P \in G$ let $L_P\colon G \to G$ and $M_P\colon \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$ be the left-multiplication maps; so the following diagram commutes:

$$\begin{tikzcd} G \arrow[r, "L_p"] \arrow[d, "F"] & G \arrow[d, "F"] \\ \mathrm{GL}(n,\mathbb{R}) \arrow[r, "M_p"] & \mathrm{GL}(n,\mathbb{R}) \end{tikzcd}$$

Let X and Y be the left-invariant vector fields on G with $X_I = A$ and $Y_I = B$ (or, more precisely, with $F_* X_I = A$ and $F_* Y_I = B$). Let U and V be the left-invariant vector fields on $\mathrm{GL}(n,\mathbb{R}) \subseteq M_n(\mathbb{R})$ with $U_I = A$ and $V_I = B$ (we now suppress the identification).

1. Need to show that $X_P = PA$ (or more precisely that $F_* X_P = PA$). Indeed, we have

$$F_* X_P = DFx_P = DFDL_P X_I = D(F \circ L_P) X_I = D(M_P \circ F) X_I = DM_P DF X_I = DM_P A = U_P = PA$$

2. Need to show that $[A,B] = AB - BA$; more precisely, that $F_*[X,Y]_I = AB - BA$. Since $F_* X_P = PA = U_P = U_{F(P)}$ and $F_* Y_P = PB = V_P = V_{F(P)}$, Theorem 3.27 shows that indeed

$$F_*[X,Y]_I = [U,V]_I = AB - BA$$

Case 3. Suppose G is a Lie subgrape of $\mathrm{GL}(n,\mathbb{C})$ or $\mathrm{GL}(n,\mathbb{H})$. Consider $\mathrm{GL}(n,\mathbb{C})$ as a Lie subgrape of $\mathrm{GL}(2n,\mathbb{R})$ using the injective homomorphism $F\colon \mathrm{GL}(n,\mathbb{C}) \to \mathrm{GL}(2n,\mathbb{R})$ given by

$$A + iB \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$$

Likewise we consider $\mathrm{GL}(n,\mathbb{H})$ as a Lie subgrape of $\mathrm{GL}(2n,\mathbb{C})$ using the map $F\colon \mathrm{GL}(n,\mathbb{H}) \to \mathrm{GL}(2n,\mathbb{C})$ given by

$$A + Cj \mapsto \begin{pmatrix} A & -\overline{C} \\ C & \overline{A} \end{pmatrix}$$

for $A, B \in M_n(\mathbb{R})$; i.e.

$$A + Bi + (C + Di)j \mapsto \begin{pmatrix} A + Bi & -C + Di \\ C + Di & A - Bi \end{pmatrix}$$

□ Theorem 3.37

Aside 3.38. $\det \begin{pmatrix} A & -B \\ B & A \end{pmatrix} = \det_{\mathbb{C}}(A + iB)$.

Alternative proof of Case 1.

1. Suppose $G = \mathrm{GL}(n,\mathbb{R})$; suppose $A \in \mathfrak{g} = M_n(\mathbb{R})$. Let U be the left-invariant vector field on G with $U_I = A$; that is $U_P = DL_P A$ (where $L_P\colon G \to G$ is $X \mapsto PX$). Since L_P is linear we get that $DL_P(X) = L_P$ as a linear map. So $U_P a = DL_P A = L_P A = PA$.

2. Now suppose $A, B \in \mathfrak{g} = M_n(\mathbb{R})$ and let U and V be the corresponding left-invariant vector fields with $U_I = A$ and $V_I = B$. Recall that $[U,V](X) = DV(X)U(X) - DU(X)V(X)$. We have $V(X) = V_X = XB$. So $V = R_B$ which is linear, and $DV(X) = R_B$ as a linear map. Similarly $DU(X) = R_A$. So we have

$$[U,V](X) = DV(X)U(X) - DU(X)V(X) = R_B(XA) - R_A(XB) = XAB - XBA = X(AB - BA)$$

Thus the Lie bracket of A and B in $\mathfrak{g}$ is

$$[A,B] = [U,V](I) = I(AB - BA) = AB - BA$$

□

4 Exponential map

Theorem 4.1 (Existence and uniqueness of solutions to ODEs)**.** *Suppose $U \subseteq \mathbb{R}^n$ be open with $p \in U$; suppose $I \subseteq \mathbb{R}$ is open with $s \in I$. Suppose $F\colon U \times I \to \mathbb{R}^n$ is smooth.*

1. *Suppose $J \subseteq I$ and $K \subseteq I$ are intervals with $s \in J \cap K$; suppose $\alpha\colon J \to U$ is smooth with $\alpha(s) = p$ and $\alpha'(t) = F(\alpha(t), t)$ for all $t \in J$ and $\beta\colon K \to U$ is smooth with $\beta(s) = p$ and $\beta'(t) = F(\beta(t), t)$ for all $t \in K$. Then $\alpha(t) = \beta(t)$ for all $t \in J \cap K$.*

2. *There is a unique maximal open interval $J \subseteq I$ with $s \in J$ and a (unique) smooth curve $\alpha\colon J \to U$ with $\alpha(s) = p$ and $\alpha'(t) = F(\alpha(t), t)$ for all $t \in J$.*

If we rule out time-variance of the vector field and relativize to a smooth manifold, we get:

Corollary 4.2. *Suppose X is a smooth vector field on a smooth manifold M.*

1. *Suppose $p \in M$ and $I, J \subseteq \mathbb{R}$ are two intervals with $0 \in I \cap J$. Suppose $\alpha\colon I \to M$ is smooth with $\alpha(0) = p$ and $\alpha'(t) = X_{\alpha(t)}$ for all $t \in I$ and $\beta\colon J \to M$ is smooth with $\beta(0) = p$ and $\beta'(t) = X_{\alpha(t)}$ for all $t \in J$. Then $\alpha(t) = \beta(t)$ for all $t \in I \cap J$.*

2. *For all $p \in M$ there is a unique maximal parameter interval $I \subseteq \mathbb{R}$ with $0 \in I$ and a (unique) smooth curve $\alpha\colon I \to M$ such that $\alpha(0) = p$ and $\alpha'(t) = X_{\alpha(t)}$ for all $t \in I$. We call this α the* integral curve *for X on M at p.*

Example 4.3.

1. Find the integral curve for $u(x, y) = (1, y)$ on $(-1, 1) \times (0, 2)$ at $p = (0, 1)$.

   We need $\alpha(t) = (x(t), y(t))$ such that $\alpha'(t) = u(\alpha(t))$; i.e.

   $$(x'(t), y'(t)) = u(x(t), y(t)) = (1, y(t))$$

   so $x'(t) = 1$ and $y'(t) = y(t)$. We also want $\alpha(0) = p = (0, 1)$; i.e. $x(0) = 0$ and $y(0) = 1$.

   To get $x'(t) = 1$ and $x(0) = 0$ we need $x(t) = t + c$ and $0 = 0 + c$; hence $x(t) = t$. To get $y'(t) = y$ and $y(0) = 1$, we write $\frac{\mathrm{d}y}{\mathrm{d}t} = y$, rearrange and integrate to get

   $$\int \frac{1}{y} \mathrm{d}y = \int \mathrm{d}t$$

   and hence $\ln(y) = t + c$, so $y = \exp(t + c)$. To get $y(0) = 1$ we get $y(t) = \exp(t)$. Thus the integral curve is $\alpha(t) = (x(t), y(t)) = (t, \exp(t))$ with maximal parameter interval $I = (-1, \ln(2))$.

2. Find the integral curve for $v(x, y) = (1, y^2)$ on $\mathbb{R}^2$ at $p = (0, 1)$.

   We need $x'(t) = 1$ with $x(0) = 0$ and $y'(t) = (y(t))^2$ with $y(0) = 1$. To get $x'(t) = 1$ and $x(0) = 0$ we again get $x(t) = t$. To get $y'(t) = (y(t))^2$ we need $\frac{\mathrm{d}y}{\mathrm{d}t} = (y(t))^2$ hence

   $$\int -y^{-2} \mathrm{d}y = \int -\mathrm{d}t$$

   and $(y(t))^{-1} = c - t$. To get $y(0) = 1$ we need $c = 1$; hence $y(t) = \frac{1}{1-t}$. Thus the integral curve is

   $$\alpha(t) = (x(t), y(t)) = \left(t, \frac{1}{1 - t}\right)$$

   which in particular has an asymptote at $x = 1$; so the maximal parameter interval is $I = (-\infty, 1)$.

Theorem 4.4. *Suppose G is a Lie grape and let $\mathfrak{g} = T_e G$. Suppose $A \in \mathfrak{g}$, and let X be the left-invariant vector field on G with $X_e = A$. Let $\alpha\colon I \to G$ be the maximal integral curve for X on G at e. Then*

1. *The maximal parameter interval is $I = \mathbb{R}$.*

2. *The map $\alpha\colon \mathbb{R} \to G$ is a Lie grape homomorphism (with $\alpha'(0) = X_{\alpha(0)} = X_e = A$).*

3. *If $\varphi\colon \mathbb{R} \to G$ is a Lie grape homomorphism with $\varphi'(0) = A \in \mathfrak{g}$ then $\varphi(t) = \alpha(t)$ for all $t \in \mathbb{R}$.*

Proof. Fix $s \in I$; then for $t \in I$ such that $s+t \in I$ if we let $\beta(t) = \alpha(s+t)$ and $\gamma(t) = \alpha(s)\alpha(t) = \ell_{\alpha(s)}(\alpha(t))$, then $\beta(0) = \alpha(s)$ and $\gamma(0) = \alpha(s)\alpha(0) = \alpha(s)e = \alpha(s)$; furthermore we have

$$\begin{aligned}
\beta'(t) &= \alpha'(s+t) \\
&= X_{\alpha(s+t)} \\
&= X_{\beta(t)} \\
\gamma'(t) &= \mathrm{d}\ell_{\alpha(s)}(\alpha(t))\alpha'(t) \\
&= \mathrm{d}\ell_{\alpha(s)}(\alpha(t))X_{\alpha(t)} \\
&= X_{\alpha(s)\alpha(t)} \\
&= X_{\gamma(t)}
\end{aligned}$$

So by uniqueness of integral curves we have $\beta(t) = \gamma(t)$ for all t; i.e. $\alpha(s+t) = \alpha(s) \cdot \alpha(t)$.

1. If the maximal interval were $I = (-a, b) \subsetneq \mathbb{R}$, then we could extend the parameter interval to $J = (-2a, 2b)$ by defining $\alpha(s+t) = \alpha(s)\alpha(t)$ for any $s, t \in I$.

2. Since the formula $\alpha(s+t) = \alpha(s)\alpha(t)$ holds for all $s, t \in \mathbb{R}$ we get that α is a grape homomorphism.

3. Suppose $\varphi\colon \mathbb{R} \to G$ is a Lie grape homomorphism with $\varphi'(0) = A$. Then for fixed s we have $\varphi(s+t) = \varphi(s)\varphi(t) = \ell_{\varphi(s)}\varphi(t)$. So

$$\varphi'(s+t) = \frac{\mathrm{d}}{\mathrm{d}t}\varphi(s+t) = \mathrm{d}\ell_{\varphi(s)}(\varphi(t)) \cdot \varphi'(t)$$

and at $t = 0$ we have

$$\varphi'(s) = \mathrm{d}\ell_{\varphi(s)}(0)(A) = X_{\varphi(s)}$$

Since $\varphi(0) = e$ and $\varphi'(s) = X_{\varphi(s)}$ for all s we get $\varphi = \alpha$ by uniqueness of integral curves. □ Theorem 4.4

Aside 4.5. The 2-sphere cannot be a Lie grape; the hairy ball theorem says that there is no nowhere vanishing vector field on S^2, whereas a left-invariant vector field generated from a non-zero tangent vector at e is nowhere vanishing.

Also the tangent bundle to a Lie grape is trivial (i.e. is isomorphic to $G \times \mathbb{R}^n$), and that of the sphere is not.

Definition 4.6. Suppose G is a Lie grape, $A \in \mathfrak{g}$, and X is the left-invariant vector field on G with $X_e = A$. The *flow* of X on G is the map $F\colon G \times \mathbb{R} \to G$ given by $F(p, t) = \alpha_p(t)$, where $\alpha_p\colon \mathbb{R} \to G$ is the (unique) integral curve for X on G at p.

Definition 4.7. Suppose G is a Lie grape with Lie algebra $\mathfrak{g}$. We define the *exponential map* $\exp\colon \mathfrak{g} \to G$ as follows: given $A \in \mathfrak{g} = T_eG$ we define $\exp(A) = \varphi(1)$ where $\varphi\colon \mathbb{R} \to G$ is the unique Lie grape homomorphism with $\varphi'(0) = A$. (i.e. φ is the integral curve at e of the left-invariant vector field generated by A.)

Remark 4.8. We could have made all the above definitions and theorems using right multiplication and right-invariant vector fields. So the Lie grape homomorphism $\varphi\colon \mathbb{R} \to G$ with $\varphi'(0) = A$ (in part (3) of Theorem 4.4) is also equal to the integral curve for the unique right-invariant vector field Y on G with $Y_e = A$ (i.e. $Y_p = \mathrm{d}r_pA$, where $r_p\colon G \to G$ is $x \mapsto x \cdot p$). The vector fields X and Y may be different, but they have the same integral curve through e; this integral curve coincides with the Lie grape homomorphism with $\varphi'(0) = A$.

Aside 4.9. The simplest Lie grapes besides $\mathbb{R}^n$ to picture are the torus or the cylinder.

Recall that for $A \in M_n(\mathbb{F})$ with $\mathbb{F} \in \mathbb{R}, \mathbb{C}$, we define

$$\exp(A) = e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k = I + A + \frac{1}{2!} A^2 + \cdots$$

Note that this series converges absolutely in $M_n(\mathbb{F})$ and uniformly in any compact set; indeed, if $m = \max_{ij} |A_{ij}|$ an d$m_\ell = \max_{ij} |(A^\ell)_{ij}|$ then since $A^{\ell+1} = A^\ell \cdot A$ we must have $m_{\ell+1} \leq n \cdot m_\ell \cdot m$. So by induction we get

$$m_\ell \leq n^{\ell-1} m^\ell \leq (nm)^\ell$$

hence if

$$S_\ell = \sum_{k=0}^{\ell} \frac{1}{k!} A^k$$

we have

$$\max |S_\ell|_{ij} \leq \sum_{k=0}^{\ell} \frac{1}{k!} (nm)^k = \exp(nm)$$

Also when $A, B \in M_n(\mathbb{C})$ commute we have $\exp(A+B) = \exp(A)\exp(B)$ because

$$\exp(A+B) = \sum_{m=0}^{\infty} \frac{1}{m!} (A+B)^m = \sum_{m,k} \frac{1}{m!} \binom{m}{k} A^k B^{m-k} = \sum_{m,k} \frac{1}{k!\ell!} A^k B^{m-k}$$

and

$$\exp(A)\exp(B) = \left(\sum_k \frac{1}{k!} A^k \right) \left(\sum_\ell \frac{1}{\ell!} B^\ell \right) = \sum_{k,\ell} \frac{1}{k!\ell!} A^k B^\ell = \sum_{k,m} \frac{1}{k!\ell!} A^k B^{m-k}$$

where we substitute $m - k = \ell$. It follows that for all $A \in M_n(\mathbb{F})$ we have $\exp(A)$ is invertible (with $(\exp(A))^{-1} = \exp(-A)$); we also get that for $s, t \in \mathbb{R}$ we have $\exp((s+t)A) = \exp(sA)\exp(tA)$. Also for $P \in \mathrm{GL}(n, \mathbb{F})$ and $A \in M_n(\mathbb{F})$ we have

$$\exp(PAP^{-1}) = \sum_{k=0}^{\infty} \frac{1}{k!} (PAP^{-1})^k = \sum_{k=0}^{\infty} \frac{1}{k!} PA^kP^{-1} = P \left(\sum_{k=0}^{\infty} \frac{1}{k!} A^k \right) P^{-1} = P \exp(A) P^{-1}$$

We also have $\frac{\mathrm{d}}{\mathrm{d}t} \exp(At) = A \exp(At) = \exp(At) A$.

Our final observation:

Proposition 4.10. *For $A \in M_n(\mathbb{F})$ we have*

$$\det(\exp(A)) = \exp(\mathrm{tr}(A))$$

Proof. Choose $P \in \mathrm{GL}(n, \mathbb{C})$ so $PAP^{-1} = J \in M_n(\mathbb{C})$ is in Jordan form. So J is upper triangular and the diagonal entries are the eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ of A. Then

$$\exp(PAP^{-1}) = \exp(J) = \sum_{k=0}^{\frac{1}{k!}} J^k$$

is upper triangular with diagonal entries $\exp(\lambda_1), \ldots, \exp(\lambda_n)$; so

$$\det(\exp(A)) = \det(P\exp(A)P^{-1}) = \det(\exp(PAP^{-1})) = \det(\exp(J)) = \prod_{i=1}^{n} \exp(\lambda_i) = \exp\left(\sum_i \lambda_i \right) = \exp(\mathrm{tr}(A))$$

as desired. □ Proposition 4.10

Alternate proof. Let $\varphi\colon M_n(\mathbb{F})$ be $\varphi(X) = \det(X)$. Then for $\ell \in \{1, \ldots, n\}$ we have

$$\varphi(X) = \det(X) = \sum_{i=1}^{n} (-1)^{i+\ell} X_{i,\ell} \det(X^{(i,\ell)}) = \sum_{i=1}^{n} X_{i\ell} (\operatorname{Adj} X)_{\ell,i}$$

where $X^{(i,\ell)}$ is the matrix obtained from X by removing the i^{th} row and ℓ^{th} column, and $\operatorname{Adj}(X)$ is the adjugate matrix. (Recall that $X \cdot \operatorname{Adj}(X) = \operatorname{Adj}(X) \cdot X = \det(X) \cdot I$.)

Then since $\operatorname{Adj}(X)_{\ell i} = (-1)^{i+\ell} \det(X^{(i,\ell)})$ does not depend on $X_{i\ell}$ we get that

$$\frac{\partial \varphi}{\partial X_{k,\ell}}(X) = (\operatorname{Adj}(X))_{\ell,k}$$

Thus if we define $g(t) = \det(\exp(At))$ then

$$\begin{aligned}
g'(t) &= D\varphi(\exp(At)) \cdot \frac{\mathrm{d}}{\mathrm{d}t}(\exp(At)) \\
&= \sum_{k,\ell} (\operatorname{Adj}(\exp(At)))_{\ell,k} (\exp(At) \cdot A)_{k,\ell} \\
&= \sum_{k,\ell,i} (\operatorname{Adj}(\exp(At)))_{\ell,k} (\exp(At))_{k,i} A_{i\ell} \\
&= \sum_{\ell,i} (\det(\exp(At)) \cdot I)_{\ell i} A_{i\ell} \\
&= \sum_{\ell,i} \det(\exp(At)) \cdot \delta_{\ell,i} A_{i\ell} \\
&= \sum_{\ell} \det(\exp(At)) A_{\ell,\ell} \\
&= \det(\exp(At)) \operatorname{tr}(A) \\
&= \operatorname{tr}(A) \det(\exp(At))
\end{aligned}$$

Thus $g(t)$ is the unique solution to the differential equation $g(t) = \operatorname{tr}(A) \cdot g(t)$ with $g(0) = \det(\exp(0)) = \det(I) = 1$. So

$$g(t) = \exp(t \cdot \operatorname{tr}(A))$$

In particular when $t = 1$ we get $\det(\exp(A)) = \exp(\operatorname{tr}(A))$. □ Proposition 4.10

Corollary 4.11. *Suppose G is a Lie subgrape of $\mathrm{GL}(n, \mathbb{F})$ where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Then*

1. *For $A \in \mathfrak{g} \subseteq M_n(\mathbb{F})$ the unique grape homomorphism $\varphi\colon \mathbb{R} \to G$ with $\varphi'(0) = A$ is given by $\varphi(t) = \exp(At)$.*

2. *The Lie algebra $\mathfrak{g} \subseteq M_n(\mathbb{F})$ is given by*

$$\mathfrak{g} = \{ A \in M_n(\mathbb{F}) : \exp(At) \in G \textit{ for all } t \in \mathbb{R} \}$$

3. *The exponential map $\exp\colon \mathfrak{g} \to G$ is a local diffeomorphism (from an open set $U \subseteq \mathfrak{g}$ with $0 \in U$ to an open set $V \subseteq G$ with $I \in V$).*

Proof.

1. For $G = \mathrm{GL}(n, \mathbb{G})$ and if $\varphi\colon \mathbb{R} \to G$ is given by $\varphi(t) = \exp(At)$, we have that $\exp(At)$ is invertible, so φ indeed has codomain G. Also $\varphi(s + t) = \exp((s + t)A) = \exp(sA) \exp(tA) = \varphi(s)\varphi(t)$; so φ is a Lie grape homomorphism and $\varphi'(t) = \exp(At) \cdot A$, and in particular $\varphi'(0) = A$.

   When $G \subseteq \mathrm{GL}(n, \mathbb{R})$ with $A \in \mathfrak{g} \subseteq M_n(\mathbb{R})$, we let X and U be the left-invariant vector fields on G and $\mathrm{GL}(n, \mathbb{F})$ with $X_I = U_I = A$. Then for $P \in G$ we have $X_P = PA = U_P$ and if $\varphi\colon \mathbb{R} \to G$ and $\psi\colon \mathbb{R} \to \mathrm{GL}(n, \mathbb{R})$ are the two unique grape homomorphisms with $\varphi'(0) = A$ and $\psi'(0) = A$, then $\varphi'(t) = X_{\varphi(t)} = U_{\varphi(t)}$; thus since $\psi'(t) = U_{\psi(t)}$ the uniqueness theorem for differential equations

TODO 7. *ref*

we have $\varphi(t) = \psi(t)$ for all t. But we know from the previous paragraph that $\psi(t) = \exp(At)$; the result follows.

2. If $A \in \mathfrak{g}$ then $\varphi\colon \mathbb{R} \to G$ given by $\varphi(t) = \exp(At)$ is the Lie grape homomorphism with $\varphi'(0) = A$ (which is equal to the integral curve for the left-invariant vector field X with $X_I = A \in \mathfrak{g}$); so $\exp(At) \in G$ for all t.

   Conversely if $\exp(At) \in G$ for all t then $\alpha(t) = \exp(At)$ is a smooth curve in G with $\alpha(0) = I$. So $A = \alpha'(0) \in T_I G = \mathfrak{g}$.

3. Check that local inverse of $\exp\colon \mathfrak{g} \to G$ is $\log\colon V \subseteq G \to \mathfrak{g}$ given by

$$\log(I + A) = \sum_{k=1} \frac{(-1)^{k+1}}{k} A^k$$

   Alternatively, if we suppose that $\exp\colon \mathfrak{g} \to G$ is smooth (which isn't particularly easy to prove) then to show that exp is a local diffeomorphism by the inverse function theorem it suffices to show that $D\exp = \exp_*$ is invertible at $0 \in \mathfrak{g}$. (Here we do this for abstract Lie grapes, so exp may not have a nice concrete form.) We have that $\exp\colon \mathfrak{g} \to G$, so $\exp_*\colon T_0\mathfrak{g} \to T_I G$; under standard identifications we may write $\exp_*\colon \mathfrak{g} \to \mathfrak{g}$. The map $\exp\colon \mathfrak{g} \to G$ is defined as follows: given $A \in \mathfrak{g}$ we define $\exp(A) = \varphi(1)$ where $\varphi\colon \mathbb{R} \to G$ is the unique Lie grape homomorphism with $\varphi'(0) = A$. For $A \in \mathfrak{g}$ we defined $\exp_*(A)$ as follows: choose a smooth curve $\alpha\colon (-\varepsilon, \varepsilon) \to \mathfrak{g}$ with $\alpha(0) = 0$ and $\alpha'(0) = A$, and set $\exp_*(A) = \beta'(0)$ where $\beta(t) = \exp(\alpha(t))$. We choose $\alpha(t) = At$; so $\alpha(0) = 0$ and $\alpha'(0) = A$. Then $\exp_*(A) = \beta'(0)$ where $\beta(s) = \exp(As)$ for each $s \in \mathbb{R}$. By part (1) the unique $\psi_s\colon \mathbb{R} \to G$ with $\psi_s'(0) = As$ is given by $s \mapsto \exp(As)$.

   Note that if $\varphi_s\colon \mathbb{R} \to G$ is given by $\varphi_s(t) = \varphi(st)$ then φ_s is a Lie grape homomorphism, and $\varphi_s'(t) = \varphi'(st) \cdot s$; so $\varphi_s'(0) = \varphi'(0) \cdot s = As$. So by uniqueness we get $\psi_s(t) = \varphi_s(t) = \varphi(st)$. Thus

$$\beta(s) = \psi_s(1) = \varphi(s \cdot 1) = \varphi(s)$$

   so $\beta'(s) = \varphi'(s)$ for all s. So $\exp_*(A) = \beta'(0) = \varphi'(0) = A$. Since $\exp_*(A) = A$ for all A we get that $\exp_*\colon \mathfrak{g} \to \mathfrak{g}$ is the identity map. □ Corollary 4.11

Theorem 4.12. *The Lie algebras of the classical matrix grapes are as follows:*

$$\begin{aligned}
\mathfrak{gl}(n, \mathbb{R}) &= M_n(\mathbb{R}) \\
\mathfrak{sl}(n, \mathbb{R}) &= \{\, A \in M_n(\mathbb{R}) : \operatorname{tr}(A) = 0 \,\} \\
\mathfrak{o}(n, \mathbb{R}) &= \{\, A \in M_n(\mathbb{R}) : A^t + A = 0 \,\} \\
\mathfrak{so}(n, \mathbb{R}) &= \{\, A \in M_n(\mathbb{R}) : A^t + A = 0, \operatorname{tr}(A) = 0 \,\} \\
&= \{\, A \in M_n(\mathbb{R}) : A^t + A = 0 \,\} \\
&= \mathfrak{o}(n, \mathbb{R}) \\
\mathfrak{gl}(n, \mathbb{C}) &= M_n(\mathbb{C}) \\
\mathfrak{sl}(n, \mathbb{C}) &= \{\, A \in M_n(\mathbb{C}) : \operatorname{tr}(A) = 0 \,\} \\
\mathfrak{u}(n, \mathbb{C}) &= \{\, A \in M_n(\mathbb{C}) : A^* + A = 0 \,\} \\
\mathfrak{su}(n, \mathbb{C}) &= \{\, A \in M_n(\mathbb{C}) : A^* + A = 0, \operatorname{tr}(A) = 0 \,\} \\
\mathfrak{gl}(n, \mathbb{H}) &= M_n(\mathbb{H}) \\
\mathfrak{sp}(n, \mathbb{H}) &= \{\, A \in M_n(\mathbb{H}) : A^* A = I \,\}
\end{aligned}$$

Proof.

TODO 8. *Description? Transpose case?*

For $\mathfrak{sl}(n, \mathbb{R})$ we have

$$\mathfrak{sl}(n, \mathbb{R}) = \{\, A \in M_n(\mathbb{R}) : \exp(At) \in \mathrm{SL}(n, \mathbb{R}) \text{ for all } t \,\}$$

But

$$\begin{aligned}
& \exp(At) \in \mathrm{SL}(n, \mathbb{R}) \text{ for all } t \\
\Longrightarrow\ & \det(\exp(At) = 1 \text{ for all } t \\
\Longrightarrow\ & \exp(\mathrm{tr}(A)t = 1 \text{ for all } t \\
\Longrightarrow\ & \mathrm{tr}(A) \exp(\mathrm{tr}(A)t) = 0 \text{ for all } t
\end{aligned}$$

(taking derivatives). So $\mathrm{tr}(A) = 0$ (taking $t = 0$). Conversely

$$\begin{aligned}
& \mathrm{tr}(A) = 0 \\
\Longrightarrow\ & \mathrm{tr}(A)t = 0 \text{ for all } t \\
\Longrightarrow\ & \exp(\mathrm{tr}(A)t) = 1 \text{ for all } t \\
\Longrightarrow\ & \det(\exp(At)) = 1 \text{ for all } t \\
\Longrightarrow\ & \exp(At) \in \mathrm{SL}(n, \mathbb{R}) \text{ for all } t
\end{aligned}$$

FOr $\mathfrak{o}(n, \mathbb{R})$ we have

$$\mathfrak{o}(n, \mathbb{R}) = \{\, A \in M_n(\mathbb{R}) : \exp(At) \in \mathrm{O}(n, \mathbb{R}) \text{ for all } t \,\}$$

Now

$$\begin{aligned}
& \exp(At) \in \mathrm{O}(n, \mathbb{R}) \text{ for all } t \\
\Longleftrightarrow\ & (\exp(At))^T (\exp(At)) = I \text{ for all } t \\
\Longleftrightarrow\ & (\exp(A^T t)) \exp(At) = I \text{ for all } t \\
\Longrightarrow\ & \exp(A^T t) A^T \exp(At) + \exp(A^T t) \exp(At) A = 0 \text{ for all } t \\
\Longrightarrow\ & A^T + A = 0
\end{aligned}$$

Conversely

$$\begin{aligned}
& A^T + A = 0 \\
\Longrightarrow\ & A^T t = -At \text{ for all } t \\
\Longrightarrow\ & \exp(A^T t) = \exp(-At) = (\exp(At))^{-1} \text{ for all } t \\
\Longrightarrow\ & (\exp(At))^T (\exp(At)) = I \text{ for all } t \\
\Longrightarrow\ & \exp(At) \in \mathrm{O}(n, \mathbb{R}) \text{ for all } t
\end{aligned}$$

One does the rest oneself. ☐ Theorem 4.12

Corollary 4.13. *We easily obtain the dimensions of all classical matrix grapes.*

$$\begin{aligned}
\dim(\mathrm{GL}(n, \mathbb{R})) &= n^2 \\
\dim(\mathrm{SL}(n, \mathbb{R})) &= n^2 - 1 \\
\dim(\mathrm{O}(n, \mathbb{R})) &= \dim(\mathrm{SO}(n, \mathbb{R})) \\
&= \frac{n^2 - n}{2}
\end{aligned}$$

etc.

5 Connectedness

Theorem 5.1. *Suppose G is a Lie grape; let H be the connected component containing e. Then H is a Lie subgrape of G and the Lie algebra $\mathfrak{h}$ of H is equal to the Lie algebra $\mathfrak{g}$ of G. Also the connected components of G are the cosets of H (all of which are diffeomorphic to H).*

Proof. It suffices to show that H is a subgrape. Let $m\colon G \times G \to G$ be $m(a,b) = ab$ and $v\colon G \to G$ be $v(a) = a^{-1}$; so m, v are smooth. Since v is a diffeomorphism (equal to its own inverse) we get that $v(H)$ is a connected component containing e; so $v(H) = H$. So H is closed under inversion. Also if $a \in H$ the map $\ell_a\colon G \to G$ given by $\ell_a(x) = ax$ is a diffeomorphism (with inverse $\ell_{a^{-1}}$); so $\ell_a(H) = aH$ is a connected component containing e (since we showed above that $a^{-1} \in H$). So $aH = H$; so H is closed under multiplication (for all $b \in H$ we have $ab \in H$). □ Theorem 5.1

We recall Frobenius' theorem. Consider a simple PDE

$$\frac{\partial u}{\partial x} = F(x,y,u)$$
$$\frac{\partial u}{\partial y} = G(x,y,u)$$

One might try to solve this by separately solving each; this doesn't always work.

More generally, suppose M is a smooth manifold with $\dim(M) = m$. Suppose $X_1, \ldots, X_n$ are smooth vector fields on M. Let $V_p = \operatorname{span}\{\, X_1(p), \ldots, X_n(p) \,\}$. Suppose that $\dim(V_p) = n$ for all p. Then in order to have an n-dimensional manifold $N \subseteq M$ with $T_pN = V_p$ at all $p \in N$ we must have $[X_k, X_\ell]_p \in T_pN = V_p$ for all k, ℓ and all p. (By the theorem that $f_*[X,Y]_p = [U,V]_{f(p)}$.)

TODO 9. *ref*

(We can turn the previous problem into an instance of this by defining $X = (1, 0, F(x,y,u))$ and $Y = (0, 1, G(x,y,u))$.)

Theorem 5.2 (Frobenius' theorem)**.** *Suppose M is a smooth manifold and $X_1, \ldots, X_n$ are smooth vector fields on M. For $p \in M$ we let $V_p = \operatorname{span}\{\, X_1(p), \ldots, X_n(p) \,\}$. Suppose for $p \in M$ we have $\dim(V_p) = n$ and $[X_k, X_\ell]_p \in V_p$ for all k, ℓ. Then for each $q \in M$ there is a unique maximal connected smooth submanifold $N \subseteq M$ with $q \in M$ such that $T_pN = V_p$ for all $p \in N$.*

Such V_p are called *distributions*, such X_i are called *involutive*, and if such N exists it is the *integral submanifold* of the distribution.

Theorem 5.3. *Suppose G is a Lie subgrape of $\mathrm{GL}(n, \mathbb{F})$ for $\mathbb{F} \in \{\, \mathbb{R}, \mathbb{C} \,\}$. Then there is a bijective correspondence between connected (real) Lie subgrapes H of G and (real) Lie subalgebras $\mathfrak{h}$ of $\mathfrak{g}$.*

Proof. When H is a subgrape of G we have $\mathfrak{h} \subseteq \mathfrak{g} \subseteq M_n(\mathbb{F})$. (Indeed, we have $\mathfrak{h} = \{\, \alpha'(0) \in M_n(\mathbb{F}) \mid \alpha\colon (-\varepsilon, \varepsilon) \to H \subseteq G \subseteq M_n(\mathbb{F}) \,\}$.)

Suppose we are given a (real) Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g} \subseteq M_n(\mathbb{F})$. Pick a basis $\{\, A_1, \ldots, A_\ell \,\}$ for $\mathfrak{h}$ over $\mathbb{R}$. Let $X_1, \ldots, X_\ell$ be the left-invariant vector fields on $\mathrm{GL}(n, \mathbb{F})$ with $X_k(P) = PA_k$. (These restrict to left-invariant vector fields on G.) Then since $[A_i, A_j] = A_iA_j - A_jA_i \in \mathfrak{h}$ for all i, j. So $[X_i, X_j]_P = P[A_i, A_j] \in P\mathfrak{h}$ and $[X_i, X_j]_P \in \operatorname{span}\{\, PA_1, \ldots, PA_\ell \,\} = \operatorname{span}\{\, X_1(P), \ldots, X_\ell(P) \,\}$. So by Frobenius' theorem there is a unique maximal smooth submanifold of G with $I \in H$ and

$$T_PH = \operatorname{span}\{\, X_1(P), \ldots, X_\ell(P) \,\} = P\mathfrak{h}$$

for all $P \in H$.

To show that H is a Lie subgrape, it suffices to show that H is closed under multiplication and inversion. This is the same as the proof of Theorem 5.1. Indeed, let $v\colon\ \mathrm{GL}(n, \mathbb{F}) \to \mathrm{GL}(n, \mathbb{F})$ be $A \mapsto A^{-1}$. Note that v is a diffeomorphism, and note that for each vector field X_k we have $DL_PX_k = X_k$ for all $P \in \mathrm{GL}(n, \mathbb{F})$. So $DL_{P^{-1}}X_k = X_k$ for all $P \in \mathrm{GL}(n, \mathbb{F})$. It follows that $v(H)$ is also a maximal connected integral submanifold of G containing I, and hence $H = v(H)$ by uniqueness. A similar argument shows that H is closed under multiplication. □ Theorem 5.3

Example 5.4.

- $\mathrm{GL}(n,\mathbb{R})$ is not connected because it is the disjoint union of the two open subsets

$$\mathrm{GL}_+(n,\mathbb{R}) = \{\, A \in M_n(\mathbb{R}) : \det(A) > 0 \,\}$$
$$\mathrm{GL}_-(n,\mathbb{R}) = \{\, A \in M_n(\mathbb{R}) : \det(A) < 0 \,\}$$

  Note that $\mathrm{GL}_+(n,\mathbb{R})$ is a Lie subgrape of $\mathrm{GL}(n,\mathbb{R})$ of index 2, and its non-identity coset is $\mathrm{GL}_-(n,\mathbb{R})$.

- Similarly $\mathrm{O}(n,\mathbb{R})$ is the disjoint union of

$$\mathrm{O}_+(n,\mathbb{R}) = \{\, A \in \mathrm{O}(n,\mathbb{R}) : \det(A) = 1 \,\}$$
$$\mathrm{O}_-(n,\mathbb{R}) = \{\, A \in \mathrm{O}(n,\mathbb{R}) : \det(A) = -1 \,\}$$

  (Note $\mathrm{O}_+(n,\mathbb{R}) = \mathrm{SO}(n,\mathbb{R})$ is a Lie subgrape of $\mathrm{O}(n,\mathbb{R})$.)

Theorem 5.5. *The matrix grapes*

$$\mathrm{GL}_+(n,\mathbb{R}), \mathrm{SL}(n,\mathbb{R}), \mathrm{SO}(n,\mathbb{R}), \mathrm{GL}(n,\mathbb{C}), \mathrm{SL}(n,\mathbb{C}), \mathrm{U}(n,\mathbb{C}), \mathrm{SU}(n,\mathbb{C}), \mathrm{GL}(n,\mathbb{H}), \mathrm{Sp}(n,\mathbb{H})$$

are all connected.

Proof. Note that $\mathrm{GL}_+(n,\mathbb{R}) \cong \mathrm{SL}(n,\mathbb{R}) \times \mathbb{R}^+$ (as Lie grapes) with an isomorphism $\varphi\colon \mathrm{SL}(n,\mathbb{R}) \times \mathbb{R}^+ \to \mathrm{GL}_+(n,\mathbb{R})$ given by $(A,t) \mapsto tA$; hence the former is connected if and only if the latter is. Given $A \in \mathrm{GL}_+(n,\mathbb{R})$ with $\det(A) = a$ we can define a path $\alpha\colon [0,1] \to \mathrm{GL}_+(n,\mathbb{R})$ given by $\alpha(t) = (1 + (b-1)t)A$ is a path from A to $B = bA$ where $b = \frac{1}{\sqrt[n]{a}}$. Given $A \in \mathrm{SL}(n,\mathbb{R})$ we can perform the Gram-Schmidt procedure setting

$$\begin{aligned} v_1 &= u_1 \\ v_2 &= u_2 - \frac{u_2 \cdot v_1}{|v_1|^2} v_1 \\ v_k &= u_k - \sum_{j=1}^{k-1} \frac{u_k \cdot f_j}{|v_j|^2} v_j \end{aligned}$$

In particular, when expressing these operations as matrices, we get that $B = (I+U)A$ where $U = U(A)$ is strictly upper triangular. A path from A to B in $\mathrm{SL}(n,\mathbb{R})$ is given by $\alpha(t) = (I + tU)A$, where $B \in \mathrm{SL}(n,\mathbb{R})$ has determinant 1 and orthogonal columns; equivalently $B^T B = \mathrm{diag}(d_1, \dots, d_n)$ with $d_k > 0$ for all k and $\prod d_i = 1$. (Note that the set $\mathcal{A}$ of such matrices is not a grape.) Given $A \in \mathcal{A}$ we can scale the lengths of the columns: for $A = (v_1, \dots, v_n)$ if we let $b_k = \ln\|v_k\|$, we can define $\alpha\colon [0,1] \to \mathrm{SL}(n,\mathbb{R})$ by

$$\alpha(t) = \begin{pmatrix} \exp(-tb_1) & & \\ & \ddots & \\ & & \exp(-tb_n) \end{pmatrix} A$$

(Note that $\sum b_k = 0$, so $\sum -tb_k = 0$.) So α is a path from $A \in \mathcal{A}$ to $B \in \mathrm{SO}(n,\mathbb{R})$ (the columns of B are a positively oriented orthonormal basis for $\mathbb{R}^r$). Finally, it remains to show that given $A \in \mathrm{SO}(n,\mathbb{R})$ we can find a path from A to I in $\mathrm{SO}(n,\mathbb{R})$. We can do this using n rotations $R_1, \dots, R_n$. Say $A = (u_1, \dots, u_n)$, and let the e_i be the standard basis vectors. If $u_1 = e_1$, let $R_1 = I$; else let $R_1 = R_1(\theta)$ be the rotation in the plane spanned by u_1 and e_1 by the angle θ between u_1 and e_1; then $\alpha_1(t) = R_1(t\theta)A$ is a path from A to $B = (v_1, \dots, v_n$, where $v_k = R_1(\theta)u_k$ (so $v_1 = e_1$).

If $v_2 = e_2$, let $R_2 = I$; else let $R_2 = R_2(\theta_2)$ be the rotation in the plane spanned by v_2 and e_2 by the angle θ_2 between v_2 and e_2. Note that $R_2(\theta_2)$ fixes e_1 since e_1 is perpendicular to both e_2 and v_2. Repeat the procedure.

Note that

$$\begin{aligned} \mathrm{GL}_+(n, \mathbb{R}) &\cong \mathrm{SL}(n, \mathbb{R}) \times \mathbb{R}^+ \\ \mathrm{SL}(n, \mathbb{R}) &\cong \mathcal{A} \times \mathcal{U} \\ \mathcal{A} &\cong \mathrm{SO}(n, \mathbb{R}) \times D(n, \mathbb{R}) \end{aligned}$$

where $\mathcal{U}$ is the set of strictly upper triangular matrices and

$$D(n, \mathbb{R}) = \left\{ \operatorname{diag}(d_1, \ldots, d_n) : d_i > 0, \prod d_i = 1 \right\}$$

(The map $\varphi\colon \mathcal{A} \times \mathcal{U} \to \mathrm{SL}(n, \mathbb{R})$ is given by $\varphi(A, U) = (I + U)A$.) □ Theorem 5.5

6 Fundamental grape, simple connectedness, covering spaces

Definition 6.1. Suppose M is a smooth manifold and $a, b \in M$. A *path* from a to b is a continuous map $\alpha\colon [0,1] \to M$ with $a = \alpha(0)$ and $b = \alpha(1)$. A *loop* at a is a path from a to a. Given paths α, β from a to b in M, a *homotopy* from α to β in M is a continuous map $F\colon [0,1] \times [0,1] \to M$ such that for all s and t we have

- $F(0, t) = \alpha(t)$
- $F(1, t) = \beta(t)$
- $F(s, 0) = a$
- $F(s, 1) = b$

When such F exists we say α and β are *homotopic* in M, and write $\alpha \sim \beta$ in M.

Exercise 6.2. Check that $\sim$ is an equivalence relation.

Definition 6.3. For $a \in M$ let $\kappa = \kappa_a$ be the constant loop $\kappa_a(t) = a$ for all t. For a path α from a to b define α^{-1} be the corresponding path from b to a, given by $t \mapsto \alpha(1 - t)$. Given a path α from a to b and a path β from b to c we let $\alpha * \beta$ be the path from a to c given by

$$t \mapsto \begin{cases} \alpha(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ \beta(2t - 1) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

Exercise 6.4. Check that

- If $\alpha_1 \sim \alpha_2$ then $\alpha_1^{-1} \sim \alpha_2^{-1}$.
- If $\alpha_1 \sim \alpha_2$ and $\beta_1 \sim \beta_2$ then $\alpha_1 * \beta_1 \sim \alpha_2 * \beta_2$.
- $\alpha * \kappa \sim \alpha$
- $\kappa * \alpha \sim \alpha$
- $\alpha * \alpha^{-1} \sim \alpha^{-1} * \alpha \sim \kappa$
- $(\alpha * \beta) * \gamma \sim \alpha * (\beta * \gamma)$.

Definition 6.5. Suppose M is a (topological) manifold and $a \in M$. The *fundamental grape* of M (or *first homotopy grape*) of M at a, denoted $\pi_1(M, a)$, is the set of loops at a in M modulo homotopy equivalence.

Theorem 6.6 (Properties of the fundamental grape)**.**

1. *If M is a convex set in $\mathbb{R}^n$ and $a \in M$ then $\pi_1(M, a) = \{\kappa\} \cong 0$.*

2. *If N and M are path-connected then $\pi_1(N \times M, (a, b)) \cong \pi_1(N, a) \times \pi_1(M, b)$.*

3. *If $f\colon N \to M$ isa homeomorphism with $f(a) = b$ then $\pi_1(N, a) \cong \pi_1(M, b)$.*

4. *If γ is a path in M from a to b then γ induces an isomorphism $\varphi\colon \pi_1(M, a) \to \pi_1(M, b)$ given by $\varphi(\alpha) = \gamma^{-1} * \alpha * \gamma$.*

5. *$\pi_1(\mathbb{S}^1, 1) = \pi_1(\mathbb{C}^*, 1) = \{\alpha_n : n \in \mathbb{N}\} \cong \mathbb{Z}$ where $\alpha_n(t) = \exp(2\pi i n t)$.*

These are easy to prove except for the fifth; covering spaces may be the easiest way to see that. The Seifert-van Kampen theorem is another way to see it.

Definition 6.7. A topological manifold M is called *simply connected* when M is path connected (which is equivalent to connected for manifolds) and for some (hence for any) $a \in M$ we have $\pi_1(M, a) \cong 0$.

Definition 6.8. Suppose M, N are smooth manifolds. A map $\varphi\colon N \to M$ is called a *(smooth) covering map* when for every $p \in M$ there is an open neighbourhood $U \subseteq M$ of p such that $\varphi^{-1}(U)$ is a disjoint union

$$\varphi^{-1}(U) = \bigcup_{a \in A} V_\alpha$$

where each V_α is open in N and the restricted map $\varphi \restriction V_\alpha$ is a diffeomorphism $V_\alpha \to U$. A *(smooth) covering manifold* of M is a manifold N together with a smooth covering map $\varphi\colon N \to M$.

Definition 6.9. Suppose $\varphi\colon N \to M$ and $\psi\colon L \to M$ are (smooth) covering maps, a *homomorphism of covering spaces* from N to L is a smooth map $f\colon N \to L$ such that the following diagram commutes:

$$\begin{tikzcd} N \arrow[rr, "f"] \arrow[dr, "\varphi"'] & & L \arrow[dl, "\psi"] \\ & M & \end{tikzcd}$$

It's an *isomorphism* when f is a diffeomorphism.

Theorem 6.10. *Every connected smooth manifold M has a simply connected smooth covering manifold $\widetilde{M}$, which is unique up to covering space isomorphism. This $\widetilde{M}$ is called the* universal cover *of M.*

Theorem 6.11. *Suppose M is a smooth manifold and let $\widetilde{M}$ be the smooth universal cover with smooth covering map $\varphi\colon \widetilde{M} \to M$. Suppose N is a simply connected manifold N and $f\colon N \to M$ is a smooth map; suppose $a \in N$ and $c \in \varphi^{-1}(f(a))$. Then there is a unique smooth map $\widetilde{f}\colon N \to \widetilde{M}$ such that $\varphi \circ \widetilde{f} = f$ and $\widetilde{f}(a) = c$.*

Example 6.12.

1. The map $\varphi\colon \mathbb{S}^1 \to \mathbb{S}^1$ given by $\varphi(z) = z^n$ (i.e. $\exp(i\theta) \mapsto \exp(in\theta)$) is an n-to-1 covering map. The universal cover of $\mathbb{S}^1$ is $\mathbb{R}$ with the covering map $\varphi\colon \mathbb{R} \to \mathbb{S}^1$ given by $\varphi(\theta) = \exp(i\theta)$ (or $\exp(i2\pi\theta)$ if you prefer).

2. The map $\varphi\colon \mathbb{C}^* \to \mathbb{C}^*$ given by $\varphi(z) = z^n$ is a covering map. Note that $\mathbb{C}^* \cong \mathbb{R}^+ \times \mathbb{S}^1$ (with an isomorphism $f\colon \mathbb{R}^+ \times \mathbb{S}^1 \to \mathbb{C}^*$ given by $f(r, \exp(i\theta)) = r\exp(i\theta)$). The universal cover of $\mathbb{C}^*$ is $\mathbb{R}^+ \times \mathbb{R}$ with covering map $\varphi\colon \mathbb{R}^+ \times \mathbb{R} \to \mathbb{C}^*$ given by $\varphi(r, \theta) = r\exp(i\theta)$ (or $r\exp(2\pi i\theta)$).

Fact 6.13. *Homotopic loops are lifted to paths with the same endpoint, and any homotopy of said loops lifts to a homotopy of the lifted paths.*

TODO 10. *Is there some uniqueness I missed in the definition of the grape operation in the following?*

Theorem 6.14. *Suppose G is a connected Lie grape and $\widetilde{G}$ its universal cover. We can define operations on $\widetilde{G}$ making it into a Lie grape such that the covering map $\varphi\colon \widetilde{G} \to G$ is a Lie grape homomorphism. Given such grape operations we have that* $\ker(\varphi)$ *is a discrete subgrape of $Z(\widetilde{G})$ with $\pi_1(G) = \pi_1(G, e) \cong$* $\ker(\varphi)$.

Proof. In order for $\varphi\colon \widetilde{G} \to G$ to be a grape homomorphism, we require that $\varphi(\widetilde{e}) = e$ where $\widetilde{e} = e_{\widetilde{G}}$ and $e = e_G$; we also require that for $\widetilde{a}, \widetilde{b} \in \widetilde{G}$ if we choose paths $\widetilde{\alpha}, \widetilde{\beta}$ in $\widetilde{G}$ from $\widetilde{e}$ to the points $\widetilde{a}$ and $\widetilde{b}$ and we let $\alpha = \varphi \circ \widetilde{\alpha}$ and $\beta = \varphi \circ \widetilde{\beta}$ then we need

$$\varphi(\widetilde{\alpha}(t) \cdot \widetilde{\beta}(t)) = \varphi(\widetilde{\alpha(t)}) \cdot \varphi(\widetilde{\beta}(t)) = \alpha(t) \cdot \beta(t)$$

So if we let $\gamma = \alpha \cdot \beta$ (so $\gamma(t) = \alpha(t) \cdot \beta(t)$ for all t) and if we let $\widetilde{\gamma} = \widetilde{\alpha} \cdot \widetilde{\beta}$ then we need $\varphi(\widetilde{\gamma}(t)) = \gamma(t)$ for all t; i.e. we need that $\widetilde{\gamma}$ is the (unique) lift of γ at $\widetilde{e}$.

So we define multiplication on $\widetilde{G}$ as follows: choose $\widetilde{e} \in \varphi^{-1}(e)$, and then given $\widetilde{a}, \widetilde{b} \in \widetilde{G}$ we choose a path $\widetilde{\alpha}$ from e to a in $\widetilde{G}$ and a path $\widetilde{\beta}$ from e to b in $\widetilde{G}$. We then let $\alpha = \varphi \circ \widetilde{\alpha}$ and $\beta = \varphi \circ \widetilde{\beta}$; we then let $\gamma = \alpha \cdot \beta$ and $\widetilde{\gamma}$ be the unique lift of γ at $\widetilde{e}$ in $\widetilde{G}$ and then define $\widetilde{a} \cdot \widetilde{b} = \widetilde{\gamma}(1)$ in $\widetilde{G}$.

One checks that $\widetilde{a} \cdot \widetilde{b}$ does not depend on the choice of $\widetilde{\alpha}$ and $\widetilde{\beta}$. One also checks that this makes $\widetilde{G}$ into a grape and that φ is a grape homomorphism (and hence by smoothness φ is a morphism of Lie grapes). Finally one checks that this multiplication is smooth.

Claim 6.15. *$K = \ker(\varphi)$ is a discrete subgrape of $Z(\widetilde{G})$.*

Proof. From the definition of a covering, it is clear that the kernel is discrete. Also K is a normal subgrape as the kernel of a grape homomorphism. So for all $a \in \widetilde{G}$ and $k \in K$ we have $aka^{-1} \in K$. Fix $k \in K$; define $g\colon \widetilde{G} \to K$ by $g(a) = aka^{-1}$. Since g is continuous and $\widetilde{G}$ is connected the image $g(\widetilde{G})$ is connected in K. But K is disrete; so $g(\widetilde{G})$ is a singleton. But $g(\widetilde{e}) = k$; so $g(\widetilde{G}) = \{ k \}$. So $aka^{-1} = k$ for all $a \in \widetilde{G}$, and $k \in Z(\widetilde{G})$. So $K \subseteq Z(\widetilde{G})$. □ Claim 6.15

It remains to check that $\pi_1(G) \cong K = \ker(\varphi)$. Define $\lambda\colon \pi_1(G)\colon K$ by $\lambda(\alpha) = \widetilde{\alpha}(1)$ where $\widetilde{\alpha}$ is the unique lift of α at $\widetilde{e}$ in $\widetilde{G}$. One checks that this is an isomorphism of grapes. □ Theorem 6.14

Missing stuff.

continued. Recall that given $\widetilde{a}, \widetilde{b} \in \widetilde{G}$ we choose paths $\widetilde{\alpha}, \widetilde{\beta}$ in $\widetilde{G}$ from $\widetilde{e}$ to $\widetilde{a}$ and $\widetilde{b}$ and then we let $\alpha = \varphi \circ \widetilde{\alpha}$ and

something something

Claim 6.16. *The lifting map $\lambda\colon \pi_1(G) \to K = \ker(\varphi)$ given by $\lambda(\alpha) = \widetilde{\alpha}(1)$ is a grape isomorphism.*

Proof. λ is well-defined since if α an $\mathrm{d}\beta$ are loops at e in G with $\alpha \sim \beta$ in G then a homotopy F from α to β lifts to a homotopy $\widetilde{F}$ from $\widetilde{\alpha}$ to $\widetilde{\beta}$ in $\widetilde{G}$, so $\widetilde{\alpha} \sim \widetilde{\beta}$ in $\widetilde{G}$. So we have $\widetilde{\alpha}(1) = \widetilde{\beta}(1)$ in $\widetilde{G}$ and also $\varphi(\widetilde{\alpha}(1)) = \alpha(1) = e$; hence $\lambda(\alpha) = \widetilde{\alpha}(1) \in \ker(\varphi)$.

λ is surjective because $\widetilde{G}$ is path-connected. (So given $\widetilde{\alpha} \in K = \ker(\varphi) = \varphi^{-1}(e)$ we can choose a path $\widetilde{\alpha}$ from $\widetilde{e}$ to $\widetilde{a}$ in $\widetilde{G}$ and then for $\alpha = \varphi \circ \widetilde{\alpha}$ we have that α is a loop at e and $\lambda(\alpha) = \widetilde{\alpha}(1) = \widetilde{a}$.)

λ is injective because $\widetilde{G}$ is simply connected. (So if $\lambda(\alpha) = \widetilde{\alpha}(1) = \widetilde{\beta}(1) = \lambda(\beta)$ then $\widetilde{\alpha} \sim \widetilde{\beta}$ in $\widetilde{G}$ and a homomotopy $\widetilde{F}$ from $\widetilde{\alpha}$ to $\widetilde{\beta}$ gives a homotopy $F = \varphi \circ \widetilde{F}$ from α to β in G; so $\alpha \sim \beta$, and $\alpha = \beta$ in $\pi_1(G)$.)

Finally, note that λ is a grape homomorphism because given loops α, β at e in G we have $\alpha \cdot \beta \sim (\alpha * \kappa) \cdot (\kappa * \beta) = \alpha * \beta$; hence $\lambda(\alpha * \beta) = \lambda(\alpha) \cdot \lambda(\beta) = \widetilde{\alpha}(1) \cdot \widetilde{\beta}(1)$. □ Claim 6.16

□

Theorem 6.17. *Suppose G is connected. Suppose $\varphi\colon H \to G$ is a Lie grape homomorphism. Then φ is a covering map if and only if $\varphi_* = \mathrm{d}\varphi$ is invertible (say at $e \in H$).*

Proof.

($\implies$) If φ is a covering map then φ is a local diffeomorphism; so φ_* is invertible.

($\impliedby$) Suppose φ_* is invertible (at e); so φ is a local diffeomorphism (by the inverse function theorem). Suppose $U_0 \subseteq G$ is open with $e_G \in U_0$; then we can pick open $V_0 \subseteq H$ with $e_H \in V_0$ such that the restriction $\varphi\colon V_0 \to U_0$ is a diffeomorphism. Choose $U \subseteq U_0$ contianing e_G such that U is connected and open with $U \cdot U^{-1} = \{ ab^{-1} : a, b \in U \} \subseteq U_0$ and let $V = \varphi^{-1}(U) \cap V_0$ (so V is connected and open and $VV^{-1} \subseteq V_0$).

Claim 6.18. *$\varphi^{-1}(U)$ is the disjoint union*

$$\varphi^{-1}(U) = \bigsqcup_{k \in K} k \cdot V$$

where $K = \ker(\varphi) = \varphi^{-1}(e_G)$.

Proof.

($\subseteq$) Suppose $a \in \varphi^{-1}(U)$ so $u = \varphi(a) \in U$; so there is (unique) $v \in V$ such that $\varphi(v) = u$. Then $\varphi(av^{-1}) = \varphi(a)\varphi(v)^{-1} = uu^{-1} = e_G$; so $k = av^{-1} \in K = \ker(\varphi)$, and hence $a = kv \in kV$.

($\supseteq$) If $b = kv$ for some $k \in K$ and $v \in V$ then $\varphi(b) = \varphi(kv) = \varphi(k)\varphi(v) = e_G\varphi(v) = \varphi(v) \in U$.

(Disjoint) If $kv = \ell w$ for $k, \ell \in K$ and $v, w \in V$ then $\varphi(v) = \varphi(kv) = \varphi(\ell w) = \varphi(w)$, and $v = w$; hence $kv = \ell v$ and $k = v$. □ Claim 6.18

Claim 6.19. *φ is surjective.*

Proof. Let $L = \langle U \cap U^{-1} \rangle$ be the subgrape of G generated by $U \cap U^{-1}$; that is

$$\langle U \cap U^{-1} \rangle = \{\, u_1 u_2 \cdots u_n : n \in \mathbb{Z}^+,\ \text{each } u_k \in U \cap U^{-1} \,\} = \bigcup_{n=1}^{\infty} (U \cap U^{-1})^n$$

Note that $\langle U \cap U^{-1} \rangle$ is open in G, and the cosets $a\langle U \cap U^{-1} \rangle$ are also open in G, and G is the disjoint union of the cosets. But G is connected; so there is only one coset. Thus $\langle U \cap U^{-1} \rangle = G$. Given $b \in G$ we can choose $u_1, \ldots, u_n \in U \cap U^{-1}$ so that $b = u_1 \cdots u_n$; for each k choose $v_k \in V$ with $\varphi(v_k) = u_k \in U$. So $\varphi(v_1 v_2 \cdots v_n) = \varphi(v_1) \cdots \varphi(v_n) = u_1 \cdots u_n = b$. So φ is surjective as claimed. □ Claim 6.19

Finally one checks that given $b \in G$ if we choose $a \in H$ so that $\varphi(a) = b$ then $\varphi^{-1}(b \cdot U)$ is the disjoint union

$$\varphi^{-1}(bU) = \bigsqcup_{k \in K} k \cdot aV$$

The result follows. □ Theorem 6.17

TODO 11. *Add the following to an earlier theorem?*

Theorem 6.20 (Another property of the exponential map)**.** *Suppose H, G are matrix Lie grapes; suppose $\varphi \colon H \to G$ is a homomorphism of Lie grapes. Then the following diagram commutes:*

$$\begin{array}{ccc} \mathfrak{h} & \xrightarrow{\varphi_*} & \mathfrak{g} \\ \downarrow{\scriptstyle\exp} & & \downarrow{\scriptstyle\exp} \\ H & \xrightarrow{\varphi} & G \end{array}$$

i.e. $\exp \circ \varphi_* = \varphi \circ \exp$ as maps $\mathfrak{h} \to G$.

Proof. We need to show that $\varphi(\exp(A)) = \exp(D\varphi \cdot A)$ for $A \in \mathfrak{h}$. We shall show that

$$\varphi(\exp(tA)) = \exp(t \cdot D\varphi \cdot A)$$

for $A \in \mathfrak{h}$ and $t \in \mathbb{R}$. (Here $D\varphi = \mathrm{d}\varphi$ but we're working with matrices so it's capital.) We shall do this by showing that $\gamma(t) = \varphi(\exp(tA))$ is the integral curve of the left-invariant vector field X on G with $X_I = D\varphi(I) \cdot A$. By a previous theorem

TODO 12. *ref*

this integral curve is $\exp(t \cdot d\Phi \cdot A)$, so this will suffice.

We need to show that $\gamma'(t) = X_{\gamma(t)} = \gamma(t) D\varphi(I) \cdot A$. For all s, t we have

$$\varphi(\exp((s+t)A)) = \varphi(\exp(sA)\exp(tA) = \varphi(\exp(sA))\varphi(\exp(tA))$$

So for fixed s we have

$$D\varphi(\exp((s+t)A))\exp((s+t)A) \cdot A = \varphi(\exp(sA))D\varphi(\exp(tA)) \cdot \exp(tA) \cdot A$$

for all t. In particular putting in $t = 0$ gives $D\varphi(\exp(sA) \cdot \exp(sA)A = \varphi(\exp(sA)D\varphi(I)A$; that is $\gamma'(s) = \gamma(s)D\varphi(I)A$ as required. $\square$ Theorem 6.20

Theorem 6.21. *Suppose H and G are matrix Lie grapes. Suppose H is simply connected. Then there is a bijective correspondence between Lie grape homomorphisms $\varphi\colon H \to G$ and Lie algebra homomorphisms $\psi\colon \mathfrak{h} \to \mathfrak{g}$ such that when φ and ψ correspond we have $\psi = \varphi_* = \mathrm{d}\varphi = D\varphi$.*

Proof. Suppose $\varphi\colon H \to G$ is a homomorphism of Lie grapes. Suppose $A, B \in \mathfrak{h}$. Let X, Y be the left-invariant vector fields on H with $X_I = A$ and $Y_I = B$. Let U, V be the left-invariant vector fields on G with $U_I = \varphi_* A$ and $V_I = \varphi_* B$. (More precisely $U_I = \varphi_{*,I}(A) = D\varphi(I)A$, but we'll omit the I unless we need it.) We show that $\varphi_* X_p = D\varphi X_p = U_{\varphi(p)}$.

We have $D\varphi \cdot X_p = D\varphi \cdot PA = D\varphi \cdot DL_P A = D(\varphi \circ L_P)A$. But

$$\varphi(L_P(Q)) = \varphi(PQ) = \varphi(P)\varphi(Q) = L_{\varphi(P)}(\varphi(Q)) = (L_{\varphi(P)} \circ \varphi)(Q)$$

So

$$D\varphi \cdot X_p = D(L_{\varphi(P)} \circ \varphi)A = (DL_{\varphi(P)} \circ D\varphi)A = DL_{\varphi(P)}U_I = U_{\varphi(P)}$$

Since $D\varphi \cdot X_P = U_{\varphi(P)}$ and $D\varphi \cdot Y_P = V_{\varphi(P)}$ for all $P \in H$ we have

$$D\varphi[X, Y]_P = [U, V]_{\varphi(P)}$$

TODO 13. *ref*

So in particular $D\varphi[X, Y]_I = [U, V]_I$. Thus $\varphi_* = D\varphi$ is a Lie algebra homomorphism.

Claim 6.22. *Given two Lie grape homomorphisms $\varphi_1, \varphi_2\colon H \to G$ if $D\varphi_1(I) = D\varphi_2(I)$ then $\varphi_1 = \varphi_2$.*

Proof. We use the fact that the following diagram commutes:

$$\begin{tikzcd}
\mathfrak{h} \arrow[r, "{D\varphi_1, D\varphi_2}"] \arrow[d, "\exp"] & \mathfrak{g} \arrow[d, "\exp"] \\
H \arrow[r, "{\varphi_1, \varphi_2}"] & G
\end{tikzcd}$$

Suppose $D\varphi_1 = D\varphi_2$ (at $I \in H$). Then we have

$$\varphi_1(\exp(A)) = \exp(D\varphi_1 \cdot A = \exp(D\varphi_2 \cdot A = \varphi_2(\exp(A)$$

for all $h \in \mathfrak{h}$. If we had $\exp\colon \mathfrak{h} \to H$ surjective then we would be done; alas, this is not necessarily the case.

When $\exp\colon \mathfrak{h} \to H$ is not surjective, the image $\exp(\mathfrak{h})$ still generates H. Indeed, since $\exp\colon \mathfrak{h} \to H$ is a local diffeomorphism if we choose open $U \subseteq \mathfrak{h}$ and $V \subseteq H$ with $0 \in U$ and $I \in V$ such that $\exp\colon U \to V$ is a diffeomorphism then (as seen previously) the grape $\langle V \cap V^{-1} \rangle$ is an open subgrape of H with all cosets open, and hence is equal to H since H is connected.

It follows that when $D\varphi_1 = D\varphi_2$. Indeed, if $P \in H$ we can choose $A_1, \ldots, A_n \in \mathfrak{h}$ such that $P = \exp(A_1) \cdots \exp(A_n)$; then

$$\begin{aligned}
\varphi_1(P) &= \varphi_1\left(\prod_k \exp(A_k)\right) \\
&= \prod_k \varphi_1(\exp(A_k) \\
&= \prod_k \exp(D\varphi_1 \cdot A_k) \\
&= \prod_k \exp(D\varphi_2 \cdot A_k) \\
&= \prod_k \varphi_2(\exp(A_k) \\
&= \varphi_2\left(\prod_k \exp(A_k)\right)
\end{aligned}$$

as desired. □ Claim 6.22

Finally, we check that our correspondence is surjective. Suppose $\psi \colon \mathfrak{h} \to \mathfrak{g}$ is a Lie algebra homomorphism. Say $H \subseteq \mathrm{GL}(n, \mathbb{F})$ and $G \subseteq \mathrm{GL}(m, \mathbb{F})$ where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Then

$$H \times G \approx \left\{ \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} : P \in H, Q \in G \right\} \subseteq \mathrm{GL}(n+m, \mathbb{F})$$

has Lie algebra

$$\mathfrak{h} \oplus \mathfrak{g} \approx \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : A \in \mathfrak{h}, B \in \mathfrak{g} \right\} \subseteq M_{n+m}(\mathbb{F})$$

Let

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & 0 \\ 0 & \psi A \end{pmatrix} : A \in H \right\} \subseteq \mathfrak{h} \oplus \mathfrak{g}$$

Note that $\mathfrak{k}$ is a Lie subalgebra because

$$\begin{aligned}
\left[\begin{pmatrix} A & 0 \\ 0 & \psi A \end{pmatrix}, \begin{pmatrix} B & 0 \\ 0 & \psi B \end{pmatrix} \right] &= \begin{pmatrix} [A, B] & 0 \\ 0 & [\psi A, \psi B] \end{pmatrix} \\
&= \begin{pmatrix} [A, B] & 0 \\ 0 & \psi[A, B] \end{pmatrix}
\end{aligned}$$

Let K be the unique connected Lie subgrape of $H \times G$ with Lie algebra $\mathfrak{k}$. Let $\varphi_H \colon K \to H$ and $\varphi_G \colon K \to G$ be the projection maps; i.e.

$$\begin{aligned}
\varphi_H \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} &= P \\
\varphi_G \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} &= Q
\end{aligned}$$

Since φ_H and φ_G are linear, they are equal to their derivatives (as linear maps). So

$$\begin{aligned}
D\varphi_H \begin{pmatrix} A & 0 \\ 0 & \psi A \end{pmatrix} &= A \\
D\varphi_G \begin{pmatrix} A & 0 \\ 0 & \psi A \end{pmatrix} &= \psi A
\end{aligned}$$

Note that $D\varphi_H$ is invertible, and thus

TODO 14. *ref*

$\varphi_H\colon K \to H$ is a covering map. Since $\pi_1(H) = 0$ we get $\ker(\varphi_H) = \pi_1(H) = 0$.

Editor's note 6.23. I don't think this follows formally from Theorem 6.14, since we don't yet know that φ_H is universal. However I think one can check directly that being a covering space of a simply connected space (like H) implies simple connectedness.

So $\varphi_H\colon K \to H$ is an isomorphism of Lie grapes.

We define $\varphi\colon H \to G$ to be $\varphi_G \circ \varphi_H^{-1}$. Then $D\varphi = D\varphi_G \circ D\varphi_H^{-1}$; i.e.

$$D\varphi(A) = D\varphi_G(D\varphi_H^{-1}(A)) = D\varphi_G \begin{pmatrix} A & 0 \\ 0 & \psi A \end{pmatrix} = \psi A$$

So $D\varphi = \psi$. □ Theorem 6.21

6.1 Fundamental grapes of classical matrix grapes

We know that $\mathrm{GL}_+(n,\mathbb{R})$ retracts $\mathrm{SL}(n,\mathbb{R})$ and that $\mathrm{SL}(n,\mathbb{R})$ retracts (using Gram-Schmidt) $\mathrm{SO}(n)$. Indeed we have diffeomorphisms

$$\begin{aligned} \mathrm{GL}_+(n,\mathbb{R}) &\cong \mathrm{SL}(n,\mathbb{R}) \times \mathbb{R}^+ \\ (n,\mathbb{R}) &\cong \mathrm{SO}(n) \times \mathbb{R}^{\frac{n^2-n}{2}+(n-1)} \end{aligned}$$

It follows that

$$\pi_1(\mathrm{GL}(n,\mathbb{R})) \cong \pi_1(\mathrm{GL}_+(n,\mathbb{R})) \cong \pi_1(\mathrm{SL}(n,\mathbb{R})) \cong \pi_1(\mathrm{O}(n)) = \pi_1(\mathrm{SO}(n))$$

So we compute the fundamental grapes of $\mathrm{SO}(n)$.

$$\begin{aligned}
\mathrm{SO}(1) &= \{\, 1 \,\} \\
\pi_1(\mathrm{SO}(1)) &= 0 \\
\mathrm{SO}(2) &= \left\{\, R_\theta \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R}/2\pi\mathbb{Z} \,\right\} \\
&\cong \mathbb{R}/2\pi\mathbb{Z} \\
&\cong \mathbb{S}^1 \\
\pi_1(\mathrm{SO}(2)) &\cong \pi_1(\mathbb{S}^1) \\
&\cong \mathbb{Z} \\
\mathrm{SO}(3) &= \{\, R_{u,\theta} : |u| = 1, \theta \in [0,\pi], R_{u,0} = I \text{ and } R_{u,\pi} = R_{-u,\pi} \text{ for all } u \,\} \\
&\cong \overline{B}(0,\pi)/\sim \ \text{ where when } |u| = \pi \text{ we have } u \sim -u \\
&\cong \mathbb{P}^3(\mathbb{R}) \\
\pi_1(\mathrm{SO}(3)) &\cong \pi_1(\mathbb{P}^3(\mathbb{R})) \\
&\cong \mathbb{Z}/2\mathbb{Z}
\end{aligned}$$

The last fact is hard to see without some more algebraic topology. The Seifert-van Kampen theorem helps. Alternatively, the 3-sphere is apparently a 2-to-1 covering space for $\mathbb{P}^3(\mathbb{R})$, which we can find some clever way to endow with a Lie grape structure.

Aside 6.24. Note that the map $\exp\colon \mathfrak{so}(3) \to \mathrm{SO}(3)$ sends

$$\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \mapsto R_{\widehat{u},\theta}$$

where

$$u = \begin{pmatrix} -c \\ b \\ -a \end{pmatrix}$$

and $\widehat{u} = \frac{u}{\|u\|}$ and $\theta = |u|$.

If $n \geq 3$ then $\mathrm{SO}(n+1)$ acts on $\mathbb{R}^{n+1}$ and if $A = (u_1, \dots, u_{n+1}) \in \mathrm{SO}(n+1)$ then $Ae_{n+1} = u_{n+1} \in \mathbb{S}^n$. Then

$$\begin{aligned}
\mathrm{orb}(e_{n+1}) &= \mathbb{S}^n \\
\mathrm{stab}(e_{n+1}) &= \{\, A = (u_1, \dots, u_{n+1}) \in \mathrm{SO}(n+1) : u_{n+1} = e_{n+1} \,\} \\
&= \left\{ \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} : B \in \mathrm{SO}(n) \right\}
\end{aligned}$$

By the orbit stabilizer theorem we have $\mathrm{SO}(n+1)/\mathrm{SO}(n) \cong \mathbb{S}^n$. This gives a fibre bundle

$$\begin{tikzcd}
\mathrm{SO}(n) \arrow[r] & \mathrm{SO}(n+1) \arrow[d] \\
& \mathbb{S}^n
\end{tikzcd}$$

From the fibre bundle we obtain a long exact sequence of homotopy grapes

$$\cdots \to \pi_2(\mathrm{SO}(n)) \to \pi_2(\mathrm{SO}(n+1)) \to \underbrace{\pi_2(\mathbb{S}^n)}_{=0} \to \pi_1(\mathrm{SO}(n)) \to \pi_1(\mathrm{SO}(n+1)) \to \underbrace{\pi_1(\mathbb{S}^n)}_{=0} \to \pi_0(\mathrm{SO}(n)) \to \cdots$$

The Hurewicz isomorphism gives $\pi_q(\mathbb{S}^n) = 0$ for $1 \leq q < n$ and $\pi_n(\mathbb{S}^n) = \mathbb{Z}$. For $n \geq 3$ we use the above sequence to see that $\pi_1(\mathrm{SO}(n)) \cong \pi_1(\mathrm{SO}(n+1))$; hence $\pi_1(\mathrm{SO}(n)) \cong \mathbb{Z}/2\mathbb{Z}$ for $n \geq 3$.

Similarly we have diffeomorphisms $\mathrm{GL}(n, \mathbb{C}) \cong \mathrm{SL}(n, \mathbb{C}) \times \mathbb{C}^*$ and $\mathrm{U}(n) \cong \mathrm{SU}(n) \times \mathbb{S}^1$ and by Gram-Schmidt we have $\mathrm{SL}(n, \mathbb{C}) \cong \mathrm{SU}(n) \times \mathbb{R}^m$.

TODO 15. *what?*

So we have $\pi_1(\mathrm{GL}(n, \mathbb{C})) = \pi_1(\mathrm{U}(n)) = \pi_1(\mathrm{SU}(n)) \times \mathbb{Z}$ and $\pi_1(\mathrm{SL}(n, \mathbb{C})) = \pi_1(\mathrm{SU}(n))$. So we solve for $\pi_1(\mathrm{SU}(n))$.

($n = 1$) $\mathrm{SU}(1) = \{\, 1 \,\}$.

($n = 2$) We have

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} a & c \\ b & d \end{pmatrix} : a, b, c, d \in \mathbb{C}, |a|^2 + |b|^2 = |c|^2 + |d|^2 = 1, ad - b = 1, a\overline{c} + b\overline{d} = 0 \right\}$$

For $\begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \mathrm{SU}(2)$ we have

$$\begin{pmatrix} -b & a \\ \overline{a} & \overline{b} \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

So

$$\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} -\overline{b} & a \\ \overline{a} & b \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\overline{b} \\ \overline{a} \end{pmatrix}$$

So

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} |a|^2 + |b|^2 = 1 \right\} \cong \mathbb{S}^3$$

We have $\pi_1(\mathrm{SU}(2)) = \pi_1(\mathbb{S}^3) = 0$.

(Larger n) We have $\mathrm{SU}(n+1)$ acts on $\mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$. For $A = (u_1, \dots, u_{n+1})$ we have $Ae_{n+1} = u_{n+1} \in \mathbb{S}^{2n+1}$; and $Ae_{n+1} = e_{n+1}$ if and only if $u_{n+1} = e_{n+1}$. So

$$\mathrm{stab}(e_{n+1}) = \left\{ \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} : B \in \mathrm{SU}(n) \right\}$$

Hence $\mathrm{SU}(n+1)/\mathrm{SU}(n) \cong \mathbb{S}^{2n+1}$ and from the fibre bundle we obtain the long exact sequence. For $n \geq 1$ we have

$$0 = \pi_2(\mathbb{S}^{2n+1}) \to \pi_1(\mathrm{SU}(n)) \to \pi_1(\mathrm{SU}(n+1)) \to \pi_1(\mathbb{S}^{2n+1}) = 0$$

so that $\pi_1(\mathrm{SU}(n+1)) = \pi_1(\mathrm{SU}(n))$. So $\pi_1(\mathrm{SU}(n)) = 0$ for $n \geq 1$.

What of $\mathrm{Sp}(n)$?

$(n = 1)$ $\mathrm{Sp}(1) = \{ u \in \mathbb{H}^1 : |u| = 1 \} = \mathbb{S}^3$; so $\pi_1(\mathrm{Sp}(1)) = 0$.

$(n \geq 2)$ We have a fibre bundle $\mathrm{Sp}(n) \hookrightarrow \mathrm{Sp}(n+1) \to \mathbb{S}^{4n+3}$, whence we obtain an exact sequence

$$\cdots \to \underbrace{\pi_2(\mathbb{S}^{4n+3})}_{=0} \to \pi_1(\mathrm{Sp}(n)) \to \pi_1(\mathrm{Sp}(n+1) \to \underbrace{\pi_1(\mathbb{S}^{4n+3})}_{=0} \to \cdots$$

Hence $\pi_1(\mathrm{Sp}(n)) = 0$ for all $n \geq 1$.

So $\mathrm{SU}(n)$ and $\mathrm{Sp}(n)$ are simply connected, and hence are equal to their own universal covers. But for $n \geq 3$ we have $\pi_1(\mathrm{SO}(n)) \cong \mathbb{Z}/2\mathbb{Z}$. So $\mathrm{SO}(n)$ has a two-to-one universal covering space, which we call the *spin grape*, denoted $\mathrm{Spin}(n)$. When $n = 3$ we have $\mathrm{SO}(3) \cong \mathbb{P}^3$ and $\mathbb{P}^3$ has universal covering space $\mathbb{S}^3$. We also have diffeomorphisms $\mathrm{SU}(2) \cong \mathrm{Sp}(1) \cong \mathbb{S}^3$.

Exercise 6.25. Find the covering map $\varphi\colon \mathrm{SU}(2) \to \mathrm{SO}(3)$ (or $\mathrm{Sp}(1) \to \mathrm{SO}(3)$).

7 Abelian Lie grapes and abelian Lie algebras

Definition 7.1. A Lie grape G is *abelian* when $ab = ba$ for all $a, b \in G$. A Lie algebra $\mathfrak{g}$ is *abelian* when $[A, B] = 0$ for all $A, B \in \mathfrak{g}$.

Theorem 7.2. *Suppose G is a connected matrix Lie grape with Lie algebra $\mathfrak{g}$. Then G is abelian if and only if $\mathfrak{g}$ is abelian.*

Proof.

($\implies$) Suppose G is abelian; suppose $A, B \in \mathfrak{g}$. Then $\exp(sA)\exp(tB) = \exp(tB)\exp(sA)$. Differentiate with respect to s to get $\exp(sA) \cdot A\exp(tB) = \exp(tB)\exp(sA)A$; putting in $s = 0$ we get $A\exp(tB) = \exp(tB)A$. Differentiate this with respect to t to get $A\exp(tB)B = \exp(tB)BA$; putting in $t = 0$ we get $AB - BA = 0$, so $[A, B] = 0$.

($\impliedby$) Suppose $\mathfrak{g}$ is abelian. Note that for $A, B \in \mathfrak{g}$ since $AB - BA = [A, B] = 0$ we have $\exp(A)\exp(B) = \exp(A + B) = \exp(B + A) = \exp(B)\exp(A)$. But we saw in the proof of Claim 6.22 that a connected Lie grape is generated by exponentials; so G is generated by $\exp(\mathfrak{g})$. Then given $P, Q \in G$ we can choose $A_1, \ldots, A_n, B_1, \ldots, B_m \in \mathfrak{g}$ so that

$$P = \prod_k \exp(A_k)$$
$$Q = \prod_\ell \exp(B_\ell)$$

and then

$$\begin{aligned} PQ &= \prod_k \exp(A_k) \prod_\ell \exp(B_\ell) \\ &= \exp\left(\sum_k A_k + \sum_\ell B_\ell\right) \\ &= \exp\left(\sum_\ell B_\ell + \sum_k A_k\right) \\ &= \prod_\ell \exp(B_\ell) \prod_k \exp(A_k) \\ &= QP \end{aligned}$$

as desired. □ Theorem 7.2

Definition 7.3. An *(integral) lattice* in a finite dimensional vector space V over $\mathbb{R}$ is a set (a free abelian grape) of the form $\Lambda = \operatorname{span}_{\mathbb{Z}}\{\, u_1, \dots, u_\ell \,\}$ for some linearly independent (over $\mathbb{R}$) vectors $u_1, \dots, u_\ell$.

Note that every lattice in V is discrete. Indeed, the point

$$a = \sum_{i=1}^{\ell} k_i u_i$$

with $k_i \in \mathbb{Z}$ can be separated from the other points in Λ using the open set

$$U = \left\{ \sum_{i=1}^{n} t_i u_i : |t_i - k_i| < 1 \text{ for all } 1 \le i \le \ell \right\}$$

where we extend $\{\, u_1, \dots, u_\ell \,\}$ to a basis $\{\, u_1, \dots, u_n \,\}$ for V.

Theorem 7.4. *Every discrete subgrape of a finite dimensional real vector space is a lattice.*

Proof. Suppose V be a finite dimensional vector space over $\mathbb{R}$ and Γ a discrete subgrape of V.

Claim 7.5. Γ *is closed.*

Proof. Suppose not; choose $x \in \overline{\Gamma} \setminus \Gamma$. Choose an open neighbourhood U of 0 which contains no other points in Γ. Choose an open $U_0 \subseteq U$ such that $0 \in U_0$ and $a - b \in U$ for all $a, b \in U_0$. Choose distinct $y, z \in (x + U_0) \cap \Gamma$; say $y = x + a$ and $z = x + b$. Then $y - a = z - b$, and $y - z = a - b \in U \cap \Gamma = \{\, 0 \,\}$, a contradiction. □ Claim 7.5

Let $W = \operatorname{span}_{\mathbb{R}}(\Gamma) \subseteq V$; pick a basis $\{\, w_1, \dots, w_\ell \,\}$ for W with each $w_k \in \Gamma$. Let $\Lambda = \operatorname{span}_{\mathbb{Z}}\{\, w_1, \dots, w_\ell \,\} \subseteq \Gamma$. Note that W is the disjoint union of the sets $a + P$ where $a \in \Lambda$ and

$$P = \left\{ \sum_{i=1}^{\ell} t_i w_i : 0 \le t_i < 1 \right\}$$

Claim 7.6. Γ/Λ *is finite.*

Proof. Let $K = \Gamma/\Lambda$; for each $k \in K$ choose a representative $r_k \in \Gamma$ (so $\Gamma/\Lambda = \{\, r_k + \Lambda : k \in K \,\}$). For $k \in K$ write $r_k = a_k + p_k$ where $a_k \in \Lambda$ and $p_k \in P$. Since $p_k = r_k - a_k \in \Gamma$ and Γ is closed and discrete, and since $p_k \in \overline{P}$ and $\overline{P}$ is compact, it follows that there are only finitely many p_k. Also for $k, \ell \in \Gamma/\Lambda$ if we had $p_k = p_\ell$ then we would get $r_k - a_k = r_\ell - a_\ell$, so $r_k - r_\ell = a_k - a_\ell \in \Lambda$; so $r_k \in r_\ell + A$, and $k = \ell$ (since the r_k contain exactly one representative of each coset). So Γ/Λ is finite. □ Claim 7.6

Let $m = |\Gamma/\Lambda| = [\Gamma : \Lambda]$. For all $a \in \Gamma$ we have $m(a + \Lambda) = 0 + \Lambda$; so $ma \in \Lambda$ for all $a \in \Gamma$, and $m\Gamma \subseteq \Lambda$. Then $\Gamma \subseteq \frac{1}{m}\Lambda = \operatorname{span}_{\mathbb{Z}}\{\, m^{-1}u_1, \dots, m^{-1}u_\ell \,\}$. Since Γ is a subgrape of the free abelian grape Λ, we get that Γ is also a free abelian grape. So Γ is of the form $\Gamma = \operatorname{span}_{\mathbb{Z}}\{\, v_1, \dots, v_k \,\}$ for some linearly independent $v_1, \dots, v_k \in \frac{1}{m}\Lambda$. (In fact $k = \ell$.) So Γ is a lattice. □ Theorem 7.4

Definition 7.7. A *torus* is a Lie grape of the form $\mathbb{T}^n = (\mathbb{S}^1)^n$ for some $n \ge 1$.

Theorem 7.8. *Suppose G is a matrix Lie grape.*

1. *If G is connected, compact, and abelian, then $G \cong \mathbb{T}^n$ where $n = \dim(G)$.*
2. *If G is compact and abelian then $G \cong \mathbb{T}^n \times K$ where $n = \dim(G)$ and K is some finite abelian grape.*

Proof. Suppose G is compact and abelian; let H be the connected component of G containing I (so H is both open and closed).

1. We show that $H \cong \mathbb{T}^n$.

   Since H is abelian we get that $\exp\colon \mathfrak{h} \to H$ is a Lie grape homomorphism. Since $\exp_* = I$ is invertible, we get that $\exp\colon \mathfrak{h} \to H$ is a covering map; indeed, since $\pi_1(\mathfrak{h}) = 0$ (as $\mathfrak{h}$ is a vector space) we get that $\mathfrak{h}$ is the universal cover. In particular, $\exp\colon \mathfrak{h} \to H$ is surjective,

   TODO 16. *ref*

   and $\ker(\exp)$ is a discrete subgrape of $Z(\mathfrak{h}) = \mathfrak{h}$. By the previous theorem we have that $\ker(\varphi)$ is a lattice; say $\ker(\varphi) = \operatorname{span}_{\mathbb{Z}}\{ u_1, \dots, u_\ell \}$. We can extend $\{ u_1, \dots, u_\ell \}$ to a basis $\{ u_1, \dots, u_n \}$ for $\mathfrak{h}$. Since $\exp\colon \operatorname{span}_{\mathbb{R}}\{ u_1, \dots, u_n \} \to H$ is surjective we get

   $$H \cong \operatorname{span}_{\mathbb{R}}\{ u_1, \dots, u_n \} / \operatorname{span}_{\mathbb{Z}}\{ u_1, \dots, u_n \} \cong (\mathbb{R}/\mathbb{Z})^\ell \times \mathbb{R}^k \cong (\mathbb{S}^1)^\ell \times \mathbb{R}^k$$

   where $k + \ell = n$

   TODO 17. *check*

   Since H is compact we get $\ell = n$ and $k = 0$. So $H \cong (\mathbb{S}^1)^n = \mathbb{T}^n$.

2. Note that G/H is finite since the cosets are all open and closed in G and G is compact. Say $G/H \cong (\mathbb{Z}/n_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/n_\ell\mathbb{Z})$. Let $P_k \in G$ correspond (under the above isomorphism) to $e_k = (0, \dots, 0, 1, 0, \dots, 0)$ with 1 in the k^{th} position. Then $P_k^{n_k} H = (P_k H)^{n_k} = 0$; so $P_k^{n_k} \in H$. Since $\exp\colon \mathfrak{h} \to H$ is surjective we can choose $B_k \in \mathfrak{h}$ so that $\exp(B_k) = P_k^{n_k}$. Let $A_k = \frac{1}{n_k} B_k \in \mathfrak{h}$ so $\exp(n_k A_k) = P_k^{n_k}$; then let $Q_k = P_k \exp(-A_k)$. So Q_k is in the same coset as P_k and $Q_k^{n_k} = I$. One checks that the map $H \times (\mathbb{Z}/n_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/n_\ell\mathbb{Z}) \to G$ given by $(P, k_1, \dots, k_\ell) \mapsto P Q_1^{k_1} \cdots Q_\ell^{k_\ell}$ is a Lie grape isomorphism. □ Theorem 7.8

We interrupt this broadcast to bring you a special report:

Theorem 7.9 (Closed subgrape theorem)**.** *Every closed subgrape of a matrix Lie grape is a regular Lie subgrape.*

Proof. Suppose G is a matrix Lie subgrape of $\mathrm{GL}(n, \mathbb{F})$ with $\mathbb{F} \in \{ \mathbb{R}, \mathbb{C} \}$; suppose $H \subseteq G$ is a closed subgrape of G. Let $\mathfrak{h} = \{ A \in \mathfrak{g} \subseteq M_n(\mathbb{F}) : \exp(At) \in H \text{ for all } t \in \mathbb{R} \}$.

Claim 7.10. $\mathfrak{h}$ *is a subspace of* $\mathfrak{g}$.

Proof. Closure under scalar multiplication is obvious; we check closure under addition. Suppose $A, B \in \mathfrak{h}$; so $\exp(tA), \exp(tB) \in H$ for all $t \in \mathbb{R}$. Then $\exp\left(\frac{t}{n} A\right), \exp\left(\frac{t}{n} B\right) \in H$ for all $t \in \mathbb{F}$ and $n \in \mathbb{Z}^+$; hence $\exp\left(\frac{t}{n} A\right) \exp\left(\frac{t}{n} B\right) \in H$ for all t, n. From A2 we have

$$\exp(t(A + B)) = \lim_{n \to \infty} \left(\exp\left(\frac{t}{n} A \right) \exp\left(\frac{t}{n} B \right) \right)^n$$

for all $t \in \mathbb{F}$, which must lie in H since H is closed. Thus $A + B \in \mathfrak{h}$, and $\mathfrak{h}$ is a subspace of $\mathfrak{g}$. □ Claim 7.10

We will show that there is a (local) *regular* chart around I; i.e. some $\varphi\colon U \subseteq G \to \varphi(U) = V \subseteq \mathfrak{g}$. In particular our φ will be the logarithm. Then we have $\varphi(U \cap H) = V \cap \mathfrak{h}$.

TODO 18. *wording?*

Suppose there is no such regular chart. We know that $E = \exp\colon \mathfrak{g} \to G$ is a local diffeomorphism. Choose a subspace $\mathfrak{k} \subseteq \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}$ (and then E is given by $E(A + B) = \exp(A + B)$ for $A \in \mathfrak{h}, B \in \mathfrak{k}$). Also the map $F\colon \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k} \to G$ given by $F(A + B) = \exp(A) \exp(B)$ is a local diffeomorphism with $F_* = I$: indeed, using series expansions we have $\exp(A) \exp(B) = (I + A + \cdots)(I + B + \cdots) = I + (A + B) + \cdots$.

Choose $0 \in U_0 \subseteq \mathfrak{g}$ and $I \in V_0 \subseteq G$ such that $F\colon U_0 \to V_0$ is a diffeomorphism. Suppose for contradiction that there exist points in H arbitrarily close to I not in $F(H \cap U_0)$. Then there are points $A + B \in \mathfrak{h} \oplus \mathfrak{k} = \mathfrak{g}$ arbitrarily close to 0 but not in $\mathfrak{h}$ (so $B \neq 0$) with $\exp(A) \exp(B) \in H$. Note that since $\exp(A) \in H$ we have

$\exp(B) \in H$. So we can choose a sequence $B_j \in \mathfrak{k}$ with $B_j \neq 0$ and $(B_j) \to 0$ such that $\exp(B_j) \in H$ for all j. By extracting a subsequence if necessary, we may suppose that $\frac{B_j}{\|B_j\|} \to C$ for some $C \in \mathfrak{k}$ with $\|C\| = 1$.

Let $t \in \mathbb{R}$ be arbitrary, and note that $\frac{tB_j}{\|B_j\|} \to tC$ in $\mathfrak{k}$. Let $n_j = \left\lfloor \frac{t}{\|B_j\|} \right\rfloor$. Then since $\exp(B_j) \in H$ we have $\exp(n_j B_j) = \exp(B_j)^{n_j} \in H$, and $n_j B_j \to tC$ in $\mathfrak{k}$ since

$$\|n_j B_j - tC\| \leq \left\| n_j B_j - \frac{tB_j}{\|B_j\|} \right\| + \left\| \frac{tB_j}{\|B_j\|} - tC \right\| = \underbrace{\left| n_j - \frac{t}{\|B_j\|} \right|}_{\leq 1} \underbrace{\|B_j\|}_{\to 0} + + \underbrace{\left\| \frac{tB_j}{\|B_j\|} - tC \right\|}_{\to 0}$$

Hence $\exp(n_j B_j) \to \exp(tC)$. Since $\exp(n_j B_j \in H$ and H is closed, it follows that $\exp(tC) \in H$. Since $\exp(tC) \in H$ for all $t \in \mathbb{R}$, we have $C \in \mathfrak{h}$. But $C \in \mathfrak{k}$ with $\|C\| = 1$ and $\mathfrak{h} \cap \mathfrak{k} = \{ 0 \}$, a contradiction.

So we have a regular chart at $I \in H$. Given $p \in H$ there is a regular chart at p obtained using left-multiplication by p. □ Theorem 7.9

We now return to your regularly scheduled programming.

Definition 7.11. For a compact matrix Lie grape G, a *maximal torus* in G (or a *Cartan subgrape*) is a maximal compact connected abelian Lie subgrape. For a matrix Lie algebra $\mathfrak{g}$ a *Cartan subalgebra* of $\mathfrak{g}$ is a maximal abelian Lie subalgebra of $\mathfrak{g}$.

Remark 7.12. Hopefully we will later prove that in a compact

TODO 19. *connected?*

matrix Lie grape

1. The maximal tori in G are conjugate to each other.
2. G is the union of the maximal tori.

Corollary 7.13. *When G is a compact*

TODO 20. *connected?*

matrix Lie grape we have $\exp\colon \mathfrak{g} \to G$ *is surjective.*

Corollary 7.14. *The maximal tori of G have the same dimension, which we call the* rank *of G.*

Exercise 7.15. Verify that the classical compact matrix grapes have the following maximal tori and Cartan subalgebras:

- In $\mathrm{SO}(2n)$ we have the maximal torus

$$T = \left\{ \begin{pmatrix} R_{\theta_1} & & 0 \\ & \ddots & \\ 0 & & R_{\theta_n} \end{pmatrix} : \theta_k \in \mathbb{R} \right\}$$

where

$$R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

and Cartan subalgebra

$$\mathfrak{t} = \left\{ \begin{pmatrix} S_{\theta_1} & & \\ & \ddots & \\ & & S_{\theta_n} \end{pmatrix} : \theta_k \in \mathbb{R} \right\}$$

where

$$S_\theta = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}$$

- In $\mathrm{SO}(2n+1)$ we have

$$T = \left\{ \begin{pmatrix} R_{\theta_1} & & & \\ & \ddots & & \\ & & R_{\theta_n} & \\ & & & 1 \end{pmatrix} : \theta_k \in \mathbb{R} \right\}$$

$$\mathfrak{t} = \left\{ \begin{pmatrix} S_{\theta_1} & & & \\ & \ddots & & \\ & & S_{\theta_n} & \\ & & & 0 \end{pmatrix} : \theta_k \in \mathbb{R} \right\}$$

- In $\mathrm{U}(n)$ we have

$$T = \left\{ \begin{pmatrix} \exp(i\theta_1) & & \\ & \ddots & \\ & & \exp(i\theta_n) \end{pmatrix} : \theta_k \in \mathbb{R} \right\}$$

$$\mathfrak{t} = \left\{ \begin{pmatrix} i\theta_1 & & \\ & \ddots & \\ & & i\theta_n \end{pmatrix} : \theta_k \in \mathbb{R} \right\}$$

- In $\mathrm{SU}(n)$ we have

$$T = \left\{ \begin{pmatrix} \exp(i\theta_1) & & \\ & \ddots & \\ & & \exp(i\theta_n) \end{pmatrix} : \theta_k \in \mathbb{R}, \prod \exp(i\theta_k) = 1 \right\}$$

$$\mathfrak{t} = \left\{ \begin{pmatrix} i\theta_1 & & \\ & \ddots & \\ & & i\theta_n \end{pmatrix} : \theta_k \in \mathbb{R}, \sum \theta_k = 0 \right\}$$

- In $\mathrm{Sp}(n)$ if we identify

$$M_n(\mathbb{H}) = \left\{ \begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix} : A, B \in M_n(\mathbb{C}) \right\} \subseteq M_{2n}(\mathbb{C})$$

then we have

$$T = \left\{ \begin{pmatrix} \exp(i\theta_1) & & & & & \\ & \ddots & & & & \\ & & \exp(i\theta_n) & & & \\ & & & \exp(-i\theta_1) & & \\ & & & & \ddots & \\ & & & & & \exp(-i\theta_n) \end{pmatrix} : \theta_k \in \mathbb{R} \right\}$$

$$\mathfrak{t} = \left\{ \begin{pmatrix} i\theta_1 & & & & & \\ & \ddots & & & & \\ & & i\theta_n & & & \\ & & & -i\theta_1 & & \\ & & & & \ddots & \\ & & & & & -i\theta_n \end{pmatrix} : \theta_k \in \mathbb{R} \right\}$$

It follows that

$$\begin{aligned}\operatorname{rank}(\mathrm{SO}(2n)) &= \operatorname{rank}(\mathrm{SO}(2n+1)) \\ &= n \\ \operatorname{rank}(\mathrm{U}(n)) &= n \\ \operatorname{rank}(\mathrm{SU}(n)) &= n-1 \\ \operatorname{rank}(\mathrm{Sp}(n)) &= n\end{aligned}$$

Fact 7.16. *When G is a compact Lie grape and $\varphi\colon \widetilde{G} \to G$ is its universal cover and T is a maximal torus in G, we have $\widetilde{T} = \varphi^{-1}(T)$ is a maximal torus in $\widetilde{G}$ with $\widetilde{\varphi}\colon \widetilde{T} \to T$ a covering map.*

8 Representations

Definition 8.1. A *Lie grape action* of a Lie grape G on a smooth manifold M is a smooth map $F\colon G \times M \to M$, usually written $F(a,x) = a \cdot x = ax$, satisfying

1. $ex = x$ for all $x \in M$ (where $e \in G$ is the identity), and
2. $a(bx) = (ab)x$ for all $a, b \in G$ and $x \in M$.

A Lie grape action of G on a vector space V (over $\mathbb{R}$ or $\mathbb{C}$, usually $\mathbb{C}$) is called *linear* if

1. $a(x+y) = ax + ay$ for all $a \in G$ and $x + y \in V$, and
2. $a(tx) = t(ax)$ for all $a \in G$, $x \in M$, and $t \in \mathbb{C}$.

A *representation* of a Lie grape G in $\mathrm{GL}(V)$, where V is a vector space (over $\mathbb{C}$), is a Lie grape homomorphism $\rho\colon G \to \mathrm{GL}(V)$. A *linear G-module* on a Lie grape G is a vector space V (over $\mathbb{C}$) with a G-action.

Exercise 8.2. Verify that the above three concepts are equivalent.

8.1 An informal review of integration on manifolds

Integrals that you see in various parts of mathematics/physics:

$$\begin{aligned}\int_a^b f(x)\mathrm{d}x &= \int_I f\mathrm{d}L \\ \int\int_R f(x,y)\mathrm{d}x\mathrm{d}y &= \int\int_R f\mathrm{d}A \\ \int\int\int_B f(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z &= \int\int\int_B f\mathrm{d}V\end{aligned}$$

Or if C is a curve given by $\alpha\colon \mathbb{R} \to \mathbb{R}^n$ then

$$\int_C f\mathrm{d}L = \int_I f(\alpha(t))|\alpha'(t)|\mathrm{d}t$$

If σ is a function out of a rectangle in $\mathbb{R}^2$, say $\sigma(s,t) = \begin{pmatrix} x(s,t) \\ y(s,t) \\ z(s,t) \end{pmatrix}$. Then the surface integral is

$$\int_S f\mathrm{d}A = \int_R f(\sigma(s,t))|\sigma_s(s,t) \times \sigma_t(s,t)|\mathrm{d}s\ t$$

In $\mathbb{R}^2$ if $\alpha(t) = (x(t), y(t))$ and $T = \frac{\alpha'(t)}{\|\alpha'(t)\|}$ and $F = (P,Q)$ is some vector field we define

$$\int_C F \cdot T\mathrm{d}L = \int_I (P(\alpha(t)), Q(\alpha(t))) \cdot (x'(t), y'(t))\mathrm{d}t$$

so $\mathrm{d}L = |x'(t)|\mathrm{d}t$. We also let $N = \frac{(-y'(t), x'(t))}{|\alpha'(t)|}$, and then

$$\int_C F \cdot N \mathrm{d}L = \int_I (P(\alpha(t)), Q(\alpha(t))) \cdot (-y'(t), x'(t)) \mathrm{d}t$$

In $\mathbb{R}^3$ we can define

$$\int_C F \cdot T \mathrm{d}L = \int_I (P(\alpha(t)), Q(\alpha(t)), R(\alpha(t))) \cdot \alpha'(t) \mathrm{d}t = \int_I P(\alpha(t)) \cdot x'(t) + \cdots = \int_\alpha P \mathrm{d}x + Q \mathrm{d}y + R \mathrm{d}z$$

We also set

$$\begin{aligned} \int\int_S F \cdot N \mathrm{d}A &= \int\int_R (P(\sigma(s,t)), Q(\sigma(s,t)), R(\sigma(s,t))) \cdot (\sigma_s(s,t) \times \sigma_t(s,t)) \mathrm{d}s\mathrm{d}t \\ &= \int P(\sigma(s,t)) \begin{vmatrix} \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} \end{vmatrix} + Q \cdots \\ &= \int\int_\sigma P \mathrm{d}y \wedge \mathrm{d}z + Q \mathrm{d}z \wedge \mathrm{d}x + R \mathrm{d}x \wedge \mathrm{d}y \end{aligned}$$

We can also relate the integral on a boundary to the integral of some kind of derivative:

$$\int\int_S (\nabla \times F) \cdot N \mathrm{d}A = \int_{C = \partial M} \alpha$$
$$\int\int\int_B (\nabla \cdot F) \mathrm{d}V = \int_{S = \partial B} F \cdot N \mathrm{d}A$$

In general, using differential geometry:

$$\int_M \mathrm{d}\alpha = \int_{\partial M} \alpha$$

We can "define" a *k-form* on $\mathbb{R}^n$ to be an expression of the form

$$\alpha = \sum_I A_i(x) \mathrm{d}x_I$$

where $I = (i_1, i_2, \dots i_k)$ with $1 \le i_1 < \cdots < i_k \le n$. We write $\mathrm{d}x_I = \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}$. We then set

$$\int_\sigma \alpha = \sum_I \int_{R \subseteq \mathbb{R}^k} a_I(\sigma(t)) \begin{vmatrix} \frac{\partial x_{i_1}}{\partial t_1} & \cdots & \\ & \ddots & \\ & & \frac{\partial x_{i_k}}{\partial t_k} \end{vmatrix} \mathrm{d}t_1 \cdots \mathrm{d}t_k$$

For

$$\alpha = \sum_I a_I(x) \mathrm{d}x_I$$

we define

$$\mathrm{d}\alpha = \sum_I \sum_{j=1}^n \frac{\partial a_I}{\partial x_j} \mathrm{d}x_j \wedge \mathrm{d}x_I$$

using $\mathrm{d}x_j \wedge \mathrm{d}x_i = -\mathrm{d}x_i \wedge \mathrm{d}x_j$.

We still have to give a formal definition of a k-form.

Definition 8.3. For a vector space V we define $T^k V$ to be the set of k-linear maps $L\colon (V^*)^k \to \mathbb{R}$. This is $\operatorname{span}\{ u_{i_1} \otimes \cdots \otimes u_{i_k} : 1 \le i_j \le n \}$, where $\{ u_1, \dots, u_n \}$ is a basis for V and

$$(u_{i_1} \otimes \cdots \otimes u_{i_k})(f_1, \dots, f_k) = f_1(u_{i_1}) \cdot \cdots \cdot f_k(u_{i_k})$$

We then set $\Lambda^k V$ to be the set of alternating k-linear maps $L\colon (V^*)^k \to \mathbb{R}$; this is then $\operatorname{span}\{\, u_{i_1} \wedge \cdots u_{i_k} : 1 \leq i_1 < \cdots < i_k \leq n \,\}$ where

$$(u_{i_1} \wedge \cdots \wedge u_{i_k})(f_1, \ldots, f_k) = \begin{vmatrix} f_1(u_{i_1}) & \cdots & \\ & \ddots & \\ & & f_k(u_{i_k}) \end{vmatrix}$$

(and $u_j \wedge u_i = -u_i \wedge u_j$).

Definition 8.4. On $\mathbb{R}^n$, a *k-form* is a smooth map $\alpha\colon \mathbb{R}^n \to \Lambda^k(\mathbb{R}^n)^*$. For a smooth manifold M and a point $p \in M$ we let

$$\left\{ \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right\}$$

be the standard basis for T_pM identified using a chart to $\mathbb{R}^n$. We then let $\mathrm{d}x_1, \ldots, \mathrm{d}x_n$ be the dual basis for T_p^*M. (So $\mathrm{d}x_k\left(\frac{\partial}{\partial x_\ell}\right) = \delta_{k\ell}$.) Then $\Lambda^k T_p^* M$ is the set of alternating k-linear maps $\alpha\colon (T_pM)^k \to \mathbb{R}$, which is the span of $\mathrm{d}x_I$ where $I = (i_1, \ldots, i_k)$ for $1 \leq i_1 < \cdots < i_k = n$. A *(smooth differential) k-form* on M is a map

$$\alpha\colon M \to \bigcup_p \Lambda^k T_p^* M$$

with $\alpha(p) \in \Lambda^k T_p^* M$ for all $p \in M$ such that for each coordinate chart φ when we write α locally as

$$\alpha(x) = \sum_I a_I(x) \mathrm{d}x_I$$

we have that each function a_I is smooth as a map $\varphi(U) \subseteq \mathbb{R}^n \to \mathbb{R}$. So a k-form α on M is a smooth section of the vector bundle

$$\Lambda^k T^* M = \bigsqcup_{p \in M} \Lambda^k T_p^* M$$

Note that when M is n-dimensional we have

$$\Lambda^n T_p^* M = \operatorname{span}\{\, \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_n \,\}$$

so $\dim(\Lambda^n T_p^* M) = 1$. An n-form is given locally by $a(x)\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$.

Definition 8.5. We say M is *orientable* when M can be given charts such that for every transition map $\psi\varphi^{-1}$ we have $\det(D(\psi\varphi^{-1})(x)) > 0$ for all $x \in \operatorname{dom}(\varphi) \cap \operatorname{dom}(\psi)$.

Fact 8.6. *If M is n-dimensional then M is orientable if and only if M has a nowhere zero n-form.*

The proof uses partitions of unity to construct a nowhere-zero top form.

When M is oriented and ω is an n-form we can define $\int_M \omega$; the integral is given locally in a chart φ where

$$\omega = \sum a_I(x) \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$$

by

$$\int_{S \subseteq U \subseteq M} \omega = \sum_I \int_{R \subseteq \varphi(U) \subseteq \mathbb{R}^n} a_I(x) \mathrm{d}x_1 \cdots \mathrm{d}x_n$$

For a smooth map $f\colon N \to M$ with $f(p) = q$ we define the *pullback* $f^*\colon \Lambda^k T_q^* M \to \Lambda^k T_p^* N$ by $f^*(\alpha)(X_1 \ldots, X_k) = \alpha(f_*(X_1), \ldots, f_*(X_k))$ where $\alpha \in \Lambda^k T_q^* M$ and each $X_i \in T_pN$.

Theorem 8.7.

1. *For $N \xrightarrow{f} M \xrightarrow{g} L$ we have $(g \circ f)^* = f^* \circ g^*$.*

2. *For* $N \xrightarrow{f} M \xrightarrow{g} \mathbb{R}$ *and for a* k*-form* α *on* M *we have* $f^*(g \cdot \alpha) = (g \circ f) \cdot f^*\alpha$.
3. *For* $N \xrightarrow{f} M$ *we have* $f^* \circ \mathrm{d} = \mathrm{d} \circ f^*$*; that is* $f^*(\mathrm{d}\alpha) = \mathrm{d}(f^*\alpha)$ *when* α *is a* k*-form on* M.

Remark 8.8. Suppose N is oriented and k-dimensional and M is n-dimensional; suppose $f\colon N \to M$ is an immersion and α is a k-form on M. We can define

$$\int_{f(N)} \alpha = \int_N f^*\alpha$$

This is the integral that agrees with the examples we saw at the beginning of the section.

Definition 8.9. A *volume form* on an n-dimensional manifold is a nowhere-zero differential n-form.

Given a volume form on M we obtain an orientation on M, and can then define the integral of a continuous function $f\colon M \to \mathbb{R}$ with compact support.

TODO 21. *?*

Definition 8.10. Suppose G is a Lie grape. A differential form ω on G is called

- *left-invariant* when $\ell_a^*\omega = \omega$ for all $a \in G$,
- *right-invariant* when $r_a^*\omega$ for all $a \in G$, and
- *invariant under inverseion* when $v^*\omega = \omega$ (where $v\colon G \to G$ is the inversion map $v(x) = x^{-1}$).

Theorem 8.11. *Suppose* G *is a Lie grape.*

1. *There exists a left-invariant volume form* ω *on* G*, and it is unique up to multiplication by* $c \in \mathbb{R} \setminus \{\, 0 \,\}$.
2. *If* G *is compact we can also require that* $\int_G \omega = 1$*; then* ω *is unique up to multiplication by* ± 1 *(where we use the form to determine the orientation).*
3. *When* G *is compact and connected, the left-invariant form* ω *(or* $-\omega$*) with* $\int_G \omega = 1$ *is also right-invariant (so* $r_a^*\omega = \omega$ *for all* a*), and* $v^*\omega = \pm\omega$.

Proof.

1. Given $0 \neq \omega_e \in \Lambda^n Te^*G$, in order to get $\ell_a^*\omega = \omega$ for all $a \in G$ we must have $\omega_q = \ell_{a^{-1}}^*\omega_e$ (since $\ell_{a^{-1}}(a) = e$, so $\ell_{a^{-1}}^*\colon \Lambda^n T_e^*G \to \Lambda^n T_n^*G$). On the other hand, if we define ω by $\omega(a) = \omega_a = \ell_{a^{-1}}^*\omega_e$ then ω is left-invariant: if $a, b \in G$ then

   $$(\ell_a^*\omega)_b = \ell_a^*(\omega_{ab}) = \ell_a^*(\ell_{b^{-1}a^{-1}}\omega_e) = (\ell_{b^{-1}a^{-1}} \circ \ell_a)^*\omega_e = \ell_{b^{-1}}^*(\omega_e) = \omega_b$$

   Uniqueness up to non-zero multiplication is because $\Lambda^n T_e^*G$ is one-dimensional.
2. Follows from the above, (since negating the form that determines the orientation and integrating it with respect to the new orientation doesn't change the integral).
3. For $a, b \in G$ we have

   $$\ell_a^*(r_b^*\omega) = (r_b \circ \ell_a)^*\omega = (\ell_a \circ r_b)^*\omega = r_b^*(\ell_a^*\omega) = r_b^*\omega$$

   So $r_b^*\omega$ is left-invariant for every $b \in G$. Hence from uniqueness of ω up to scalar multiplication we get that $r_b^*(\omega) = c(b)\omega$ for some smooth map $c\colon G \to \mathbb{R} \setminus \{\, 0 \,\}$. Also note that

   $$c(a)c(b)\omega = r_a^*(c(b)\omega) = r_a^*(r_b^*\omega) = (r_b \circ r_a)^*\omega = r_{ab}^*\omega = c(ab)\omega$$

So $c(ab) = c(a)c(b)$. So the map $c\colon G \to \mathbb{R} \setminus \{0\}$ is a homomorphism of Lie grapes. Since G is compact, we get that $c(G)$ is compact; so $c(a) = \pm 1$ for all $a \in G$. Since G is connected either $c(a) = 1$ for all $a \in G$ or $c(a) = -1$ for all $a \in G$. Since $r_e = \mathrm{id}$ we have $r_e^*\omega = \omega$; so $c(a) = 1$ for all $a \in G$. So $r_a^*\omega = \omega$ for all $a \in G$. Also for all $a \in G$ we have

$$\ell_a^*(v^*\omega) = (v \circ \ell_a)^*\omega = (r_{a^{-1}} \circ v)^*\omega = v^*(r_{a^{-1}}^*\omega) = v^*\omega$$

Thus $v^*\omega$ is left-invariant; so $v^*\omega = c\omega$ for some $c \in \mathbb{R} \setminus \{0\}$. We must have $c = \pm 1$ since $v \circ v = \mathrm{id}$, so

$$\omega = (v \circ v)^*\omega = v^*(v^*\omega) = v^*(c\omega) = cv^*(\omega) = c^2\omega$$

as desired. □ Theorem 8.11

Definition 8.12. Suppose G is a compact Lie grape and $\pm\omega$ is the left-invariant volume-form; suppose $f\colon G \to \mathbb{R}$ is a continuous (or integrable) function $f\colon G \to \mathbb{R}$. We write

$$\int_G f = \int_G f(x)\mathrm{d}g(x) = \int_G f \cdot \omega$$

Corollary 8.13. *Suppose G is compact and $a \in G$. Then*

$$\int_G f(ax)\mathrm{d}g(x) = \int_G f(xa)\mathrm{d}g(x) = \int_G f(x^{-1}\mathrm{d}g(x) = \int_G f(x)\mathrm{d}g(x)$$

Remark 8.14. The corresponding measure on G given by

$$\mu(A) = \int_G \chi_A \mathrm{d}g(x)$$

is called the *Haar measure* on G.

8.2 Back to representations

Definition 8.15. A *representation* of a Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism $\psi\colon \mathfrak{g} \to \mathrm{End}(V)$.

We define $\mathfrak{g}$-actions and $\mathfrak{g}$-modules analogously.

Remark 8.16. When G is a Lie grape with Lie algebra $\mathfrak{g}$ we have that every Lie grape representation $\rho\colon G \to \mathrm{GL}(V)$ induces a Lie algebra representation $\psi = \rho_*\colon \mathfrak{g} \to \mathrm{End}(V)$. When G is connected we saw (Claim 6.22) that for two representations $\rho, \varphi\colon G \to \mathrm{GL}(V)$ if $\rho_* = \varphi_*$ then $\rho = \varphi$. We also saw (Theorem 6.21) that if G is simply connected then every Lie algebra representation $\psi\colon \mathfrak{g} \to \mathrm{End}(V)$ is of the form $\psi = \rho_*$ for some Lie grape representation ρ.

Definition 8.17. When a Lie grape representation $\rho\colon G \to \mathrm{GL}(V)$ is injective, we say that it is *faithful*.

Example 8.18. When G is a matrix Lie grape $G \subseteq \mathrm{GL}(n, \mathbb{C})$ we have the *standard representation* $\rho\colon G \to \mathrm{GL}(\mathbb{C}^n)$ the inclusion map. When G is any Lie grape we have the *adjoint representation* defined as follows: for $a \in G$ let $C_a\colon G \to G$ be the conjguation map $x \mapsto axa^{-1}$. Since C_a is a diffeomorphism we have that $\mathrm{d}C_a = (C_a)_*\colon \mathfrak{g} \to \mathfrak{g}$ is invertible. The map $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$ given by $\mathrm{Ad}(a) = \mathrm{d}C_a$ is called the *adjoint representation* of G. The induced representation $\mathrm{ad} = \mathrm{Ad}_*\colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is called the *adjoint representation* of $\mathfrak{g}$).

Example 8.19. Let $V_n = \mathrm{span}_{\mathbb{C}}(x^n, x^{n-1}y, \ldots, xy^{n-1}, y^n)$ be the set of homogeneous polynomials of degree n. Then $\mathrm{SU}(2)$ acts on V_n by

$$A\left(p\begin{pmatrix} x \\ y \end{pmatrix}\right) = p\left(A^{-1}\begin{pmatrix} x \\ y \end{pmatrix}\right)$$

Example 8.20. When V, W are G-modules (or equivalently when ρ and φ are representations) we can define modules (or representations) $\overline{V}, V^*, V \oplus W, V \otimes W, \mathcal{L}(V,W), T^k V, \Lambda^k V$ (or $\overline{\rho}, \rho^*, \rho \oplus \varphi, \rho \otimes \varphi$, etc.) as follows:

- $\overline{V}$ is equal to V as an abelian grape, but scalar multiplication on $\overline{V}$ is given by

$$\underbrace{c \cdot x}_{\text{in } \overline{V}} = \underbrace{\overline{c} \cdot x}_{\text{in } V}$$

  and the action of G on $\overline{V}$ is the same as the action of G on V:

$$\underbrace{a \cdot x}_{\text{in } \overline{V}} = \underbrace{a \cdot x}_{\text{in } V}$$

  for $a \in G, x \in V$.

- V^* is the set of linear maps $f \colon V \to \mathbb{C}$, and the action of G on V^* is given by $(a \cdot f)(x) = f(a^{-1} \cdot x)$.
- The action of G on $V \oplus W$ is given by $a(x,y) = (ax, ay)$.
- Consider $V \otimes W$, which we view as the set of bilinear maps $L \colon V^* \times W^* \to \mathbb{C}$, or equivalently $\operatorname{span}_{\mathbb{C}}\{ v_i \otimes w_j : i, j \}$ where the v_i are a basis for V, the w_j are a basis for W, and $(v_i \otimes w_j)(f, g) = f(v_i) g(w_j)$. The action of G on $V \otimes W$ is given by $a \cdot (v \otimes w)(f,g) = f(av)g(aw)$ (or $a \cdot (v \otimes w) = (av) \otimes (aw)$).
- The action of G on $\mathcal{L}(V,W)$ is given by $(aL)(x) = a \cdot L(a^{-1} \cdot x)$ for $a \in G$, $L \colon V \to W$, and $x \in V$. (i.e. if $\rho \colon G \to \mathrm{GL}(V)$ and $\varphi \colon G \to \mathrm{GL}(W)$ are the constituent representations then we get a representation $\psi \colon G \to \mathrm{GL}(\mathcal{L}(V,W))$ given by $(\psi(a)(L))(x) = \varphi(a)(L(\rho(a)^{-1}x))$.)

  TODO 22. *Are we calling this* $\operatorname{End}(V,W)$*?*

Definition 8.21. Suppose G is a Lie grape; suppose V and W are G-modules. A *G-module homomorphism* from V to W is a linear map $L \colon V \to W$ which is *G-invariant* (or *G-intertwining*): namely $a \cdot L(x) = L(a \cdot x)$, or writing the representation explicitly $\varphi(a)(L(x)) = L(\rho(a)(x))$. The set of such G-module homomorphisms is denoted $\hom_G(V,W)$. A *G-module isomorphism* from V to W is a bijective G-module homomorphism $L \colon V \to W$. If such an isomorphism exists we say that V and W are *isomorphic* (as G-modules) and we write $V \cong W$. When $V \cong W$ as G-modules we say the associated representations (or G-actions) are *equivalent*.

Example 8.22. Given a representation $\rho \colon G \to \mathrm{GL}(V)$ with V finite-dimensional we can choose a basis $\mathcal{U} = \{ u_1, \ldots, u_n \}$ for V; this gives a vector space isomorphism $\Phi \colon V \to \mathbb{C}^n$ (given by $\Phi(u_k) = e_k$). We then define a representation $\varphi \colon G \to \mathrm{GL}(\mathbb{C}^n) = \mathrm{GL}(n, \mathbb{C})$ that is equivalent to ρ by $\varphi(a)(e_k) = \Phi^{-1}(\rho(n)(u_k))$.

Example 8.23. Show that the standard representation σ of $\mathrm{SU}(2)$ is equivalent to the represnetation ρ of $\mathrm{SU}(2)$ on $V_1 = \operatorname{span}_{\mathbb{C}}\{ x, y \} \subseteq \mathbb{C}[x,y]$ given by

$$(\rho(A) \cdot p)(x,y) = p\left(\left(\rho(A)^{-1} \begin{pmatrix} x \\ y \end{pmatrix} \right)^T \right)$$

For

$$A = \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \in \mathrm{SU}(2)$$

we have

$$A^{-1} = \begin{pmatrix} \overline{a} & \overline{b} \\ -b & a \end{pmatrix} = A^*$$

so

$$A^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \overline{a}x + \overline{b}y \\ -bx + ay \end{pmatrix}$$

and for $p(x,y) = u \cdot x + v \cdot y$ we have

$$P\left(\left(A^{-1} \begin{pmatrix} x \\ y \end{pmatrix} \right)^T \right) = u(\overline{a}x + \overline{b}y) + v(-bx + ay) = (\overline{a}u - bv)x + (\overline{b}u + av)y$$

Thus when $\sigma(a) = A \in M_2(\mathbb{C})$ and $\rho(a) = B \in M_2(\mathbb{C})$ (with respect to $\{ e_1, e_2 \}$ for σ and $\{ x, y \}$ for ρ) and when

$$A = \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix}$$

we have

$$B = \begin{pmatrix} \overline{a} & -b \\ \overline{b} & a \end{pmatrix} = \overline{A} = (A^{-1})^T$$

(So we have $\rho = \overline{\sigma} = \sigma^*$.) To show that $\rho \cong \sigma$ we need to find a bijective linear map $L\colon \mathbb{C}^2 \to V_1$ (or $\to \mathbb{C}^2$) such that $L \cdot A = B \cdot L$ whenever $A = \sigma(a)$ and $B = \rho(a)$ for $a \in \mathrm{SU}(2)$; i.e.

$$L \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} = \begin{pmatrix} \overline{a} & -b \\ \overline{b} & a \end{pmatrix}$$

We take

$$L = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

since

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} = \begin{pmatrix} b & \overline{a} \\ -a & \overline{b} \end{pmatrix} = \begin{pmatrix} \overline{a} & -b \\ \overline{b} & a \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(We have shown that $\overline{\sigma} = \sigma^* \cong \sigma$.)

Example 8.24. Let V be a finite-dimensional G-module. Let $\mathcal{U} = \{ u_1, \ldots, u_n \}$ be a basis for V and let $\mathcal{F} = \{ f_1, \ldots, f_n \}$ be the dual basis for V^*. Determine how the matrix of $\rho^*(a)$ is related to the matrix of $\rho(a)$ (with respect to these bases).

Let

$$A = [\rho(a)]_{\mathcal{U}} = \begin{pmatrix} [\rho(a) \cdot u_1]_{\mathcal{U}} & \cdots & [\rho(a) u_n]_{\mathcal{U}} \end{pmatrix}$$

and let

$$B = [\rho^*(a)]_{\mathcal{F}} = \begin{pmatrix} [\rho^*(a) f_1]_{\mathcal{F}} & \cdots & [\rho^*(a) f_n]_{\mathcal{F}} \end{pmatrix}$$

for $a \in G$. Then $A_{k\ell}$ is the k^{th} entry of

$$[\rho(a) u_\ell]_{\mathcal{U}} = [a \cdot u_\ell]_{\mathcal{U}} = \begin{pmatrix} f_1(a u_\ell) \\ \vdots \\ f_n(a u_\ell) \end{pmatrix}$$

which is just $f_k(a u_\ell)$. (Note that $f_k(\sum c_i u_i) = \sum c_i \delta_{ki} = c_k$.) Also $B_{k\ell}$ is the k^{th} entry of

$$[\rho^*(a) f_\ell]_{\mathcal{F}} = (\rho^*(a) f_\ell)(u_k) = f_\ell(\rho(a)^{-1} u_k) = (A^{-1})_{\ell k}$$

Thus $B = (A^{-1})^T$.

Exercise 8.25. Find the relationship between the matrix of $(\rho \otimes \varphi)(a)$ and those of $\rho(a)$ and $\varphi(a)$, etc.

Exercise 8.26. Determine how $\mathfrak{g}$ acts on $\overline{V}, V^*, V \oplus W, V \otimes W, \mathcal{L}(V, W)$, etc. (in terms of the actions of $\mathfrak{g}$ on V and W).

Answers:

- $\mathfrak{g}$ acts on $\overline{V}$ using the same action as on V.
- $\mathfrak{g}$ acts on V^* by $(A \cdot f)(x) = f(-Ax)$.
- $\mathfrak{g}$ acts on $\mathcal{L}(V, W)$ by $(AL(x) = AL(x) - L(Ax)$.

Definition 8.27. Suppose G is a Lie grape and W a G-module. A *submodule* of W is a *G-invariant* subspace $U \subseteq W$ where we say $U \subseteq W$ is G-invariant when $a \cdot u \in U$ for all $a \in G$ and $u \in U$ (so that $\rho\colon G \to \mathrm{GL}(W)$ determines a representation $\rho\colon G \to \mathrm{GL}(U)$). We say that W is *reducible* when there is a non-trivial proper submodule $0 \neq U \subsetneqq W$; otherwise we say that W is *irreducible.* We say that W is *completely reducible* when it is a direct sum of irreducible submodules.

Example 8.28. When $L\colon V\to W$ is a G-module homomorphism, verify that $\ker(L)$ and $\mathrm{Ran}(L)$ are G-invariant (and are thus submodules of V and W).

Theorem 8.29 (Schur's lemma)**.** *Suppose G is a Lie grape and V,W are finite-dimensional irreducible G-modules. Then*

$$\dim(\hom_G(V,W)=\begin{cases}1 & \text{if } V\cong W\\ 0 & \text{if } V\not\cong W\end{cases}$$

In particular, $\mathrm{End}_G(V)=\hom_G(V,V)=\{\,cI : c\in\mathbb{C}\,\}$.

Proof. Suppose $0\neq L\in\hom_G(V,W)$. Since $L\neq 0$ we have $\ker(L)\neq V$; so since V is irreducible we get $\ker(L)=0$. Since $L\neq 0$ we get $\mathrm{Ran}(L)\neq 0$; so since W is irreducible we get $\mathrm{Ran}(L)=W$). So L is an isomorphism.

Suppose now that $L,M\colon V\to W$ are isomorphisms. Then $M^{-1}\circ L\colon V\to V$ is an isomorphism. Note that $M^{-1}L$ has an eigenvalue$0\neq\lambda\in\mathbb{C}$, and the eigenspace $E_\lambda=\ker(M^{-1}L-\lambda I)\subseteq V$ is G-invariant; since V is irreducible and $E_\lambda\neq 0$, we get that $E_\lambda=V$. So $M^{-1}L=\lambda I$, and $L=\lambda M$. Thus $\hom_G(V,W)=\{\,\lambda M : \lambda\in\mathbb{C}\,\}$ is one-dimensional. ☐ Theorem 8.29

Theorem 8.30. *Suppose G is a compact Lie grape. Then*

1. *Every G-module V has a G-invariant inner product $(\cdot,\cdot)$; i.e. $(ax,ay)=(x,y)$ for all $a\in G$ and $x,y\in V$.*

2. *Every n-dimensional representation on G is equivalent to a* unitary *representation; i.e. some $\rho\colon G\to\mathrm{U}(n)$.*

3. *Every finite-dimensional representation of G is completely reducible.*

Proof.

1. Suppose V is a G-module. Let $\langle\cdot,\cdot\rangle$ be any inner product on V; then define a new inner product $(\cdot,\cdot)$ by

$$(u,v)=\int_G\langle xu,xv\rangle\mathrm{d}g(x)$$

for all $u,v\in V$. Note that this is G invariant because if we let $f(x)=\langle xu,xv\rangle$ then

$$(au,av)=\int_G\langle xau,xav\rangle\mathrm{d}g(x)=\int_G f(xa)\mathrm{d}g(x)=\int_G f(x)\mathrm{d}g(x)=(u,v)$$

since integration is right-invariant.

2. We choose an orthonormal basis $\mathcal{U}=\{u_1,...,u_n\}$ for V (with respect to a G-invariant Hermitian inner product on V). Let $S:=\{e_1,...,e_n\}$ be the standard basis for $\mathbb{C}^n$. Let $L:V\to\mathbb{C}^n$ be the inner product space isomorphism with $L(u_k):=e_k$. Let $\varphi:G\to\mathrm{GL}(n,\mathbb{C})$ be given by

$$\varphi(a)u:=L(\rho(a)(L^{-1}(u)))$$

and note that L is a G-invariant isomorphism. (Indeed, omitting ρ from our notation, we can write $a\cdot u=L(a\cdot L^{-1}(u))$, so $a\cdot L(u)=L(a\cdot u)$.)

We have

$$\begin{aligned}[\varphi(a)]_S&=\underbrace{(\varphi(a)e_1,...,\varphi(a)e_n)}_{\in M_n(\mathbb{C})}\\&=(L(\rho(a)(L^{-1}(e_1))),...,L(\rho(a)(L^{-1}(e_n))))\\&=(L(\rho(a)u_1),...,L(\rho(a)u_n))\end{aligned}$$

and we have

$$\langle L(\rho(a)u_k),L(\rho(a)u_\ell)\rangle_{\mathbb{C}^n}=(\rho(a)u_k,\rho(a)u_\ell)_V=(u_k,u_\ell)=\delta_{k\ell}$$

since L preserves the inner product and the inner product is G-invariant.

So we do have $[\varphi(a)]_S\in\mathrm{U}(n)$.

3.

TODO 23. *Something along the lines of: suppose that V is not irreducible (since we'd be done if it were). Then V contains a non-trivial proper G-submodule, say U. Then you need an argument for irreducibility of U and $U^\perp$. You can do this by induction on the dimension of U.*

And we note that $U^\perp$ is also a G-submodule of V because for all $u \in U$ and $v \in U^\perp$ and $a \in G$ we have

$$\begin{aligned}(a \cdot v, u) &= (a \cdot v, a \cdot a^{-1} \cdot u) \\ &= (v, a^{-1}u) \\ &= 0\end{aligned}$$

since $v \in U^\perp$ and $a^{-1}u \in U$. $\square$ Theorem 8.30

Corollary 8.31. *Suppose G is a compact Lie grape; suppose V is a finite-dimensional G-module with associated representation $\rho\colon G \to \mathrm{GL}(V)$. Let $(\cdot,\cdot)$ be a G-invariant inner product on V. Then*

1. *V is irreducible if and only if $\mathrm{End}_G(V) = \{\, cI : c \in \mathbb{C} \,\}$.*
2. *$\overline{V} \cong V^*$. (Here $\overline{V}$ is the complex conjugate of V, not the conjugate transpose, and V^* is the dual of V.)*
3. *The G-invariant inner product on V is unique up to multiplication by a positive real number.*
4. *If U_1, U_2 are G-submodules of V*

   TODO 24. *irreducible?*

   with $U_1 \not\cong U_2$ then $U_1 \perp U_2$.

Proof.

1. If V is irreducible then $\mathrm{End}_G(V) = \{\, cI : c \in \mathbb{C} \,\}$ by Schur's lemma. If V is reducible, say $0 \neq U \subseteq V$ is a G-submodule, then $V \cong U \oplus U^\perp$, so $\dim(\mathrm{End}_G(V)) \geq 2$ (since $\mathrm{End}_G(V)$ contains $cI_U \oplus dI_{U^\perp}$ for $c, d \in \mathbb{C}$).
2. Let $\mathcal{U}$ be an orthonormal basis for V. For $a \in G$ we let $A := [\rho(a)]_{\mathcal{U}} \in \mathrm{U}(n)$. Then we have $[\overline{\rho}(a)]_{\mathcal{U}} = \overline{A}$, and if $\mathcal{J}$ is the dual basis for V^* then $[\rho^*(a)]_{\mathcal{J}} = (A^{-1})^T = \overline{A}$ (by definition of the dual representation, and since $A^*A = I$).
3. The inner product $(\cdot,\cdot)$ gives a linear isomorphism $L\colon \overline{V} \to V^*$ given by $L(u)(v) = (v, u)$ for $u \in \overline{V} = V$ and $v \in V$. Another inner product $\langle\cdot,\cdot\rangle$ gives another isomorphism $M\colon \overline{V} \to V^*$ given by $M(u)(v) = \langle v, u\rangle$. By a similar argument to the proof of Schur's lemma we get that L and M differ by a constant $c \in \mathbb{C}$; by positive definiteness, we get $c \in \mathbb{R}_{>0}$.
4. Suppose U_1, U_2 are irreducible submodules of V. Suppose that U_1 is not orthogonal to U_2; so there is $u_1 \in U_1$ and $u_2 \in U_2$ such that $(u_1, u_2) \neq 0$. Define $L\colon \overline{U_1} \to U_2^*$ by $L(u_1)(u_2) = (u_2, u_1)$. Then

   $$\ker(L) = \{\, u_1 \in \overline{U_1} = U_1 : u_1 \in U_2^\perp \,\} = U_1 \cap U_2^\perp$$

   Since U_1 is irreducible, we get that $\ker(L)$ is either 0 or U_2. But by assumption there is $u_1 \in U_1$ and $u_2 \in U_2$ such that $(u_1, u_2) \neq 0$; so $\ker(L) = 0$, and L is injective. Also L must be surjective since $L(U_1) \subseteq U_2^*$ and U_2^* is irreducible (since $U_2^* \cong \overline{U_2}$ and U_2 is irreducible). Thus $\overline{U_1} \cong U_2^* \cong \overline{U_2}$; so $U_1 \cong U_2$. $\square$ Corollary 8.31

Let G be a compact Lie grape, and let W be a finite-dimensional G-module. By the above theorem and its corollaries, W decomposes as a direct sum of irreducible submodules, and if we group together isomorphic irreducible submodules, we have

$$W = \bigoplus_{k=1}^{\ell} W_k$$

with $W_k \cong V_k^{\oplus m_k}$ where the V_k are irreducible G-modules, and when $k \neq \ell$, $V_k \not\cong V_\ell$ and $W_k \perp W_\ell$. Note that the submodules W_k are canonical (i.e., they are determined up to isomorphism by W). Indeed, W_k is equal to the sum of all submodules of W which are isomorphic to V_k because if U and V are submodules of W with $U = U_1 \oplus ... \oplus U_\ell$ with each $U_i \cong V_k$ and also $V \cong V_k$, then we have $U \cap V \subseteq V$, which is irreducible, so either $U \cap V = V$, in which case $U + V = U$, or $U \cap V = 0$, in which case $U + V = U \oplus V = U_1 \oplus ... \oplus U_\ell \oplus V$.

We use the following notation. Let $\widehat{G}$ be (a set of representatives for) the set of all isomorphism classes of irreducible finite-dimensional (unitary) representations of G. For any finite-dimensional G-module W and any $\sigma \in \widehat{G}$, we write E_σ for the G-module associated to σ (so $\sigma\colon G \to \mathrm{GL}(E_\sigma)$). Then, our the above work, we can write a decomposition

$$W = \bigoplus_{\sigma \in \widehat{G}} W_\sigma$$

where each W_σ is a G-submodule of W for which there exists an integer $m_\sigma(W)$ such that $W_\sigma \cong E_\sigma^{m_\sigma(W)}$.

Definition 8.32. The integer $m_\sigma(W)$ is called the *multiplicity* of σ in W. Note that $m_\sigma(W) = \dim(W_\sigma)/\dim(E_\sigma)$. The decomposition

$$W = \bigoplus_{\sigma \in \widehat{G}} W_\sigma$$

with $W_\sigma \cong E_\sigma^{m_\sigma(W)}$ is called the *canonical decomposition* of the G-module W, and W_σ is called the *isotypical component* for σ.

Theorem 8.33. *Let G be a compact Lie grape. Let W be a finite-dimensional G-module of G. Let $\sigma \in \widehat{G}$. The map $F\colon \hom_G(E_\sigma, W) \otimes E_\sigma \to W_\sigma$ given by $F(L, u) := L(u)$ is a G-module isomorphism; so*

$$W_\sigma \cong \hom_G(E_\sigma, W) \otimes E_\sigma$$

and

$$m_\sigma(W) = \dim(\hom_G(E_\sigma, W))$$

Proof. Note that G acts on $\hom_G(E_\sigma, W)$ by $(aL)(u) = aL(a^{-1}u)$, and when $L \in \hom_G(E_\sigma, W)$, we have $L(au) = aL(u)$. Thus

$$(aL)(u) = aL(a^{-1}u) = aa^{-1}L(u) = L(u)$$

so $aL = L$ when $L \in \hom_G(E_\sigma, W)$ (so G acts trivially on $\hom_G(E_\sigma, W)$).

We claim that F is well-defined (i.e., F does take values in W_σ, not just W). For $L \in \hom_G(E_\sigma, W)$, we have $\ker(L) \subseteq E_\sigma$, which is irreducible, so either $\ker(L) = 0$ or $\ker(L) = E_\sigma$. When $\ker(L) = 0$, we have $L(E_\sigma) \cong E_\sigma$, hence $L(E_\sigma) \subseteq W_\sigma$, since W is equal to a sum of W_σ's which are isomorphic to powers E_σ. We claim that F is G-invariant (also called G-equivariant or G-intertwining). For $L \in \hom_G(E_\sigma, W)$ and $u \in E_\sigma$ and $a \in G$, we have

$$F(a(L \otimes u)) = F(aL \otimes au) = F(L \otimes au) = L(au) = aL(u) = F(l \otimes u)$$

since $aL = L$ and since L is G-invariant. Thus F is G-invariant.

We also claim that F is surjective. We can use the same argument we used to show that F is well-defined. Let $v \in W_\sigma$. Since W_σ is isomorphic to a power of E_σ, we can choose a submodule $V \subseteq W_\sigma$ with $v \in V$ and $V \cong E_\sigma$ (as a G-module). Let $L\colon E_\sigma \to V$ be a G-module isomorphism, and let $u = L^{-1}(v)$. Then $F(L \otimes u) = L(u) = v$. This proves F is surjective.

We also claim that F is injective. We do this by counting dimensions. We have

$$F : \hom_G(E_\sigma, W) \otimes E_\sigma \to W_\sigma$$

where $W_\sigma \cong E_\sigma^{\oplus m_\sigma(W)}$. So $\dim(W_\sigma) = m_\sigma \dim(E_\sigma)$. Also,

$$\begin{aligned}
\hom_G(E_\sigma, W) &= \hom_G\left(E_\sigma, W_\sigma \oplus \bigoplus_{\tau\neq\sigma} W_\tau\right) \\
&\cong \hom_G\left(E_\sigma, E_\sigma^{m_\sigma} \oplus \bigoplus_{\tau\neq\sigma} E_\tau^{m_\tau}\right) \\
&\cong \hom_G(E_\sigma, E_\sigma)^{\oplus m_\sigma} \oplus \bigoplus_{\tau\neq\sigma} \hom_G(E_\sigma, E_\tau)^{\oplus m_\tau}
\end{aligned}$$

Then we take dimensions. The leftmost hom in the last line has dimension 1 by Schur's lemma, and the other hom's in the last line have have dimension 0. Therefore,

$$\dim(\hom_G(E_\sigma, W)) = \dim\left(\hom_G(E_\sigma, E_\sigma)^{\oplus m_\sigma} \oplus \bigoplus_{\tau\neq\sigma} \hom_G(E_\sigma, E_\tau)^{\oplus m_\tau}\right) = m_\sigma,$$

which implies that

$$\dim(\hom_G(E_\sigma, W) \otimes E_\sigma) = m_\sigma \dim(E_\sigma) = \dim(W_\sigma)$$

So F is injective. □ Theorem 8.33

9 More on maximal tori

Recall that for any Lie grape G and any representation $\rho\colon G \to \mathrm{GL}(V)$ induces a representation $\rho_*\colon \mathfrak{g} \to \mathrm{End}(V)$. Also for any Lie grape G we have the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$ given by $\mathrm{Ad}(a) = \mathrm{d}c_a$; when G is a matrix Lie grape and $P \in G, X \in \mathfrak{g}$ we have $\mathrm{Ad}(P)(X) = PX^{-1}P$. The adjoint representation on G induces the adjoint representation $\mathrm{ad} = \mathrm{Ad}_*\colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$; when G is a matrix Lie grape and $A, X \in \mathfrak{g}$ we have $\mathrm{ad}(A)(X) = [A, X]$.

Note that $\mathfrak{g}$ is a real vector space, so Ad and ad are real representations; so Schur's lemma does not hold in a simple form for real representations. But we can still construct an Ad-invariant (real) inner product: choose any inner product on $\mathfrak{g}$, and then define

$$(u, v) = \int_G \langle xu, xv\rangle \mathrm{d}g(x)$$

Example 9.1. Suppose G is a connected matrix Lie grape; show that $\ker(\mathrm{Ad}) = Z(G)$.

Note $\ker(\mathrm{Ad}) = \{\, P \in G : \mathrm{Ad}(P) = I\colon \mathfrak{g} \to \mathfrak{g} \,\}$. If $P \in Z(G)$ then $C_P = I\colon G \to G$; so $\mathrm{Ad}(P) = \mathrm{d}C_P = I\colon \mathfrak{g} \to \mathfrak{g}$; so $P \in \ker(\mathrm{Ad})$.

Suppose $P \in \ker(\mathrm{Ad})$; then $\mathrm{Ad}(P) = I\colon \mathfrak{g} \to \mathfrak{g}$; so $(\mathrm{Ad}(P))(A) = A$ for all $A \in \mathfrak{g}$, and $PAP^{-1} = A$ for all $A \in \mathfrak{g}$. Hence for $Q \in G$, if we can choose $A \in \mathfrak{g}$ such that $Q = \exp(A)$ then

$$PQP^{-1} = P\exp(A)P^{-1} = \exp(PAP^{-1}) = \exp(A) = Q$$

But G is connected; so we can choose $A_1, \dots, A_\ell$ such that $Q = \exp(A_1)\cdots\exp(A_\ell)$. Then

$$PQP^{-1} = P\exp(A_1)P^{-1}P\exp(A_2)P^{-1}\cdots P\exp(A_\ell)P^{-1} = \exp(A_1)\cdots\exp(A_\ell) = Q$$

so $P \in Z(G)$.

Theorem 9.2. *Suppose G is a matrix Lie grape with Lie algebra $\mathfrak{g}$; suppose V is a finite-dimensional G-module and $U \subseteq V$ is a subspace. If U is G-invariant (i.e. $P \cdot u = \rho(P)(u) \in U$ for all $P \in G$ and $u \in U$) then U is $\mathfrak{g}$-invariant (i.e. $Au = (\rho_* A)(u) \in U$ for all $A \in \mathfrak{g}$ and $u \in U$). If U is $\mathfrak{g}$-invariant and G is connected then U is G-invariant.*

Proof. Suppose U is G-invariant; suppose $A \in \mathfrak{g}$ and $u \in U$. Then $tA \in \mathfrak{g}$ for all $t \in \mathbb{R}$, so $\exp(tA) \in G$ and hence $\rho(\exp(tA))u \in U$ for all $t \in \mathbb{R}$. Thus

$$(\rho_* A)(u) = \frac{\mathrm{d}}{\mathrm{d}t}(\rho(\exp(tA))(u))|_{t=0} \in U$$

(since if $u(t) \in U$ for all t then $u'(t) \in U$).

Suppose that U is $\mathfrak{g}$-invariant and G is connected. Suppose $P \in G$ and $u \in U$. If $P = \exp(A)$ for some $A \in \mathfrak{g}$ then

$$\rho(P)(u) = \rho(\exp(A))(u) = \exp(\rho_* A)u = \sum_{n=0}^{\infty} \frac{1}{n!}(\rho_* A)^n u \in U$$

since $(\rho_* A)^n(u) \in U$ for all n (one checks this last by induction). In general, since G is connected, we can choose $A_1, \ldots, A_\ell$ such that $P = \exp(A_1) \cdots \exp(A_\ell)$; it follows by induction on ℓ that $\rho(P) = \rho(\exp(A_1)) \cdots \rho(\exp(A_\ell)) \in U$. ☐ Theorem 9.2

Aside 9.3 (Remarks on A3). Change 4(c) to "determine whether". 5(c) can be computationally intensive.

Theorem 9.4. *Suppose G is a matrix Lie grape with Lie algebra $\mathfrak{g}$; suppose V and W are finite-dimensional G-modules with associated representations ρ and φ. Suppose $L \in \hom(V, W)$.*

TODO 25. *I assume this is $\mathcal{L}(V, W)$?*

Then if L is G-invariant (meaning $L(\rho(P)(v)) = \varphi(P)(L(v))$ for all $P \in G$ and $v \in V$) then L is $\mathfrak{g}$-invariant (meaning that $L(\rho_*(A)(v)) = \varphi_*(A)(L(v))$ for all $A \in \mathfrak{g}$ and $v \in V$). Conversely if L is $\mathfrak{g}$-invariant and G is connected then L is G-invariant.

Proof. Assignment 3. ☐ Theorem 9.4

Theorem 9.5. *Suppose G is a compact matrix Lie grape with Lie algebra $\mathfrak{g}$; suppose V is a finite-dimensional G-module with associated representation ρ. Suppose $(\cdot, \cdot)$ is a (Hermitian or real)*

TODO 26. *?*

inner product on V. Then if $(\cdot, \cdot)$ is G-invariant (meaning that $(\rho(P)(u), \rho(P)(v)) = (u, v)$ for all $P \in G$ and $u, v \in V$) then $(\cdot, \cdot)$ is $\mathfrak{g}$-invariant (meaning $(\rho_*(A)(u), v) + (u, \rho_*(A)(v)) = 0$ for all $A \in \mathfrak{g}$ and $u, v \in V$).

Proof. Suppose $(\cdot, \cdot)$ is G-invariant. Suppose $A \in \mathfrak{g}$ and $u, v \in \mathfrak{g}$.

TODO 27. *$\in V$?*

Then

$$(u, v) = (\rho(\exp(tA))(u), \rho(\exp(tA))(v)) = (\exp(t\rho_* A)u, \exp(t\rho_* A)v)$$

for all $t \in \mathbb{R}$. Note that if we choose any basis for V so that $u(t)$ and $v(t)$ become vectors

TODO 28. *?*

and the inner product is given by a matrix, then

$$\frac{\mathrm{d}}{\mathrm{d}t}(u(t), v(t)) = \frac{\mathrm{d}}{\mathrm{d}t} v(t)^* B \cdot u(t) = (v'(t))^* B u(t) + (v(t))^* B u'(t) = (u(t), v'(t)) + (u'(t), v(t))$$

Thus

$$(\exp(t\rho_* A) \cdot \rho_* A \cdot u, \exp(t\rho_* A)v) + (\exp(t\rho_* A)u, \exp(t\rho_* A) \cdot \rho_* A \cdot v) = 0$$

so at $t = 0$ we get $(\rho_* A \cdot u, v) + (u, \rho_* A \cdot v) = 0$. ☐ Theorem 9.5

Theorem 9.6. *Suppose G is a compact matrix Lie grape with Lie algebra $\mathfrak{g}$. Suppose $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$. Then there is $A \in \mathfrak{t}$ such that $\mathfrak{t} = \mathfrak{z}_{\mathfrak{g}}(A) = \{ X \in \mathfrak{g} : [A, X] = 0 \}$.*

An element $A \in \mathfrak{g}$ such that $\mathfrak{z}(A)$ is a Cartan subalgebra is called *regular*.

Proof. Choose an Ad-invariant inner product $(\cdot,\cdot)$ on $\mathfrak{g}$. Let $\{ A_1, \dots, A_\ell \}$ be a basis for $\mathfrak{t}$. Note that since $\mathfrak{t}$ is abelian, we have $[A_k, A_\ell] = 0$ for all k, ℓ (i.e. $A_\ell \in \ker(\mathrm{ad}(A_k))$). Also if $Y \in \mathfrak{g}$ but $Y \notin \mathfrak{t}$ then $[A_k, Y] \neq 0$ for some k since $\mathfrak{t}$ is maximal; thus

$$\mathfrak{t} = \bigcap_{k=1}^{\ell} \ker(\mathrm{ad}(A_k))$$

Claim 9.7. *For all $A, B \in \mathfrak{t}$ we can find $r \in \mathbb{R}$ so that $\ker(\mathrm{ad}(A + rB)) = \ker(\mathrm{ad}(A)) \cap \ker(\mathrm{ad}(B))$.*

Proof. Suppose $A, B \in \mathfrak{t}$. Note that since $[A, B] = 0$ it follows that $\mathrm{ad}(A)$ commutes with $\mathrm{ad}(B)$; indeed

$$\mathrm{ad}(A)(\mathrm{ad}(B)(X)) = [A, [B, X]] = [A, BX] - [A, XB] = ABX - BXA - AXB + XBA$$

and likewise

$$\mathrm{ad}(B)(\mathrm{ad}(A)(X)) = BAX - AXB - BXA + XAB = \mathrm{ad}(A)(\mathrm{ad}(B)(X))$$

since $AB = BA$. Let

$$\begin{aligned} \mathfrak{h} &= \ker(\mathrm{ad}(A)) = \{ X : [A, X] = 0 \} \\ \mathfrak{l} &= \ker(\mathrm{ad}(B)) = \{ X : [B, X] = 0 \} \end{aligned}$$

Since $\mathrm{ad}(A)$ and $\mathrm{ad}(B)$ commute, it follows that $\mathfrak{h}$ (and hence also $\mathfrak{h}^\perp$) are invariant under $\mathrm{ad}(B)$; indeed, for $X \in \mathfrak{h} = \ker(\mathrm{ad}(A))$ we have

$$\mathrm{ad}(A)(\mathrm{ad}(B)(X)) = \mathrm{ad}(B)(\mathrm{ad}(A)(X)) = \mathrm{ad}(B)(0) = 0$$

and for $Y \in \mathfrak{h}^\perp$ and $X \in \mathfrak{h}$ we have

$$((\mathrm{ad}(B)(Y), X) = -(Y, \underbrace{\mathrm{ad}(B)(X)}_{\in \mathfrak{h}})$$

since $(\cdot,\cdot)$ is $\mathfrak{g}$-invariant. It follows that $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{l}) \oplus (\mathfrak{h} \cap \mathfrak{l}^\perp)$, and thus

$$\mathfrak{g} = (\mathfrak{h} \cap \mathfrak{l}) \oplus (\mathfrak{h} \cap \mathfrak{l}^\perp) \oplus (\mathfrak{h}^\perp \cap \mathfrak{l}) \oplus (\mathfrak{h}^\perp \cap \mathfrak{l}^\perp)$$

Case 1. Suppose $\mathfrak{h}^\perp \cap \mathfrak{l}^\perp = 0$.

Subclaim 9.8. $\ker(\mathrm{ad}(A + B)) = \ker(\mathrm{ad}(A)) \cap \ker(\mathrm{ad}(B))$.

Proof. If $X \in \ker(\mathrm{ad}(A)) \cap \ker(\mathrm{ad}(B))$ then $[A, X] = [B, X] = 0$, so $[A + B, X] = 0$ and $X \in \ker(\mathrm{ad}(A + B)$. Conversely if $X \in \ker(\mathrm{ad}(A + B))$ then $[A + B, X] = 0$, so $[A, X] = -[B, X]$; but $[A, X] \in \mathfrak{h}^\perp$ since for $Y \in \mathfrak{h}$ we have

$$([A, X], Y) = ((\mathrm{ad}(A))(X), Y) = -(X, (\mathrm{ad}(A))(Y)) = -(X, 0) = 0$$

Similarly we get $[B, X] \in \mathfrak{l}^\perp$. So $[A, X] = -[B, X] \in \mathfrak{h}^\perp \cap \mathfrak{l}^\perp = 0$; so $[A, X] = [B, X] = 0$, and hence $X \in \ker(\mathrm{ad}(A)) \cap \ker(\mathrm{ad}(B))$. ☐ Subclaim 9.8

So $r = 1$ works.

Case 2. Suppose $\mathfrak{h}^\perp \cap \mathfrak{l}^\perp \neq 0$.

Exercise 9.9. Finish this. Let $L(r) \colon \mathfrak{h}^\perp \cap \mathfrak{l}^\perp \to \mathfrak{h}^\perp \cap \mathfrak{l}^\perp$ be $L(r) = \mathrm{ad}(A + rB)$ for $r \in \mathbb{R}$. Consider $f(r) = \det(L(r))$. Find $r \neq 0$ such that $f(r) \neq 0$, and show that r works.

TODO 29. *Delete the above? The following seems to subsume it.*

Use bases for each of the four space $(\mathfrak{h} \cap \mathfrak{l})$, etc. to make a basis for $\mathfrak{g}$. With respect to this basis $\mathrm{ad}(A)$ and $\mathrm{ad}(B)$ have matrices of the form

$$[\mathrm{ad}(A)] = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & C & \\ & & & D \end{pmatrix}$$

$$[\mathrm{ad}(B)] = \begin{pmatrix} 0 & & & \\ & E & & \\ & & 0 & \\ & & & F \end{pmatrix}$$

and for $r \in \mathbb{R}$ we have

$$[\mathrm{ad}(A + rB)] = \begin{pmatrix} 0 & & & \\ & rE & & \\ & & C & \\ & & & D + rF \end{pmatrix}$$

Case 1. Suppose $\mathfrak{h}^\perp \cap \mathfrak{l}^\perp = \{ 0 \}$. Then

$$[\mathrm{ad}(A + rB)] = \begin{pmatrix} 0 & & \\ & rE & \\ & & C \end{pmatrix}$$

so for all $r \neq 0$ we have $\ker(\mathrm{ad}(A + rB)) = \mathfrak{h} \cap \mathfrak{l} = \ker(\mathrm{ad}(A)) \cap \ker(\mathrm{ad}(B))$, as desired.

Case 2. Suppose $\mathfrak{h}^\perp \cap \mathfrak{l}^\perp \neq 0$. Then the map $L(r) \colon = \mathrm{ad}(A + rB) \colon \mathfrak{h}^\perp \cap \mathfrak{l}^\perp \to \mathfrak{h}^\perp \cap \mathfrak{l}^\perp$ has matrix $[L(r)] = D + rF$.

When $r = 0$ we have $[L(0)] = D$ is invertible. So if $f(r) = \det(L(r))$, then f is a polynomial in r and $f(0) \neq 0$; so $f(r) \neq 0$ for all but finitely many values of r. We can choose $r \in \mathbb{R} \setminus \{ 0 \}$ such that $f(r) \neq 0$; then $L(r)$ is invertible, so $D + rE$ is invertible. So

$$\ker(\mathrm{ad}(A + rB) = \mathfrak{h} \cap \mathfrak{l} = \ker(\mathrm{ad}(A)) \cap \ker(\mathrm{ad}(B))$$

as desired. □ Claim 9.7

We then replace A_1 by $A_1' = A_1 + rA_2$ so that $\ker(\mathrm{ad}(A_1')) = \ker(\mathrm{ad}(A_1)) \cap \ker(\mathrm{ad}(A_2))$; then replace A_1' by $A_1'' = A_1' + r' A_3$, and so on. □ Theorem 9.6

Theorem 9.10 (Cartan subalgebras)**.** *Suppose G is a compact matrix Lie grape with Lie algebra $\mathfrak{g}$. Then*

1. *If $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$ and if $A \in \mathfrak{g}$ then there is $P \in G$ such that $PAP^{-1} \in \mathfrak{t}$ (equivalently, there is $P \in G$ such that $A \in P\mathfrak{t}P^{-1}$). Equivalently*

$$\mathfrak{g} = \bigcup_{P \in G} P\mathfrak{t}P^{-1}$$

2. *If $\mathfrak{s}, \mathfrak{t}$ are two Cartan subalgebras, then there is $P \in G$ such that $\mathfrak{t} = P\mathfrak{s}P^{-1}$. i.e. G acts transitively on the set of Cartan subalgebras by conjugation.*

Proof.

1. Choose an Ad-invariant inner product $(\cdot, \cdot)$ on $\mathfrak{g}$.

   Suppose $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$; suppose $A \in \mathfrak{g}$. By previous theorem there is $B \in \mathfrak{t}$ such that $\mathfrak{t} = \mathfrak{z}_{\mathfrak{g}}(B) = \{ X \in \mathfrak{g} : [B, X] = 0 \}$.

For $P \in G$ we have

$$\begin{aligned} PAP^{-1} \in \mathfrak{t} &\iff PAP^{-1} \in \mathfrak{z}(B) \\ &\iff [PAP^{-1}, B] = 0 \\ &\iff ([PAP^{-1}, B], X) = 0 \text{ for all } X \in \mathfrak{g} \\ &\iff (\mathrm{ad}(PAP^{-1})(B), X) = 0 \text{ for all } X \in \mathfrak{g} \\ &\iff (B, \mathrm{ad}(PAP^{-1})(X)) = 0 \text{ for all } X \in \mathfrak{g} \\ &\iff (B, [PAP^{-1}, X]) = 0 \text{ for all } X \in \mathfrak{g} \end{aligned}$$

Let $f\colon G \to \mathbb{R}$ be $P \mapsto (B, PAP^{-1})$. By compactness of G there is $P \in G$ maximizing $f(P)$. Suppose now that $X \in \mathfrak{g}$. Let $g\colon \mathbb{R} \to \mathbb{R}$ be $g(t) = (B, \exp(tX)PAP^{-1}\exp(-tX))$; then by choice of P we have that $g(t)$ has a local maximum at $t = 0$. But

$$g'(t) = (B, \exp(tX)XPAP^{-1}\exp(-tX) - \exp(tX)PAP^{-1}\exp(-tX)X$$

So

$$0 = g'(0) = (B, XPAP^{-1} - PAP^{-1}X) = (B, [X, PAP^{-1}])$$

Thus for all X we have $(B, [PAP^{-1}, X]) = 0$; so $PAP^{-1} \in \mathfrak{t}$.

2. Suppose $\mathfrak{s}$ and $\mathfrak{t}$ are Cartan subalgebras. Choose $A \in \mathfrak{s}$ such that $\mathfrak{s} = \mathfrak{z}_{\mathfrak{g}}(A) = \{ X \in \mathfrak{g} : [A, X] = 0 \}$. Choose $P \in G$ such that $PAP^{-1} \in \mathfrak{t}$. We will show that $P\mathfrak{s}P^{-1} = \mathfrak{t}$.

   For $X \in \mathfrak{g}$ we have

$$\begin{aligned} X \in P\mathfrak{s}P^{-1} &\iff P^{-1}XP \in \mathfrak{s} = \mathfrak{z}_{\mathfrak{g}}(A) \\ &\iff [P^{-1}XP, A] = 0 \\ &\iff [X, PAP^{-1}] = 0 \\ &\iff X \in \mathfrak{z}_{\mathfrak{g}}(PAP^{-1}) \end{aligned}$$

   So $P\mathfrak{s}P^{-1} = \mathfrak{z}_{\mathfrak{g}}(PAP^{-1})$. So since $PAP^{-1} \in \mathfrak{t}$ and $\mathfrak{t}$ is abelian we have $\mathfrak{t} \subseteq \mathfrak{z}_{\mathfrak{g}}(PAP^{-1}) = P\mathfrak{s}P^{-1}$. So since $\mathfrak{t}$ and $P\mathfrak{s}P^{-1}$ are maximal abelian subalgebras of $\mathfrak{g}$ with $\mathfrak{t} \subseteq P\mathfrak{s}P^{-1}$, they are equal. □ Theorem 9.10

Theorem 9.11. *Suppose G is a connected compact matrix Lie grape with Lie algebra $\mathfrak{g}$. Then*

1. *If S and T are maximal tori in G then there exists $P \in G$ such that $PSP^{-1} = T$. Equivalently, G acts transitively on the set of maximal tori by conjugation.*

2. *$\exp(\mathfrak{g}) = G$.*

3. *If T is a maximal torus in G and $Q \in G$ then there exists $P \in G$ such that $PQP^{-1} \in T$; equivalently, if T is a maximal torus in G then*

$$G = \bigcup_{P \in G} PTP^{-1}$$

Proof.

1. Suppose S and T are maximal tori. Let $\mathfrak{s}$ and $\mathfrak{t}$ be their Lie algebras, which are Cartan subalgebras.

   TODO 30. *?*

   Choose $P \in G$ such that $P\mathfrak{s}P^{-1} = \mathfrak{t}$. Then for $Q \in S$ since $\exp(\mathfrak{s}) = S$ we can choose $B \in \mathfrak{s}$ such that $Q = \exp(B)$. Then

$$PQP^{-1} = P\exp(B)P^{-1} = \exp(\underbrace{PBP^{-1}}_{\in \mathfrak{t}}) \in T$$

   Thus $PSP^{-1} \subseteq T$; so since T and PSP^{-1} are maximal tori, we get $PSP^{-1} = T$.

2. We shall show that $\exp(\mathfrak{g})$ is open and closed in G, and hence $\exp(\mathfrak{g}) = G$ since G is connected. If we fix a maximal torus T and let $\mathfrak{t}$ be its Lie algebra, then

$$\mathfrak{g} = \bigcup_{P \in G} P\mathfrak{t}P^{-1}$$

So

$$\exp(\mathfrak{g}) = \bigcup_{P \in G} PTP^{-1}$$

(since exp is surjective on tori). So if $F\colon G \times T \to G$ is $(P, X) \mapsto PXP^{-1}$ then $\exp(\mathfrak{g}) = F(G \times T)$ is closed, since $G \times T$ is compact.

It remains to show that $\exp(\mathfrak{g})$ is open. Suppose $Q \in \exp(\mathfrak{g})$, say $Q = \exp(B)$ for $B \in \mathfrak{g}$. Let $\mathfrak{h} = \mathfrak{z}_{\mathfrak{g}}(Q) = \{ X \in \mathfrak{g} : QXQ^{-1} = X \}$; let H be the connected component of $Z_G(Q)$ containing I. One checks that H is a closed Lie subgrape of G with Lie algebra $\mathfrak{h}$. Note that $Q = \exp(B) \in \exp(\mathfrak{h}) \subseteq H$.

TODO 31. *We changed from $\mathfrak{z}_{\mathfrak{g}}(B)$ and $Z_G(B)$ to $\mathfrak{z}_{\mathfrak{g}}(Q)$ and $Z_G(Q)$; make sure this doesn't change anything in case 1 below.*

Case 1. Suppose $\mathfrak{h} = \mathfrak{g}$; so $\mathfrak{z}_g(B) = \mathfrak{g}$. Then $Q \in Z(G)$ since for $P \in G$ we have $PQP^{-1} = P\exp(B)P^{-1} = \exp(PBP^{-1}) = \exp(B) = Q$.

TODO 32. *?*

Choose a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ so that $B \in \mathfrak{t} \subseteq \mathfrak{z}(B)$. For $X \in \mathfrak{g}$ choose $P \in G$ so that $P^{-1}XP \in \mathfrak{t}$; say $P^{-1}XP = Y \in \mathfrak{t}$ so $X = PYP^{-1}$. Then

$$\begin{aligned} Q\exp(X) &= Q\exp(PYP^{-1}) = QP\exp(Y)P^{-1} \\ &= PQ\exp(Y)P^{-1} \\ &= P\exp(B)\exp(Y)P^{-1} \\ &= \exp(P(B+Y)P^{-1}) \in \exp(\mathfrak{g}) \end{aligned}$$

Hence $Q\exp(\mathfrak{g}) \subseteq \exp(\mathfrak{g})$. But $\exp(\mathfrak{g})$ contains an open neighbourhood of I; so $Q\exp(\mathfrak{g})$ contains an open neighbourhood of Q. So $\exp(\mathfrak{g})$ contains an open neighbourhood of Q, as required.

Case 2. Suppose $\mathfrak{h} \subsetneqq \mathfrak{g}$. We can suppose, inductively on $\dim(\mathfrak{g})$, that $\exp(\mathfrak{h}) = H$ (since H is connected and compact). Consider $F\colon \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp} \to G$

TODO 33. *I think we said we're using an* Ad*-invariant inner product? It was something-invariant.*

given by $F(X, Y) = Q^{-1}\exp(Y)Q\exp(X)\exp(-Y)$ for $X \in \mathfrak{h}$ and $Y \in \mathfrak{h}^{\perp}$. Note that

$$\begin{aligned} F(X, Y) &= Q^{-1}(I + Y + \frac{1}{2}Y^2 + \cdots)Q(I + X + \frac{1}{2}X^2 + \cdots)(I - Y + \frac{1}{2}Y^2 - \cdots) \\ &= I + (Q^{-1}YQ + X - Y) + \text{higher} - \text{orderterms} \end{aligned}$$

So at $0 = (0, 0) \in \mathfrak{g}$ we have $DF_{(0,0)}(X, Y) = X, Q^{-1}YQ - Y)$; i.e. $DF = I_{\mathfrak{h}} \oplus (\mathrm{Ad}(Q^{-1}) - I)_{\mathfrak{h}^{\perp}}$. Note that

$$\mathfrak{h} = \mathfrak{z}_{\mathfrak{g}}(Q) = \{ X \in \mathfrak{g} : QXQ^{-1} = X \} = \{ X \in \mathfrak{g} : X = Q^{-1}XQ \} = \ker(\mathrm{Ad}(Q^{-1}) - I)$$

So $\mathrm{Ad}(Q^{-1}) - I\colon \mathfrak{h}^{\perp} \to \mathfrak{h}^{\perp}$ is invertible; so at $0 \in \mathfrak{g}$ we have that DF is invertible. So $F\colon \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp} \to G$ is a local diffeomorphism. So

$$\{ Q^{-1}\exp(Y)Q\exp(X)\exp(-Y) : X \in \mathfrak{h}, Y \in \mathfrak{h}^{\perp} \}$$

contains an open neighbourhood of I in G. Hence, multiplying on the left by Q, we get that

$$\{ \exp(Y)Q\exp(X)\exp(-Y) : X \in \mathfrak{h}, Y \in \mathfrak{h}^{\perp} \}$$

contains an open neighbourhood of Q. But $\exp(X) \in H$, and $Q\exp(X) \in H$; so

$$\{\, \exp(Y)Q\exp(X)\exp(-Y) : X \in \mathfrak{h} Y \in \mathfrak{h}^\perp \,\} \subseteq \bigcup_{Y\in\mathfrak{h}^\perp} \exp(Y)H\exp(-Y) \subseteq \bigcup_{P\in G} PHP^{-1}$$

For $R \in H$ we can write $R = \exp(X)$ for some $X \in \mathfrak{h}$ (since by induction hypothesis we get that $H = \exp(\mathfrak{h})$); then $PRP^{-1} = P\exp(X)P^{-1} = \exp(PXP^{-1} \in \exp(\mathfrak{g})$. Thus

$$\{\, \exp(Y)Q\exp(X)\exp(-Y) : X \in \mathfrak{h}, Y \in \mathfrak{h}^\perp \,\} \subseteq \bigcup_{P\in G} PHP^{-1} \subseteq \exp(\mathfrak{g})$$

Thus $\exp(\mathfrak{g})$ contains an open neighbourhood of Q in G, as required.

3. Suppose T is a maximal torus in G; suppose $Q \in G$. By above we get $\exp(\mathfrak{g}) = G$, so we can pick $B \in \mathfrak{g}$ such that $\exp(B) = Q$. Let $\mathfrak{t}$ be the Lie algebra of T. Choose $P \in G$ such that $PBP^{-1} \in \mathfrak{t}$. Then $PQP^{-1} = P\exp(B)P^{-1} = \exp(PBP^{-1}) \in T$. □ Theorem 9.11

Corollary 9.12. *Suppose G is a connected and compact matrix Lie grape; suppose T is a maximal torus in G. Then*

1. $Z_G(T) = T$.
2. $Z(G) = \bigcap_{P\in G} PTP^{-1}$.

Proof.

1. Since T is abelian we get $T \subseteq Z_G(T)$. Conversely, suppose $Q \in Z_G(T)$. Since $\exp(\mathfrak{g}) = G$ we can write $Q = \exp(B)$ for some $B \in \mathfrak{g}$. Let H be the connected component of $Z_G(Q)$ containing I. Note that $Q \in H$ since $Q \in Z_G(Q)$ and $\alpha(t) = \exp(tB)$ is a path in $Z_G(Q)$ from I to Q. (Note $\exp(tB) \in Z_G(Q)$ since $\exp(tB)$ commutes with $Q = \exp(B)$.) Also $T \subseteq H$ since $Q \subseteq Z_G(T)$ so $T \subseteq Z_G(Q)$, and T is connected and contains I. Thus T is a maximal torus in H; so by theorem there is $P \in H$ such that $PQP^{-1} \in T$. But $P \in H \subseteq Z_G(Q)$; so $Q = PQP^{-1} \in T$.
2. Follows from previous item and theorem. □ Corollary 9.12

10 Weights and roots

A representation $\rho\colon G \to \mathrm{GL}(V)$ induces $\rho_*\colon \mathfrak{g} \to \mathrm{End}(V)$, which we can extend (by $\mathbb{C}$-linearity) to $\rho_*\colon \mathfrak{g}_\mathbb{C} \to \mathrm{End}(V)$.

Aside 10.1. Given a real vector space U we obtain a complex vector space $U_\mathbb{C} = U \otimes_\mathbb{R} \mathbb{C}$ the set of bilinear maps $L\colon U^* \times \mathbb{C}^* \to \mathbb{R}$; so $U_\mathbb{C} = \mathrm{span}\{\, u \otimes c : u \in U, c \in \mathbb{C} \,\} = \mathrm{span}\{\, u_1 \otimes 1, \dots, u_n \otimes 1, u_1 \otimes i, \dots, u_n \otimes i \,\}$ where $\mathcal{U} = \{\, u_1, \dots, u_n \,\}$ is a basis for U over $\mathbb{R}$ (and using the fact that $\{\, 1, i \,\}$ is a basis for $\mathbb{C}$ over $\mathbb{R}$). The scalar multiplication in $U_\mathbb{C}$ is given by $a \cdot (u \otimes b) = u \otimes (ab)$ where $a, b \in \mathbb{C}$ and $u \in U$. For $u \in U$ we write $u = u \otimes 1$ and $iu = u \otimes i$; so

$$U_\mathbb{C} = \mathrm{span}_\mathbb{R}\{\, u_1, \dots, u_n, iu_1, \dots, iu_n \,\} = \mathrm{span}_\mathbb{C}\{\, u_1, \dots, u_n \,\} = \mathrm{span}_\mathbb{C}\{\, iu_1, \dots, iu_n \,\}$$

We often write $U_\mathbb{C} = U \oplus iU$ (where U is identified with $\{\, u \otimes 1 : u \in U \,\}$).

When $G \subseteq \mathrm{U}(n)$ we have $\mathfrak{g} \subseteq \mathfrak{u}(n) = \{\, A \in M_n(\mathbb{C}) : A^* + A = 0 \,\}$. We can then identify $\mathfrak{u}(n)_\mathbb{C} = \mathfrak{u}(n) \oplus i\,\mathfrak{u}(n)$ with $\mathrm{GL}(n, \mathbb{C})$ as follows: given $A, B \in \mathfrak{u}(n)$ (so $A^* = -A$ and $B^* = -B$) we have $(iB)^* = -iB^* = iB$ and $A + iB \in M_n(\mathbb{C})$. On the other hand given $C \in M_n(\mathbb{C})$ we can write $C = A + iB$ with

$$A = \frac{C - C^*}{2}$$
$$B = \frac{C + C^*}{2i}$$

Exercise 10.2. Verify that $\mathfrak{su}(n)_{\mathbb{C}} = \mathfrak{su}(n) \oplus i\,\mathfrak{su}(n)$ can be identified with $\mathfrak{sl}(n,\mathbb{C}) = \{ A \in M_n(\mathbb{C}) : A \in M_n(\mathbb{C}) : \operatorname{tr}(A) = 0 \}$ and that $\mathfrak{so}(n)_{\mathbb{C}} = \mathfrak{so}(n) \oplus i\,\mathfrak{so}(n)$ can be identified with $\mathfrak{so}(n,\mathbb{C}) = \{ A \in M_n(\mathbb{C}) : A^T = -A \}$.

Example 10.3. Consider the action of $\mathrm{SU}(2)$ on the space $V_n \subseteq \mathbb{C}[x,y]$ of homogeneous polynomials of degree n. The action is given by

$$(P \cdot f)(x,y) = f\left(\left(P^{-1} \begin{pmatrix} x \\ y \end{pmatrix} \right)^T \right)$$

When

$$P = \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix}$$

we have

$$P^{-1} = \begin{pmatrix} \overline{a} & \overline{b} \\ -b & a \end{pmatrix}$$

so

$$P^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \overline{a}x + \overline{b}y \\ -bx + ay \end{pmatrix}$$

So

$$P \cdot (x^k y^\ell) = (\overline{a}x + \overline{b}y)^k (-bx + ay)^\ell$$

This induces an action $\rho_* \colon \mathfrak{su}(2) \to \operatorname{End}(V_n)$; this action is given by $(A \cdot f)(x,y) = \frac{\mathrm{d}}{\mathrm{d}t}(\exp(tA \cdot f)(x,y))|_{t=0}$. For

$$A = \begin{pmatrix} ir & -u + iv \\ u + iv & -ir \end{pmatrix} = \begin{pmatrix} ir & -\overline{w} \\ w & -ir \end{pmatrix} \in \mathfrak{su}(2)$$

we have $\det(A - xI) = (x^2 + r^2) + |w|^2 = x^2 + \theta^2$ where $\theta = \sqrt{r^2 + u^2 + v^2}$; so $A^2 = -\theta^2 I$.

Thus

$$\begin{aligned}
\exp(tA) &= I + tA + \frac{1}{2!}t^2A^2 + \frac{1}{3!}t^3A^3 + \cdots \\
&= I + tA - \frac{1}{2!}t^2\theta^2 I - \frac{1}{3!}t^3\theta^2 A + \frac{1}{4!}t^4\theta^4 I + \frac{1}{5!}t^4\theta^4 A - \cdots \\
&= (1 - \frac{1}{2!}t^2\theta^2 + \frac{1}{4!}t^4\theta^4 - \cdots)I + (t - \frac{1}{3!}t^3\theta^2 + \frac{1}{5!}t^5\theta^4 - \cdots)A \\
&= \cos(t\theta)I + \theta^{-1}\sin(t\theta)A \\
&= \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix}
\end{aligned}$$

where

$$\begin{aligned}
a &= \cos(t\theta) + ir\theta^{-1}\sin(t\theta) \\
b &= w\theta^{-1}\sin(t\theta) \\
&= (u + iv)\theta^{-1}\sin(t\theta)
\end{aligned}$$

Thus $A \cdot (x^k y^\ell) = \frac{\mathrm{d}}{\mathrm{d}t}(\overline{a}x + \overline{b}y)^k(-bx + ay)^\ell|_{t=0}$. So we compute

$$\frac{\mathrm{d}}{\mathrm{d}t}(\overline{a}x + \overline{b}y)^k(-bx + ay)^\ell = k(\overline{a}x + \overline{b}y)^{k-1}(\overline{a}'x + \overline{b}'y)(-bx + ay)^\ell + \ell(\overline{a}x + \overline{b}y)^k(-bx + ay)^{\ell-1}(-b'x + a'y)^\ell$$

Going back to our formulas for a and b we find

$$\begin{aligned}
a' &= -\theta\sin(t\theta) + ir\cos(t\theta) \\
b' &= w\cos(t\theta) \\
a(0) &= 1 \\
b(0) &= 0 \\
a'(0) &= ir \\
b'(0) &= w
\end{aligned}$$

So, putting these together, we find that

$$\begin{aligned}
A(x^k y^\ell) &= kx^{k-1}(-irx + \overline{w}y)y^\ell + \ell x^k y^{\ell-1}(-wx + iry) \\
&= -kirx^k y^\ell + k\overline{w}x^{k-1}y^{\ell+1} - \ell w x^{k+1} y^{\ell-1} + \ell irx^k y^\ell \\
&= (\ell - k)irx^k y^\ell + k\overline{w}x^{k-1}y^{\ell+1} - \ell w x^{k+1} y^{\ell-1}
\end{aligned}$$

We can extend the action $\rho_* \colon \mathfrak{su}(2) \to \mathrm{End}(V_n)$ to $\rho_* \colon \mathfrak{su}_{\mathbb{C}}(2) = \mathfrak{sl}(2,\mathbb{C}) \to \mathrm{End}(V_n)$. We have

$$\mathfrak{su}(2) = \mathrm{span}_{\mathbb{R}}\left\{ \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\}$$

and

$$\mathfrak{su}(2)_{\mathbb{C}} = \mathfrak{sl}(2,\mathbb{C}) = \mathrm{span}_{\mathbb{C}}\left\{ \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\} = \mathrm{span}_{\mathbb{C}}\{\, H, E, F \,\}$$

where

$$\begin{aligned}
H &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
E &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
F &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
\end{aligned}$$

We have

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -i \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$$

so

$$H \cdot (x^k y^\ell) = (\ell - k)x^k y^\ell = (n - 2k)x^k y^{n-k}$$

Also

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}$$

So when we decompose E we get $v = -\frac{1}{2}i$ and $u = -\frac{1}{2}$; so

$$E \cdot (x^k y^\ell) = -kx^{k-1}y^{\ell+1}$$

Finally we have

$$F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$$

so $v = -\frac{1}{2}i$ and $u = \frac{1}{2}$. So

$$F \cdot (x^k y^\ell) = -\ell x^{k+1} y^{\ell-1}$$

With respect to the basis $\{ x^n, x^{n-1}y, \dots, y^n \}$ for V_n we have

$$\begin{aligned}
H &= \rho_*(H) \\
&= [\rho_* H)]_{\mathcal{U}} \\
&= \operatorname{diag}(-n, -n+2, \dots, n-2, n) \\
E &= \rho_*(E) \\
&= [\rho_* E]_{\mathcal{U}} \\
&= \begin{pmatrix} 0 & & & & \\ -n & 0 & & & \\ & -n+1 & 0 & & \\ & & & \ddots & \\ & & & -1 & 0 \end{pmatrix} \\
F &= \begin{pmatrix} 0 & -1 & & & \\ & 0 & -2 & & \\ & & & \ddots & \\ & & & 0 & -n \\ & & & & 0 \end{pmatrix}
\end{aligned}$$

By Schur's lemma we have ρ (or V_n) is irreducible when $\operatorname{End}_G(V_n) = \{ cI : c \in \mathbb{C} \}$ and for $L \in \operatorname{End}(V_n)$ we have L is G-invariant if and only if L is $\mathfrak{g}$-invariant if and only if L is $\mathfrak{g}_{\mathbb{C}}$-invariant; meaning that $L \cdot \rho_*(A) = \rho_*(A) \cdot L$ for all $A \in \mathfrak{g}$ (or all $A \in \mathfrak{g}_{\mathbb{C}}$). Using the basis for V_n to identify $L \in \operatorname{End}(V_n)$ with its matrix, we have

$$\begin{aligned}
LH = HL &\iff (LH)_{k\ell} = (HL)_{k\ell} \text{ for all } k, \ell \\
&\iff (n - 2\ell)L_{k\ell} = (n - 2k)L_{k\ell} \text{ for all } k, \ell \\
&\iff L_{k\ell} = 0 \text{ for all } k \neq \ell \\
&\iff L \text{ is diagonal}
\end{aligned}$$

Also for $L = \operatorname{diag}(c_0, c_1, \dots, c_n)$ we have

$$\begin{aligned}
LE = EL &\iff \begin{pmatrix} 0 & & & & \\ -nc_1 & 0 & & & \\ & (-n+2)c_2 & 0 & & \\ & & & \ddots & \\ & & & -c_n & 0 \end{pmatrix} = \begin{pmatrix} 0 & & & & \\ -nc_0 & 0 & & & \\ & (-n+2)c_1 & 0 & & \\ & & & \ddots & \\ & & & -c_{n-1} & 0 \end{pmatrix} \\
&\iff c_0 = c_1 = \cdots = c_n
\end{aligned}$$

So $L \in \operatorname{End}(V_n)$ is G-invariant if and only if $L = cI$ for some $c \in \mathbb{C}$; so by Schur's lemma ρ is irreducible.

Aside 10.4 (Hint for 5c on the assignment). The domain of the chart is an open dense subset; hence to integrate on the entire manifold it suffices to integrate on one chart. Also if $P = P(\theta, \varphi, \psi) = A(\theta)B(\varphi)A(\psi)$, it's useful to compute

$$P^{-1}\frac{\partial P}{\partial \theta}, P^{-1}\frac{\partial P}{\partial \varphi}, P^{-1}\frac{\partial P}{\partial \psi}$$

10.1 Weights

Suppose G is a compact matrix Lie grape and $\rho \colon G \to \operatorname{GL}(V)$ is a finite-dimensional represnetation of G. This gives $\rho_* \colon \mathfrak{g} \to \operatorname{End}(V)$ which extends to $\rho_* \colon \mathfrak{g}_{\mathbb{C}} \to \operatorname{End}(V)$. Fix a G-invariant inner product $(\cdot, \cdot)$ on V; fix a maximal torus $T \subseteq G$ and let $\mathfrak{t}$ be its Lie algebra. We restrict ρ_* to $\rho_* \colon \mathfrak{t}_{\mathbb{C}} \to \operatorname{End}(V)$. For

$A, B \in \mathfrak{t}$ (so $A + iB \in \mathfrak{t}_{\mathbb{C}}$) we have that $\rho_*(A + iB)$ is a normal operator. Using an orthonormal basis we have $(\rho_*(A)u, v) = -(u, \rho_*(A)v)$; so $A = \rho_*(A)$ is skew-Hermitian, and $A^* = -A$. Likewise we get $B^* = -B$; so

$$\begin{aligned}(A + iB)^*(A + iB) &= (A^* - iB^*)(A + iB)\\ &= A^*A + iA^*B - iB^*A + B^*B\\ &= -A^2 - iAB + iBA - B^2\\ &= -A^2 - B^2\end{aligned}$$

since $AB = BA$. Also

$$\begin{aligned}(A + iB)(A + iB)^* &= (A + iB)(A^* - iB^*)\\ &= AA^* - iAB^* + iBA^* + BB^*\\ &= -A^2 + iAB - iBA - B^2\\ &= -A^2 - B^2\end{aligned}$$

So $A + iB$ is normal, and thus unitarily diagonalizable. Also for $A, B \in \mathfrak{t}_{\mathbb{C}}$ since $[A, B] = 0$ we have $\rho_*A \cdot \rho_*B - \rho_*B\rho_*A = \rho_*[A, B] = \rho_*0 = 0$; so $\rho_*(A)$ and $\rho_*(B)$ commute.

TODO 34. *Weren't we using this to show normality?*

Thus $S = \{ \rho_*(A) : A \in \mathfrak{t}_{\mathbb{C}} \}$ is a set of commuting normal (hence diagonalizable) operators on V.

Proposition 10.5. *If S is a set of commuting normal operators $V \to V$ then the elements of S can be simultaneously diagonalized.*

Proof. Note that if $L, M : V \to V$ commute and λ is an eigenvalue of L then M preserves the eigenspace $E_\lambda = \ker(L - \lambda I)$: indeed, if $v \in E_\lambda$ then $LMv = MLv = M\lambda v = \lambda Mv$, so $Mv \in E_\lambda$. If we extend E_λ to a basis for V then M has the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

with A, B normal.

We now show that S is simultaneously diagonalizable using induction on $\dim(V)$. When $\dim(V) = 1$ this is immediate.

If every $L \in S$ is a constant multiple of I then they are already diagonalized, and we're done; so assume we have $L \in S$ that is not a constant multiple of I. Pick an eigenvalue λ for L; then $0 \neq E_\lambda \subsetneq V$. So by induction hypothesis there is a non-trivial subspace $0 \neq U \subseteq E_\lambda$ such that all the operators restricted to U act as constant multiples of the identity. □ Proposition 10.5

Thus our $S = \{ \rho_*(A) : A \in \mathfrak{t}_{\mathbb{C}} \}$ is simultaneously diagonalizable (using an orthonormal basis). So

$$V = \bigoplus_{\alpha \in W} V_\alpha$$

where $W = W(\rho) = W(V)$ is a finite set of functions $\alpha : \mathfrak{t}_{\mathbb{C}} \to \mathbb{C}$ where

$$V_\alpha = \{ v \in V : \rho_*(B)(v) = \alpha(B) \cdot v \text{ for all } B \in \mathfrak{t}_{\mathbb{C}} \}$$

Note that these $\alpha : \mathfrak{t}_{\mathbb{C}} \to \mathbb{C}$ are linear (so $\alpha \in \mathfrak{t}_{\mathbb{C}}^*$); indeed, if $v \in V_\alpha$, $c \in \mathbb{C}$, and $A, B \in \mathfrak{t}_{\mathbb{C}}$ we have

$$\alpha(A + cB)(v) = \rho_*(A + cB)(v) = \rho_*(A)(v) + c\rho_*(B)(v) = \alpha(A)v + c\alpha(B)v = (\alpha(A) + c\alpha(B))v$$

Definition 10.6. The elements $\alpha \in W \subseteq \mathfrak{t}_{\mathbb{C}}^*$ are called the *weights* of ρ; the space V_α is called the *weight space* of α.

(So $\alpha \in \mathfrak{t}_{\mathbb{C}}^*$ is a weight of ρ if and only if $V_\alpha \neq 0$.)

Example 10.7. Consider the action of $\mathrm{SU}(2)$ on V_n. Recall that $\rho_*\colon \mathfrak{su}(2)_{\mathbb{C}} = \mathrm{End}(V_n)$ is given by

$$\begin{aligned}\rho_*(H)\cdot x^k y^{n-k} &= (n-2k)x^k y^{n-k}\\ \rho_*(E)\cdot x^k y^\ell &= -kx^{k-1}y^{\ell+1}\\ \rho_*(F)\cdot x^k y^\ell &= -\ell x^{k+1}y^{\ell-1}\end{aligned}$$

We wish to restrict ρ_* to $\mathfrak{t}_{\mathbb{C}}$; we choose $T = \{\, \mathrm{diag}(\exp(i\theta), \exp(-i\theta)) : \theta \in \mathbb{R} \,\}$, so $\mathfrak{t} = \{\, \mathrm{diag}(i\theta, -i\theta) : \theta \in \mathbb{R} \,\} = \mathrm{span}_{\mathbb{R}}\{\, \mathrm{diag}(i, -i) \,\}$ and $\mathfrak{t}_{\mathbb{C}} = \mathrm{span}_{\mathbb{C}}\{\, \mathrm{diag}(1, -1) \,\} = \mathrm{span}_{\mathbb{C}}\{\, H \,\}$. So the restriction $\rho_*\colon \mathfrak{t}_{\mathbb{C}} \to \mathrm{End}(V_n)$ is determined by $\rho_*(H)$ given by $\rho_*(H)\cdot x^k y^\ell = (\ell - k)x^k y^\ell$ (or by $H = \rho_*(H) = \mathrm{diag}(-n, -n+2, \dots, n)$). The weights are $W = \{\, \alpha_{-n}, \alpha_{-n+2}, \dots, \alpha_n \,\}$ where $\alpha_{-n+2k}\colon \mathfrak{t}_{\mathbb{C}} \to \mathbb{C}$ is given by $\alpha_{-n+2k}(H) = -n + 2k$ and the weight spaces are $V_{\alpha_{n-2k}} = \mathrm{span}_{\mathbb{C}}\{\, x^k y^{n-k} \,\}$.

Remark 10.8. Since for $A \in \mathfrak{t}$ (so $iA \in i\mathfrak{t}$) we have that $\rho_*(A)$ is skew-Hermitian, so its eigenvalues are purely imaginary; also $\rho_*(iA)$ is Hermitian, so its eigenvalues are real. So for all of the weights $\alpha \in W = W(p)$ we have $\alpha(A) \in i\mathbb{R}$ and $\alpha(iA) \in \mathbb{R}$ (i.e. $\alpha(\mathfrak{t}) \subseteq i\mathbb{R}$ and $\alpha(i\mathfrak{t}) \subseteq \mathbb{R}$).

Remark 10.9. When $P \in T \subseteq G$ and $P = \exp(B)$ with $B \in \mathfrak{t}$, and when $v \in V_\alpha$ where $\alpha \in W$, we have

$$\rho(P)(v) = \rho(\exp(B))(v) = \exp(\rho_*(B))(v) = \sum_{n=0}^{\infty} \frac{1}{n!}(\rho_*(B))^n(v) = \sum_{n=0}^{\infty} \frac{1}{n!}\alpha(B)^n \cdot v = \exp(\alpha(B)) \cdot v$$

Thus the weight spaces V_α are also common eigenspaces for all the operators $\rho(P)$ where $P \in T$.

Definition 10.10. The *roots* of G (or the roots of $\mathfrak{g}$ or $\mathfrak{g}_{\mathbb{C}}$) are the non-zero weights of the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g}) \subseteq \mathrm{GL}(\mathfrak{g}_{\mathbb{C}})$ (where we extend $L\colon \mathfrak{g} \to \mathfrak{g}$ to $L\colon \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ by complex linearity); so $\mathrm{ad}\colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g}) \subseteq \mathrm{End}(\mathfrak{g}_{\mathbb{C}})$, which we extend to $\mathrm{ad}\colon \mathfrak{g}_{\mathbb{C}} \to \mathrm{End}(\mathfrak{g}_{\mathbb{C}})$ and then restrict to $\mathrm{ad}\colon \mathfrak{t}_{\mathbb{C}} \to \mathrm{End}(\mathfrak{g}_{\mathbb{C}})$. So

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus_{\alpha \in R \cup \{\, 0 \,\}} \mathfrak{g}_\alpha$$

where

$$\mathfrak{g}_\alpha = \{\, A \in \mathfrak{g}_{\mathbb{C}} : \mathrm{ad}(B)(A) = \alpha(B)\cdot A \text{ for all } B \in \mathfrak{t}_{\mathbb{C}} \,\} = \{\, A \in \mathfrak{g}_{\mathbb{C}} : [B, A] = \alpha(B)\cdot A \text{ for all } B \in \mathfrak{t}_{\mathbb{C}} \,\}$$

Remark 10.11. Note that

$$\mathfrak{g}_0 = \{\, A \in \mathfrak{g}_{\mathbb{C}} : [B, A] = 0 \text{ for all } B \in \mathfrak{t}_{\mathbb{C}} \,\} = \mathfrak{z}(\mathfrak{t}_{\mathbb{C}}) = \mathfrak{t}_{\mathbb{C}}$$

So

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

So the roots of G are the $\alpha \in R = W(\mathrm{ad}) \setminus \{\, 0 \,\}$; the *root spaces* are the weight spaces $\mathfrak{g}_\alpha$.

Remark 10.12. Suppose $\rho\colon G \to \mathrm{GL}(V)$ is a representation. If $\alpha \in R$, $\beta \in W$, $v \in V_\beta$, and $A \in \mathfrak{g}_\alpha$, then $\rho_*(A)(v)$. Indeed, given $B \in \mathfrak{t}_{\mathbb{C}}$ we have

$$\begin{aligned}\rho_*(B)\rho_*(A)(v) &= \rho_*(B)\rho_*(A)(v) - \rho_*(A)\rho_*(B)(v) + \rho_*(A)\rho_*(B)(v)\\ &= \rho_*([B, A])(v) + \rho_*(A)\rho_*(B)(v)\\ &= \rho_*(\alpha(B)\cdot A)(v) + \rho_*(A)(\beta(B)\cdot v)\\ &= \alpha(B)\cdot \rho_*(A)(v) + \beta(B)\rho_*(A)(v)\\ &= (\alpha + \beta)(B)\cdot(\rho_*(A)(v))\end{aligned}$$

In particular, taking $\rho = \mathrm{Ad}$, if we let $\alpha, \beta \in R$ and $A \in \mathfrak{g}_\alpha, B \in \mathfrak{g}_\beta$, then $[A, B] \in \mathfrak{g}_{\alpha+\beta}$.

Example 10.13. Find the roots and root spaces of $\mathrm{U}(n)$.

Let $T = \{ \operatorname{diag}(\exp(i\theta_1), \dots, \exp(i\theta_n)) : \theta_k \in \mathbb{R} \}$; so $\mathfrak{t} = \{ \operatorname{diag}(i\theta_1, \dots, i\theta_n) : \theta_k \in \mathbb{R} \}$ and $\mathfrak{t}_{\mathbb{C}} = \{ \operatorname{diag}(c_1, \dots, c_n) : c_k \in \mathbb{C} \}$. Let $E_{k\ell} \in M_n(\mathbb{C})$ be the matrix with a 1 in posiiton (k, ℓ) and 0 elsewhere; let $E_k = E_{k,k}$. So $\mathfrak{t}_{\mathbb{C}} = \operatorname{span}\{ E_1, \dots, E_n \}$. Let $\varepsilon_1, \dots, \varepsilon_n$ be the dual basis for $\mathfrak{t}_{\mathbb{C}}^*$ (so $\varepsilon_k(E_\ell) = \delta_{k,\ell}$). Note that $\mathfrak{u}(n)_{\mathbb{C}} = \mathfrak{u}(n) \oplus i\,\mathfrak{u}(n) = \mathfrak{gl}(n, \mathbb{C}) = M_n(\mathbb{C})$. Let $0 \neq A \in \mathfrak{gl}(n, \mathbb{C})$ be a common eigenvector for the maps $\operatorname{ad}(B)$ for $B \in \mathfrak{t}_{\mathbb{C}}$. So

$$\begin{aligned} & \operatorname{ad}(B)(A) = \alpha(B) \cdot A \text{ for all } B \in \mathfrak{t}_{\mathbb{C}} \\ \implies & \quad BA - AB = \alpha(B) \cdot A \text{ for all } B \in \mathfrak{t}_{\mathbb{C}} \\ \implies & (BA - AB)_{k\ell} = \alpha(B) A_{k\ell} \text{ for all } B, k, \ell \\ \implies & \ (b_k - b_\ell) A_{k\ell} = \alpha(B) A_{k\ell} \text{ for all} k, \ell, B = \operatorname{diag}(b_1, \dots, b_n) \end{aligned}$$

Since $A \neq 0$ we can choose k, ℓ so that $A_{k,\ell} \neq 0$. So we must have $\alpha(B) = b_k - b_\ell$ for all $B = \operatorname{diag}(b_1, \dots, b_n)$. Thus we must have $\alpha = \varepsilon_k - \varepsilon_\ell$. When $\alpha = \varepsilon_k - \varepsilon_\ell$ we also need $(b_i - b_j) A_{ij} = (b_k - b_\ell) A_{ij}$ for all $(i, j) \neq (k, \ell)$ and all $B = \operatorname{diag}(b_1, \dots, b_n)$. For any $(i, j) \neq (k, \ell)$ we can choose B so that $b_i - b_j \neq b_k - b_\ell$; so we must have $A_{ij} = 0$ for all $(i, j) \neq (k, \ell)$. Thus when $\alpha = \varepsilon_k - \varepsilon_\ell$ we have $\mathfrak{g}_\alpha = \operatorname{span}_{\mathbb{C}}\{ E_{k\ell} \}$. So the set of roots is $R = \{ \varepsilon_k - \varepsilon_\ell : k \neq \ell \}$.

Example 10.14. For SU(3) with respect to the basis $\{ \varepsilon_1, \varepsilon_2, \varepsilon_3 \}$ for $\mathfrak{t}_{\mathbb{C}}^*$ we have

$$R = \left\{ \pm \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \pm \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \pm \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$$

Note that

$$\cos(\theta((1, -1, 0), (0, 1, -1))) = \frac{(1, -1, 0) \cdot (0, 1, -1)}{\|(1, -1, 0)\| \|0, 1, -1\|} = \frac{-1}{2}$$

Since the roots live in a two-dimensional space, we can draw them on the plane; by the above they end up in a hexagon.

Example 10.15. For SU(4) we have

$$R = \left\{ \pm \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \pm \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \pm \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \pm \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \pm \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \pm \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}$$

So $\operatorname{span}_{\mathbb{R}}(R) = \{ x \in \mathbb{R}^4 : x_1 + x_2 + x_3 + x_4 = 0 \}$. We can use $(1, -1, 0, 0), (0, 0, 1, -1), (1, 1, -1, -1)$ as a basis and draw this in $\mathbb{R}^3$; you get the vertices of a polyhedron that you get by "cutting off the corners" of a cube.

11 Stuff we didn't get to

Characters

For a finite-dimensional representation $\rho \colon G \to \mathrm{GL}(V)$ the *character* of ρ is the map $\chi_\rho = \chi_V \colon G \to \mathbb{C}$ given by $P \mapsto \operatorname{tr}(\rho(P))$. If V, W are irreducible, then

$$\int_G \chi_V \overline{\chi_W(x)} \mathrm{d}g(x) = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{else} \end{cases}$$

In general

$$\int_G \chi_V(x) \chi_W(x) \mathrm{d}g(x) = \dim \hom_G(V, W)$$

Killing form

For $A, B \in \mathfrak{g}_{\mathbb{C}}$ we define

$$\mathcal{B}(A, B) = \mathrm{tr}(\mathrm{ad}(A) \circ \mathrm{ad}(B))$$

On $\mathfrak{t}_{\mathbb{C}}$ we obtain a Hermitian form $(A + iB, C + iD) = \mathcal{B}(A + iB, C - iD)$. One can use this to obtain symmetry properties of the roots.

$\mathfrak{u}(n)$ has the same roots as $\mathfrak{su}(n)$.