Course notes for PMATH 965

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Lectures by Matthew Satriano, Winter 2017

Contents

1 Introduction

My thanks to Anthony McCormick and Nickolas Rollick for the use of their notes when I was absent.

Sources:

- Fulton's *Toric varieties*
- Cox, Little, Schenck

Toric varieties:

- Broad class of varieties
- Combinatorics meets algebraic geometry.

  In particular, we get a dictionary between combinatorics and geometry, with toric varieties corresponding to *fans*. The main focus of the course will be building this dictionary.
- Good testing ground for conjectures; easy to run computations on.
- One cool application: mirror symmetry.
- Possibly in class we'll discuss toric degeneration. The idea is that you can go from a general variety to a toric one.

We think of toric varieties as the "geometry of monoids".

Definition 1.1. A *monoid* is a set P equipped with a binary operation $+\colon P \times P \to P$ which is comnmutative, associative, and has an identity element.

Definition 1.2. A *morphism of monoids* is a map $f\colon P \to Q$ between two monoids such that $f(p_1 + p_2) = f(p_1) + f(p_2)$ for all $p_1, p_2 \in P$ and such that $f(0) = 0$.

Example 1.3.

- $\mathbb{N} = \{ 0, 1, 2, \dots \}$ with the usual addition.
- $\mathbb{N}^n = \underbrace{\mathbb{N} \oplus \cdots \oplus \mathbb{N}}_{n \text{ times}}$.
- Given monoids P, Q we get a new monoid $P \oplus Q$ where $(p, q) + (p', q') = (p + p', q + q')$ and $0 = (0, 0)$.
- $P = \mathbb{N}^3/((1,0,0) + (0,1,0) = (0,0,2))$. We can identify P with the lattice (i.e. integer) points on the plane lying in the cone generated by $(1, 2)$ and $(1, 0)$ by identifying

$$(1,0,0) \approx (1,0)$$
$$(0,1,0) \approx (1,2)$$
$$(0,0,1) \approx (1,1)$$

  Then P is generated by $(1,0)$, $(1,2)$, and $(1,1)$ with the relation $(1,0)+(1,2) = 2(1,1)$, as desired.

Monoids get *really* pathological; for example, if P is finitely generated (meaning it is generated as a monoid by finitely many elements) and $Q \subseteq P$ is a submonoid, it need not be the case that Q is finitely generated.

Example 1.4. Take $P = \mathbb{N}^2$ and let $Q = \mathbb{N}^2 \setminus \{ (0, b) : b \in \mathbb{Z}^+ \}$. Then any generating set for Q must contain every point of the form $(1, b)$ for each $b \in \mathbb{N}$.

Another bad thing: given a monoid P, we can construct a new monoid $P \cup \{ \infty \}$ where $+$ extends addition on P by declaring $p + \infty = \infty + \infty = \infty$.

This is an example of a *sink*.

Definition 1.5. An element $q \in Q$ is a *sink* if $q \neq 0$ and $q' + q = q$ for all $q' \in Q$.

Our first goal will be to put hypotheses on monoids to avoid pathologies.

Our 0^{th} goal: given a monoid, construct a ring.

Definition 1.6. If P is a monoid and R is a commutative ring, then we define an R-algebra called the *monoid algebra*, denoted $R[P]$. We define

$$R[P] = \left\{ \sum_{p \in P} a_p x^p : a_p \in R, \text{ all but finitely many } a_p = 0 \right\}$$

(where x^p is a formal symbol (i.e. variable)) with the relations $x^p \cdot x^q = x^{p+q}$. i.e.

$$R[P] = R[x^p : p \in P]/(x^p x^q = x^{p+q})$$

More explicitly

$$\sum_{p \in P} a_p x^p + \sum_{p \in P} b_p x^p = \sum_{p \in P} (a_p + b_p) x^p$$

and

$$\left(\sum_{p \in P} a_p x^p \right) \cdot \left(\sum_{p \in P} b_p x^p \right) = \sum_{p, q \in P} a_p b_p x^{p+q} = \sum_{r \in P} \left(\sum_{p+q=r} a_p b_q \right) x^r$$

Example 1.7. $R[\mathbb{N}] = R[x^0, x^1, x^2, \dots]/(x^n \cdot x^m = x^{n+m})$. In fact, we don't need $|\mathbb{N}|$-many variables since $x^n = (x^1)^n$; so we just need one variable. (Note that $x^0 = 1$ in $R[\mathbb{N}]$.) So $R[\mathbb{N}] \cong R[x]$ is a polynomial ring.

Similarly, we get $R[\mathbb{N}^n] \cong R[x_1, \dots, x_n]$.

Note that abelian grapes are monoids that happen to have additive inverses. So $\mathbb{Z}$ is a monoid under both $+$ and $\times$. Note that under multiplication, 0 is a sink.

Example 1.8. Consider $\mathbb{Z}$ under addition; consider

$$R[\mathbb{Z}] = R[x^1, x^{-1}]/(x^1 \cdot x^{-1} = 1) = R[x, y]/(xy = 1) = R[x^{\pm 1}]$$

Similarly, $R[\mathbb{Z}^n] = R[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$.

Exercise 1.9. Compute $R[\mathbb{Z}/n\mathbb{Z}]$, $R[(\mathbb{Z},\times)]$, and $R[P\cup\{\infty\}]$.

Note that $R[P]$ is a monoid under both $+$ and $\times$. We get a morphism of monoids $f\colon P\to R[P]$ given by $p\mapsto x^p$. (Note that

$$f(p+q)=x^{p+q}=x^px^q=f(p)\cdot f(q)$$

and f is indeed a morphism of monoids if we treat $R[P]$ as a monoid under multiplication.) Sometimes f is called the *exponential map*; some authors write $f=\exp=e$, and e^p in place of x^p.

Aside 1.10. The study of maps $f\colon P\to R$ where P is a monoid, R is a ring, and f is a morphism of monoids (where R is considered a monoid under $\times$) is the subject of logarithmic algebraic geometry; it leads to an alternative look at toric varieties.

Now we can look at the geometry of a monoid P by studying the geometry of the corresponding ring $R[P]$. We now turn to the first goal above: put hypotheses on monoids to avoid pathologies.

Definition 1.11. A monoid P is *finitely-generated* if there are $p_1,\dots,p_n\in P$ such that for all $p\in P$ there are $a_i\in\mathbb{N}$ such that

$$p=\sum_{i=1}^n a_ip_i$$

Equivalently, if there is a surjective morphism of monoids $\mathbb{N}^n\twoheadrightarrow P$.

Example 1.12. $\mathbb{Z}$ is a finitely-generated monoid under $+$ since every integer m can be written as a non-negative linear combination of 1 and -1. By contrast, $\mathbb{Z}$ is not finitely generated under $\times$.

Definition 1.13. A monoid P is *integral* (or *cancellative*) if whenever $p,q,r\in P$ satisfy $p+r=q+r$, we must have $p=q$.

Example 1.14. Monoids with sinks are not integral, since $\infty+\infty=0+\infty$ but $0\neq\infty$.

Definition 1.15. If P is a monoid, we define the *associated grape* (or *grape-ification*) P^{gp} to be P^2 modulo the relation $(p,q)\sim(p',q')$ if there is $r\in P$ such that $p+q'+r=p'+q+r$. We use $[p,q]$ to denote the equivalence class of (p,q). We think of $[p,q]$ as being "$p-q$".

Intuitively, we'd like $p-q=p'-q'$ if $p+q'=p'+q$. The r helps for non-integral monoids. (Compare with localization in rings with zero divisors.)

Why is P^{gp} a grape? The only thing we need to check is that every element has an additive inverse. Intuitively, $p-q$ has inverse $q-p$. Formally, we see that

$$[p,q]+[q,p]=[p+q,p+q]=[0,0]$$

since $p+q+0=p+q+0$.

Lemma 1.16. *P is integral if and only if the canonical map*

$$\begin{aligned}\iota\colon P&\to P^{\mathrm{gp}}\\ p&\mapsto[p,0]\end{aligned}$$

is injective.

Proof.

($\implies$) Suppose P is integral; suppose $\iota(p)=\iota(q)$. Then $[p,0]=[q,0]$, so there is $r\in P$ such that $p+0+r=q+0+r$; since P is integral, we get that $p=q$. So ι is injective.

($\impliedby$) Suppose ι is injective. Suppose $p+r=q+r$; we will check that $p=q$. By definition we get that $[p,0]=[q,0]$, so $\iota(p)=\iota(q)$; by injectivity, we then get that $p=q$, as desired. So P is integral. □ Lemma 1.16

Example 1.17. A special case: this is how we get $\mathbb{Q}$ from $\mathbb{Z}$. Almost... $\mathbb{Z}$ has a sink, so really it's how we get $\mathbb{Q}^\times$ from $\mathbb{Z}\setminus 0$. More concretely, let $P=\mathbb{Z}\setminus 0$ considered as a monoid under $\times$. Note that lots of elements have no multiplicative inverse. We then set $\mathbb{Q}^\times=(\mathbb{Z}\setminus 0)^{\mathrm{gp}}$; we interpret $\frac{p}{q}$ as $[p,q]$.

In fact, grape-ification defines a functor from the category of monoids to the category of abelian grapes. In particular, given a morphism of monoids $f\colon P \to Q$, we get $f^{\mathrm{gp}}\colon P^{\mathrm{gp}} \to Q^{\mathrm{gp}}$ given by $[p_1, p_2] \mapsto [f(p_1), f(p_2)]$.

Lemma 1.18. *$\iota\colon P \to P^{\mathrm{gp}}$ is the universal map from P to an abelian grape. i.e. for all $g\colon P \to A$ where A is an abelian grape there is a unique morphism of grapes $f\colon P^{\mathrm{gp}} \to A$ such that the following diagram commutes:*

$$\begin{tikzcd} P \arrow[r, "\iota"] \arrow[dr, "g"'] & P^{\mathrm{gp}} \arrow[d, dashed, "h"] \\ & A \end{tikzcd}$$

i.e. grape-ification is left adjoint to the forgetful functor from abelian grapes to monoids.

Proof. We let $h([p, q]) = g(p) - g(q)$. Then the diagram commutes:

$$h(\iota(p)) = h([p, 0]) = g(p) - g(0) = g(p)$$

We now check that h is well-defined. Suppose $[p, q] = [p', q']$; i.e. suppose there is r such that $p+q'+r = p'+q+r$. Then $g(p) + g(q') + g(r) = g(p') + g(q) + g(r)$. Since A is an abelian grape, we can cancel to get that $g(p) - g(q) = g(p') - g(q')$; so h is well-defined.

Uniqueness is left as an exercise. □ Lemma 1.18

Corollary 1.19. *P is integral if and only if there is an abelian grape A and an injection $g\colon P \hookrightarrow A$.*

Proof.

($\Longrightarrow$) Suppose P is integral; then we simply let $A = P^{\mathrm{gp}}$.

($\Longleftarrow$) Suppose we have such an A and g. Then by Lemma 1.18 there is a unique morphism of grapes $h\colon P^{\mathrm{gp}} \to A$ such that the following diagram commutes:

$$\begin{tikzcd} P \arrow[r, "\iota"] \arrow[dr, hook, "g"'] & P^{\mathrm{gp}} \arrow[d, dashed, "h"] \\ & A \end{tikzcd}$$

Since g is injective, we get that so too is ι. So, by Lemma 1.16, we get that P is integral. □ Corollary 1.19

We now work over $k = \overline{k}$ an algebraically closed field.

Fact 1.20. *If P is finitely generated, the so is P^{gp}.*

Let's assume P^{gp} is free and P is integral. Then $\iota\colon P \hookrightarrow P^{\mathrm{gp}}$ induces $k[P] \hookrightarrow k[P^{\mathrm{gp}}] \cong k[x_1^{\pm}, \dots, x_n^{\pm}]$. But $k[x_1^{\pm}, \dots, x_n^{\pm}]$ is a finitely generated k-algebra; so $k[P]$ is as well.

Thus:

Fact 1.21. *If P is integral, finitely generated, and has P^{gp} a free grape, then $k[P]$ is a finitely generated k-algebra.*

Hence we get a variety $\operatorname{Spec}(k[P])$.

We'll later characterize the integral, finitely generated monoids with a free grape-ification using convex geometry.

Example 1.22. Consider $P = \mathbb{N}^n$. Then $k[\mathbb{N}^n] = k[x_1, \dots, x_n]$, so $\operatorname{Spec}(k[\mathbb{N}^n]) = \mathbb{A}^n$.

Example 1.23. Consider $P = \mathbb{Z}$. Then $\operatorname{Spec}(k[\mathbb{Z}]) = \operatorname{Spec}(k[x^{\pm}]) = \operatorname{Spec}(k[x, y]/(xy - 1))$; via projection, we get that this is isomorphic to $\mathbb{A} \setminus 0$. This is a very important variety, called the 1-*dimensional torus*, sometimes written $\mathbb{G}_{\mathrm{m}}$, the "multiplicative grape". (Keep in mind the points of $\mathbb{G}_{\mathrm{m}}$ are just the elements of k^*.)

Example 1.24. Consider $P = \mathbb{Z}^n$. Then $\operatorname{Spec}(k[\mathbb{Z}^n]) = (k^*)^n = \mathbb{G}_{\mathrm{m}}^n$ is the *n-dimensional torus.*

Example 1.25. Consider $P = \mathbb{N}^3/((0,0,2) = (1,0,0) + (0,1,0))$. A good exercise is to check that $k[P] = k[x,y,z]/(xy - z^2)$. Then $\mathrm{Spec}(k[P]) = V(xy - z^2)$, which looks like some kind of double cone, and in particular has a singular point at the origin.

Aside 1.26. The last item of Example 1.3 could also be embedded in the plane by identifying

$$\begin{aligned}(1,0,0) &\approx (2,0)\\ (0,1,0) &\approx (0,2)\\ (0,0,1) &\approx (1,1)\end{aligned}$$

We will eventually see that the monoid that are integral and finitely generated and have free grape-ification (the "toric varieties") are exactly those that arise from taking a cone $\sigma \subseteq \mathbb{R}^n$ and looking at $\sigma \cap \mathbb{Z}^n$; i.e. those that can be described as the lattice points of some cone. The embedding immediately above does not take this form, so we prefer the original embedding.

We have seen that given P integral and finitely generated with P^{gp} free, we have $X_P = \mathrm{Spec}(k[P])$ is a variety. What are the points of X_P? They're exactly surjective k-algebra homomorphisms $k[P] \twoheadrightarrow k$.

But recall that $(P, +) \to (k[P], \cdot)$ is a morphism of monoids. Hence we can compose $P \to k[P] \twoheadrightarrow k$ to get a monoid map $P \to k$. Conversely, given $f\colon P \to k$ we get $k[P] \to k$ by the universal property; this is surjective because f is a morphism of monoids, and hence $f(0) = 1$. So the map $k[P] \to k$ satisfies $1 = x^0 \mapsto 1$, and is thus surjective. Putting these together, we see that the (closed) points of X_p correspond to the monoid morphisms $P \to k$.

Recall we defined the *n-dimensional torus over k* to be $\mathbb{G}_{\mathrm{m}}^n = (k^*)^n = \mathrm{Spec}(k[\mathbb{Z}^n]) = \mathrm{Spec}(k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}])$. In fact, $T = \mathbb{G}_{\mathrm{m}}^n$ is a *grape variety*; there is a map $\mu\colon T \times T \to T$ satisfying the grape multiplication properties. What is the map? It is

$$\begin{aligned}(k^*)^n \times (k^*)^n &\to (k^*)^n\\ ((x_i : i < n), (y_i : i < n)) &\mapsto (x_i y_i : i < n)\end{aligned}$$

What is the map in terms of rings? Well, $T = \mathrm{Spec}(k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}])$. The corresponding map is then

$$\begin{aligned}k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] &\to k[y_1^{\pm 1}, \ldots, y_n^{\pm 1}] \otimes_k k[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]\\ x_i &\mapsto y_i \otimes_k z_i\end{aligned}$$

(Recall that the coproduct in the category of k-algebras is the tensor product.) This corresponds to multiplication $T \times T \to T$, but on the ring side it's a *comultiplication map* $k[\mathbb{Z}^n] \to k[\mathbb{Z}^n] \otimes_k k[\mathbb{Z}^n]$.

Editor's note 1.27. I think this corresponds on the monoid side to

$$\begin{aligned}\mathbb{Z}^n &\to \mathbb{Z}^n \oplus \mathbb{Z}^n\\ p &\mapsto (p, p)\end{aligned}$$

(Note that taking the monoid algebra preserves colimits, and in particular coproducts, since it is a left adjoint; hence this does indeed induce a map $k[\mathbb{Z}^n] \to k[\mathbb{Z}^n] \otimes_k k[\mathbb{Z}^n]$.)

Aside 1.28. $G = \mathrm{Spec}(R)$ is a grape variety if and only if R is a commutative Hopf algebra.

We now return to $X = \mathrm{Spec}(k[P])$. Now, P^{gp} is a free abelian grape, so $P^{\mathrm{gp}} \cong \mathbb{Z}^n$, and $\mathrm{Spec}(k[P^{\mathrm{gp}}]) \cong \mathbb{G}_{\mathrm{m}}^n$.

Fact 1.29. *There is an action of $T = \mathbb{G}_{\mathrm{m}}^n$ on X, that is, a map $\tau\colon T \times X \to X$ satisfying the properties of a grape action.*

Explicitly, we're looking for a map $k[P] \to k[P^{\mathrm{gp}}] \otimes_k k[P]$; a natural choice is $x^p \mapsto x^p \otimes_k x^p$. i.e. the map on rings induced by the monoid map

$$\begin{aligned}P &\to P^{\mathrm{gp}} \oplus P\\ p &\mapsto (p, p) = ([p, 0], p)\end{aligned}$$

Editor's note 1.30. When writing commutative diagrams, I will generally use τ, μ, and ι to refer to grape action, grape multiplication, and inclusion, respectively, on the levels of varieties, k-algebras, and monoids; the ambient category will dictate which level is meant.

Notice we also have $\iota\colon P \hookrightarrow P^{\mathrm{gp}}$ which yields a map $T \to X$. These two things, a map $T \to X$ and a T-action, are what we'll call a *toric variety*.

Let's check that our map $T \times X \to X$ is a T-action; we need to show that the following diagram commutes:

$$\begin{tikzcd} T \times T \times X \arrow[r, "\mathrm{id} \times \tau"] \arrow[d, "\mu \times \mathrm{id}"] & T \times X \arrow[d, "\tau"] \\ T \times X \arrow[r, "\tau"] & X \end{tikzcd}$$

i.e. given $g, h \in T$ and $x \in X$ we need $(g \cdot h)x = g \cdot (h \cdot x)$. On the level of monoids, we require the following diagram commutes:

$$\begin{tikzcd} P^{\mathrm{gp}} \oplus P^{\mathrm{gp}} \oplus P & P^{\mathrm{gp}} \oplus P \arrow[l, "\mathrm{id} \times \tau"'] \\ P^{\mathrm{gp}} \oplus P \arrow[u, "\mu \times \mathrm{id}"] & P \arrow[l, "\tau"] \arrow[u, "\tau"'] \end{tikzcd}$$

But this does commute: going down and right, we find

$$p \mapsto (p, p) \mapsto (p, p, p)$$

and going right and down, we find

$$p \mapsto (p, p) \mapsto (p, p, p)$$

This commutes on the level of monoids, and hence commutes on the monoid algebras.

Recall that $P \hookrightarrow P^{\mathrm{gp}}$ yields a map $T \to X$.

Proposition 1.31. *$T \to X$ is an open (dense) immersion. In fact, $T \subseteq X$ is a principal affine open; i.e. $k[P^{\mathrm{gp}}]$ is the localization at an element of $k[P]$.*

Proof. Let $p_1, \ldots, p_n$ be generators of P; then the p_i also generate P^{gp} as an abelian grape, and P^{gp} is generated as a monoid by $p_1, \ldots, p_n$ and $-\sum_{i=1}^n p_i$.

So $k[P^{\mathrm{gp}}]$ is "the same as" $k[P]$ after allowing the new elemen $x^{-\sum p_i}$; i.e. it's $k[P]$ with denominators obtained by inverting $x^{-\sum p_i}$. So we just inverted the single element

$$x^{\sum p_i} = \prod x^{p_i}$$

So $k[P^{\mathrm{gp}}] = k[P]_{\prod x^p}$ is the localization at a single element of $k[P]$.

The statement then follows from the fact that the spectrum of a ring is open in the spectrum of its localization at an element. □ Proposition 1.31

Corollary 1.32. $\dim(X) = \operatorname{rank}(P^{\mathrm{gp}})$.

Proof. Since $\dim(X) = \dim(T) = \operatorname{rank}(P^{\mathrm{gp}})$. □ Corollary 1.32

Notice we have the following diagram:

$$\begin{tikzcd} T \times T \arrow[r, "\mu"] \arrow[d, hook, "\mathrm{id} \times \iota"] & T \arrow[d, hook, "\iota"] \\ T \times X \arrow[r, "\tau"] & X \end{tikzcd}$$

This commutes because the following diagram commutes:

$$\begin{tikzcd} P^{\mathrm{gp}} \oplus P^{\mathrm{gp}} & P^{\mathrm{gp}} \arrow[l] \\ P^{\mathrm{gp}} \oplus P \arrow[u] & P \arrow[l] \arrow[u] \end{tikzcd}$$

commutes; this is because going up and left we find

$$p \mapsto p \mapsto (p, p)$$

and going left and up we find $p \mapsto (p, p) \mapsto (p, p)$.

In summary, we have found:

Theorem 1.33. *If P is integral and finitely generated and P^{gp} is free, then $X = \mathrm{Spec}(k[P])$ is a variety, $T = \mathrm{Spec}(k[P^{\mathrm{gp}}])$ is a torus, $T \subseteq X$ is open principal affine, and the T-action on itself (via grape multiplication) extends to a T-action on X.*

Definition 1.34. A (not necessarily normal) *toric variety* is a variety X together with a torus $T \subseteq X$ that is open and dense in X such that the action $T \times T \to T$ extends to $T \times X \to X$ in such a way that the following diagram commutes:

$$\begin{tikzcd} T \times T \arrow[r, "\mu"] \arrow[d, hook] & T \arrow[d, hook] \\ T \times X \arrow[r, "\tau"] & X \end{tikzcd}$$

Exercise 1.35. The T-action on X in the definition of a variety is unique if it exists. (Uses basic algebraic geometry.)

Example 1.36. Consider $\mathbb{A}^1 \supseteq \mathbb{G}_{\mathrm{m}}$. The torus multiplication is $\mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}$ given by $(s,t) \mapsto st$; this extends to the $\mathbb{G}_{\mathrm{m}} \times \mathbb{A}^1 \to \mathbb{A}^1$ via $(s,x) \mapsto sx$. (Note that the action does indeed restrict to grape multiplication on $\mathbb{G}_{\mathrm{m}}$.)

Note that the action of $\mathbb{G}_{\mathrm{m}}$ on itself has no fixed points; i.e. there's a single orbit. But the action of $\mathbb{G}_{\mathrm{m}}$ on $\mathbb{A}^1$ has a fixed point, namely 0: $s \cdot 0 = 0$ for all $s \in \mathbb{G}_{\mathrm{m}}$.

Example 1.37. $\mathbb{P}^1$ is also a toric variety, by noting that $\mathbb{G}_{\mathrm{m}} \subseteq \mathbb{A}^1 \subseteq \mathbb{P}^1$ (and all inclusions are open and dense). The action of $\mathbb{G}_{\mathrm{m}}$ on $\mathbb{P}^1$ is given by $s \cdot [a : b] = [sa : b]$.

Definition 1.38. Suppose (T, X) and (T', X') are toric varieties. A *toric morphism* $(T, X) \to (T', X')$ is a morpism of varieties $f\colon X \to X'$ such that $f \restriction T\colon T \to T'$ is a grape homomorphism and f respects the T- and T'-actions.

Example 1.39. Consider $P = \mathbb{N} \setminus 1$; this is generated by 2 and 3. Then $k[P] = k[x^2, x^3] = k[x,y]/(x^3 - y^2)$; its geometry $X = \mathrm{Spec}(k[P])$ is the cuspidal cubic. Then $\mathbb{G}_{\mathrm{m}}$ sits inside X as X without the origin. The action of T on X is given by $t \cdot (x,y) = (t^2x, t^3y)$.

Theorem 1.40. *The functor F from the category of integral and finitely generated monoids with free grape-ifications to the category of affine (not necessarily normal) toric varieties given by $P \mapsto \mathrm{Spec}(k[P])$ is an equivalence of categories.*

Proof. To prove an equivalence of categories, we need to check

Essential surjectivity We must show that every affine not necessarily normal toric variety comes from a monoid.

Suppose X is a toric variety; then we have a torus $T \subseteq X$, say $T = \mathrm{Spec}(k[M])$. Suppose $X = \mathrm{Spec}(R)$; then since $T \subseteq X$ we get $R \hookrightarrow k[M]$ from elementary algebraic geometry. Now, we are given that the following diagram commutes:

$$\begin{tikzcd} T \times T \arrow[r, "\mu"] \arrow[d, hook] & T \arrow[d, hook] \\ T \times X \arrow[r, "\tau"] & X \end{tikzcd}$$

On the level of rings, the co-action is $R \to R \otimes_k k[M] = R[M]$; we further know that this is computed by following the embedding $R \hookrightarrow k[M]$ and then using the map $k[M] \to k[M] \otimes_k k[M]$ given by the grape multiplication $T \times T \to T$. But this last is just $k[M] \to k[M] \otimes_k k[M]$ given by $x^u \mapsto x^u \otimes_k x^u$ on the level of rings; this then determines the co-action map $R \to R \otimes_k k[M]$.

If

$$\sum_{u \in M} \alpha_u x^u \in R$$

then apply the co-action map, and we get

$$\sum_{u \in M} \alpha_u x^u \otimes_k x^u \in R[M]$$

But if

$$\sum_{u \in M} r_u x^u \in R[M]$$

then $r_u \in R$. For us, the $r_u = \alpha_u x^u$; hence $\alpha_u x^u \in R$.

If $\alpha_u \neq 0$, then scale, so $x^u \in R$; we've thus produced a subset $P \subseteq M$ such that

$$R = \bigoplus_{u \in P} k \cdot x^u$$

But $R \subseteq k[M]$ is a subring; so, if $u, v \in P$, then $x^u, x^v \in R$, so $x^{u+v} = x^u x^v \in R$, and $u + v \in P$. So $P \subseteq M$ is a submonoid. So $R = k[P]$ where $P \subseteq M$ is a monoid.

Now, P is a submonoid of an abelian grape, so P is integral. Also, R is a finitely generated k-algebra, since X is a variety and

$$R = \bigoplus_{u \in P} k \cdot x^u$$

So P is a finitely generated monoid. Furthermore, P^{gp} is free since $P^{\mathrm{gp}} \subseteq M$ and M is a free abelian grape.

It remains to show that $P^{\mathrm{gp}} \cong M$, so that $X = F(P)$. Well, $T \subseteq X$ with $\mathrm{rank}(P^{\mathrm{gp}}) = \dim(X) = \dim(T) = \mathrm{rank}(M)$. If $P^{\mathrm{gp}} \ncong M$, then we have $P^{\mathrm{gp}} \hookrightarrow M$ with finite cokernel; we thus get a finite map $T = \mathrm{Spec}(k[M]) \to \mathrm{Spec}(k[P^{\mathrm{gp}}])$ of degree $|M/P^{\mathrm{gp}}|$. But $T \subseteq X$ is an open immersion; so the degree is 1. So $P^{\mathrm{gp}} \cong M$.

Full faithfulness We must show that the map $\hom(P, P') \to \hom(X', X)$ is bijective.

Say

$$X = \mathrm{Spec}(k[P])$$
$$X' = \mathrm{Spec}(k[P'])$$

Suppose $f\colon X \to X'$ is a toric morphism; we need to show that there is a unique morphism of monoids $\varphi\colon P' \to P$ inducing f.

Let $M = P^{\mathrm{gp}}$ and $M' = (P')^{\mathrm{gp}}$. Let $g = f \restriction T\colon T \to T'$; then g is a morphism of grapes and the following diagram commutes:

$$\begin{array}{ccc} k[M'] & \xrightarrow{\mu'} & k[M'] \otimes_k k[M'] \\ \big\downarrow{\scriptstyle g^*} & & \big\downarrow{\scriptstyle g^* \otimes_k g^*} \\ k[M] & \xrightarrow{\mu} & k[M] \otimes_k k[M] \end{array}$$

If $u' \in M'$ and

$$g^*(x^{u'}) = \sum_{u \in M} \alpha_u x^u$$

then the diagram yields

$$\begin{aligned} \sum_{u \in M} x^u \otimes_k x^u &= g^*(x^{u'}) \otimes_k g^*(x^{u'}) \\ &= \left(\sum_{u \in M} \alpha_u x^u\right) \otimes_k \left(\sum_{v \in M} \alpha_v x^v\right) \\ &= \sum_{u,v \in M} \alpha_u \alpha_v x^u \otimes_k x^v \end{aligned}$$

Comparing, we find that $\alpha_u \alpha_v = 0$ if $u \neq v$ and $\alpha_u^2 = \alpha_u$; hence all $\alpha_u \in \{0, 1\}$, and if $\alpha_u = 1$, then $\alpha_v = 0$ for all $v \neq u$. Hence either $g^*(x^{u'}) = 0$ or $g^*(x^{u'}) = x^u$ for some $u \in M$.

Let $\varphi\colon M' \to M$ be

$$u' \mapsto \begin{cases} 0 & \text{if } g^*(x^{u'}) = 0 \\ u & \text{if } g^*(x^{u'}) = x^u \end{cases}$$

This is a morphism of monoids because g^* is.

We have now shown:

Claim 1.41. *If M and M' are free abelian grapes of finite rank then* $\hom_{\mathbb{Z}}(M, M') \cong \hom(T', T)$*.*

This is called *Cartier duality.*

Then g is induced by the monoid morphism φ. But since f is a toric morphism, we have the following diagram commutes:

$$\begin{tikzcd} T \times X \arrow[r, "\tau"] \arrow[d, "f\restriction T \times f"] & X \arrow[d, "f"] \\ T' \times X' \arrow[r, "\tau"] & X' \end{tikzcd}$$

So the following diagram commutes:

$$\begin{tikzcd} k[P'] \arrow[r] \arrow[d] & k[P'] \otimes_k k[M'] \arrow[d, "f^* \times_k (f\restriction T)^*"] \\ k[P] \arrow[r] & k[P] \otimes_k k[M] \end{tikzcd}$$

A similar argument yields that this diagram is induced by the following diagram commutes:

$$\begin{tikzcd} P' \arrow[r] \arrow[d] & P' \oplus M' \arrow[d] \\ P \arrow[r] & P \oplus M \end{tikzcd}$$

$\square$ Theorem 1.40

Definition 1.42. If R is an integral domain, we say $\operatorname{Spec}(R)$ is *normal* if R is integrally closed (in its field of fractions).

Definition 1.43. If T is a torus, we define a *character* to be a grape homomorphism $T \to \mathbb{G}_{\mathrm{m}}$; we define a 1-*parameter subgrape* to be a grape homomorphism $\mathbb{G}_{\mathrm{m}} \to T$.

Remark 1.44. If $T = \operatorname{Spec}(k[M])$, then the characters are in bijection with $\hom_{\mathbb{Z}}(\mathbb{Z}, M) \cong M$, and the one parameter subgrapes are in bijection with $\hom(M, \mathbb{Z}) = M^*$ (frequently called N) the dual free abelian grape.

We have a map

$$\begin{aligned} M \times N &\to \mathbb{Z} \\ (m, n) &\mapsto n(m) = \langle m, n \rangle \end{aligned}$$

In terms of characters and 1-parameter subgrapes, we take the composition

$$\mathbb{G}_{\mathrm{m}} \xrightarrow{n} T \xrightarrow{m} \mathbb{G}_{\mathrm{m}}$$

But every grape homomorphism $\mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}$ is $t \mapsto t^a$ for some $a \in \mathbb{Z}$; the composition is then

$$t \mapsto t^{\langle m, n \rangle}$$

This yields a geometric description of $M \times N \to \mathbb{Z}$.

1.1 Invariant subvarieties of affine toric varieties

Example 1.45. Consider the action of $\mathbb{G}_m^2 = T$ on $\mathbb{A}^2$. Given $(\lambda, \mu) \in T = k^* \times k^*$ and $(x, y) \in \mathbb{A}^2 = k \times k$, we have $(\lambda, \mu) \cdot (x, y) = (\lambda x, \mu y)$.

Figure out the orbits: consider $1 \in T$ (i.e. $(1, 1) \in \mathbb{A}^2$) the identity element. The $(\lambda, \mu) \cdot (1, 1) = (\lambda, \mu)$; so the orbit of $(1, 1)$ is T.

Consider the orbit of $(0, 1)$; we have $(\lambda, \mu) \cdot (0, 1) = (0, \mu)$, with the condition $\mu \neq 0$.

We in fact find that the orbits are T, $\{ (0, \mu) : \mu \neq 0 \}$, $\{ (\lambda, 0) : \lambda \neq 0 \}$, and the origin. The invariant subvarieties are the the closures of the orbits: $\mathbb{A}^2$, the vertical and horizontal axes, and the origin.

Example 1.46. Consider $P = \mathbb{N}^3/((1, 0, 0) + (0, 1, 0) = (0, 0, 2))$; see Example 1.3. Then the invariant subspaces of $X = \mathrm{Spec}(k[P])$ are X, the origin, and the two lines bounding the profile of the double cone.

Let's characterize the closed T-invariant subvarieties; i.e. closed $Y \subseteq X = \mathrm{Spec}(k[P])$ with T acting on Y. Then Y corresponds to an ideal $I \subseteq k[P]$ with $\mathrm{Spec}(k[P]/I) = Y$. Now, the following diagram commutes:

$$\begin{tikzcd}
T \times Y \arrow[r] \arrow[d, hook] & Y \arrow[d, hook] \\
T \times X \arrow[r] & X
\end{tikzcd}$$

So on the level of rings we have the following diagram commutes:

$$\begin{tikzcd}
k[P] \arrow[r, "\tau"] \arrow[d, two heads] & k[P] \otimes_k k[M] \arrow[d, two heads] \\
k[P]/I \arrow[r] & k[P]/I \otimes_k k[M]
\end{tikzcd}$$

Recall we saw this diagram when showing essential surjectivity:

$$\begin{tikzcd}
k[M] \arrow[r] & k[M] \otimes_k k[M] \\
R \arrow[r, dashed] \arrow[u, hook] & R \otimes_k k[M] \arrow[u, hook]
\end{tikzcd}$$

We showed that

$$I = \bigoplus_{u \in P'} k \cdot x^u$$

and $P' \subseteq P$ is a subset. We now use I to deduce properties about $P' \subseteq P$. Suppose $u \in P'$ and $v \in P$; then $x^u \in I$ and $x^v \in k[P]$. But I is an ideal of $k[P]$; so $x^{u+v} = x^u \cdot x^v \in I$, and $u + v \in P'$. Furthermore, if I is prime, then if $u, v \in P$ have $x^u x^v \in I$, then one of x^u and x^v is in I. In terms of P', if $u, v \in P$ and $u + v \in P'$, then $u \in P'$ or $v \in P'$.

Definition 1.47. We say a subset $P' \subseteq P$ (where P is a monoid) is an *ideal* if for all $u \in P'$ and $v \in P$ we have $u + v \in P'$.

Example 1.48. Consider $P = \mathbb{N}^2$; let $P' = \{ (a, b) \in \mathbb{N}^2 : b > 0 \} \subseteq \mathbb{N}^2$. Then P' is an ideal. Consider also $(y) \subseteq k[x, y]$.

TODO 1. *Correspondence?*

Example 1.49. $(x^a y^b, x^c y^d) \subseteq k[x, y]$.

Now, since $Y = \mathrm{Spec}(k[P]/I)$, recall that Y is irreducible if and only if I is prime. In terms of P', if and only if whenever $u, v \in P$ have $u + v \in P'$, we must also have $u \in P'$ or $v \in P'$.

Definition 1.50. An ideal $P' \subseteq P$ is *prime* if $0 \notin P'$ and for all $u, v \in P$ with $u + v \in P'$ we have $u \in P'$ or $v \in P'$.

A further remark: recall that $Y = \operatorname{Spec}(k[P]/I)$ and

$$I = \bigoplus_{u \in P'} k \cdot x^u$$

and in particular

$$k[P]/I = \bigoplus_{u \in P \setminus P'} k \cdot x^u$$

Hence

$$Y = \operatorname{Spec}\left(\bigoplus_{u \in P \setminus P'} k \cdot x^u\right)$$

Definition 1.51. A submonoid $F \subseteq P$ is a *face* if whenever $u, v \in P$ and $u + v \in F$ then $u \in F$ and $v \in F$.

Lemma 1.52. *Suppose $F \subseteq P$ is a subset of a monoid P. Then F is a face if and only if $P \setminus F$ is a prime ideal.*

Proof.

($\Longrightarrow$) Suppose F is a face; we first check that $P \setminus F$ is an ideal. Suppose $v \in P$; supppose $u \notin F$. Then if $u + v \in F$, we would have $u, v \in F$, contradicting our assumption that $u \notin F$; so $u + v \notin F$.

We now check that $P \setminus F$ is prime; suppose $v, u \in P$ with $v + u \notin F$. If $v, u \in F$, then since F is a submonoid, we would have $u + v \in F$, a contradiction.

($\Longleftarrow$) Dual to the above.

□ Lemma 1.52

Example 1.53. Consider $\mathbb{A}^2 \supseteq \mathbb{A}^1$ (identified with the horizontal axis). What are the faces of $\mathbb{N}^2$? We have

- $\mathbb{N}^2$
- $\{ 0 \}$
- $\mathbb{N} \oplus 0$
- $0 \oplus \mathbb{N}$

Example 1.54. Consider a triangular cone emanating from a point. The interesting kinds of faces are

1. A ray.
2. A "face" of the cone in the classical sense.

Proposition 1.55. *If P is a finitely generated monoid then it has finitely many faces.*

Proof. Let $u_1, \ldots, u_n$ be generators of P. If $F \subseteq P$ is a face and $F \neq 0$ then there is some non-zero $\sum a_i u_i \in F$. But then for all i with $a_i \neq 0$ we have $u_i \in F$. So F is generated by a subset of $\{ u_1, \ldots, u_n \}$, of which there are finitely many. □ Proposition 1.55

1.2 Normal toric varieties

We know the finitely generated integral monoids with free grape-ifications are equivalent to the affine toric varieties; what corresponds to the normal affine toric varieties?

Definition 1.56. Suppose R is an integral domain. We say R is *normal* if it it integrally closed in $K = \operatorname{Frac}(R)$; i.e. if $\alpha \in K$ and $p(t) \in R[t]$ is monic with $p(\alpha) = 0$, then $\alpha \in R$.

Example 1.57. Consider $R = k[x,y]/(x^3 - y^2)$; let $\alpha = \frac{y}{x} \in \mathrm{Frac}(R)$. Then $\alpha^2 - x = 0$, so α is a root of a monic polynomial in $R[t]$. But $\alpha \notin R$ (exercise); so R is not normal.

In picture, $\mathrm{Spec}(R)$ is the cuspidal cubic. Note that $R = k[\mathbb{N} \setminus 1]$, and $R \cong k[t^2, t^3]$ with $\alpha \approx \frac{t^3}{t^2} = t \approx 1$. The reason R is not normal is that there's a "hole".

Remark 1.58. A curve C over k is normal if and only if C is smooth.

Definition 1.59. An integral monoid P is *saturated* if for all $u \in P^{\mathrm{gp}}$ and all $n \in \mathbb{Z}^+$ with $nu \in P$ we have $u \in P$.

Example 1.60. $P = \mathbb{N} \setminus 1$ is not saturated since $1 = 3 - 2 \in P^{\mathrm{gp}}$ and $2 \cdot 1 \in P$ but $1 \notin P$.

Proposition 1.61. *An affine toric variety $X = \mathrm{Spec}(k[P])$ is normal if and only if P is saturated.*

Proof. We have $T = \mathrm{Spec}(k[P^{\mathrm{gp}}]) \subseteq X$ open an dense. Then $k[P] \subseteq k[P^{\mathrm{gp}}] \subseteq K$ where $K = \mathrm{Frac}(k[P]) = \mathrm{Frac}(k[P^{\mathrm{gp}}])$. Now, T is normal because it is smooth, so $k[P^{\mathrm{gp}}]$ is integrally closed in K. So X is normal if and only if $k[P]$ is integrally closed in $k[P^{\mathrm{gp}}]$. So, to show that X is normal, we may assume $\alpha \in k[P^{\mathrm{gp}}]$ satisfies a monic polynomial over $k[P]$.

($\implies$) Suppose X is normal. Suppose $u \in P^{\mathrm{gp}}$ and $n \in \mathbb{Z}^+$ with $nu \in P$. We know that $k[P] \subseteq k[P^{\mathrm{gp}}] \subseteq K = \mathrm{Frac}(K[P^{\mathrm{gp}}])$ and $k[P^{\mathrm{gp}}]$ is integrally closed. Now, $(x^u)^n \in K[P]$, so x^u satisfies the monic polynomial $t^n - x^{nu} \in k[P][t]$. Since X is normal (i.e. $k[P]$ is integrally closed) and $x^u \in K$ it follows that $x^u \in k[P]$; i..e that $u \in P$.

($\impliedby$) Suppose P is saturated. Let $\widetilde{X} = \mathrm{Spec}(R)$ be the *normalization* of X; i.e. R is the integral closure of $k[P]$. Then by functoriality of normalization, if $\pi \colon \widetilde{X} \to X$ is the projection, then $\pi^{-1}(T)$ is the normalization of T. But T is normal; so $\pi^{-1}(T) \cong T$. Also by functoriality, the grape action T on X then extends to a grape action T on $\widetilde{X}$.

So $\widetilde{X}$ is a toric variety. So by Theorem 1.40 we have $\widetilde{X} = \mathrm{Spec}(k[S])$ for some monoid S. Then $P \subseteq S \subseteq S^{\mathrm{gp}} = P^{\mathrm{gp}}$. But $\widetilde{X}$ is normal; so, assuming the easy direction, we know that S is saturated. To show that X is normal, we will show $X = \widetilde{X}$; i.e. that $P = S$.

If $u \in S$ then $x^u \in R = k[S]$. Now, R is finite over $k[P]$, so there is $f \in k[P][t]$ such that $f(x^u) = 0$. Then the coefficients of f are in

$$k[P] = \bigoplus_{v \in P} k \cdot x^v$$

decomposing, we may assume f has the form

$$t^m + \alpha_1 x^{v_1} t^{m-1} + \cdots + \alpha_m x^{v_m}$$

with each $\alpha_i \in k$. Same direct sum decomposition trick but in $k[S]$. We need that if $\alpha_i \neq 0$ then $x^{v_i}(x^u)^{m-i} = x^{um}$; i.e. $um = v_i + u(m-i)$, i.e. $iu = v_i$. Now, $u \in S$ and the $v_i \in P$; since we assumed P is saturated, we then get that $u \in P$, which means $S = P$ and $R = k[P]$.

$\square$ Proposition 1.61

Note: the proof actually showed that the normalization of $X = \mathrm{Spec}(k[P])$ is $\widetilde{X} = \mathrm{Spec}(k[P^{\mathrm{sat}}])$ where

$$P^{\mathrm{sat}} = \{\, u \in P^{\mathrm{gp}} : \exists n \in \mathbb{Z}^+ \text{ such that } nu \in P \,\}$$

is the *saturation* of P.

Pictorially, look at $\mathbb{A}^2$, which corresponds to $P = \mathbb{N}^2$. To obtain a non-normal toric variety whose normalization is $\mathbb{N}^2$, we just remove finitely many points from $\mathbb{N}^2$ (while making sure the result is closed under addition).

Definition 1.62. Suppose P is a monoid. The *units* of P are the elements of $P^* = \{\, p \in P : \exists q \in P \text{ such that } p + q = 0 \,\}$. We say P is *sharp* if $P^* = 0$.

Lemma 1.63. *Suppose P is integral, finitely generated, and saturated with free grape-ification. Then P has a decomposition of the form*

$$P \cong \mathbb{Z}^{\oplus r} \oplus P'$$

where P' has no units.

Proof. We sketch the proof.

Begin with $P^* \subseteq P$ and form the quotient $\overline{P} = P/P^*$. We thus have a short exact sequence $0 \to P^* \to P^{\mathrm{gp}} \to \overline{P}^{\mathrm{gp}} \to 0$; one can show that $\overline{P}^{\mathrm{gp}}$ is free. Hence the sequence splits; one can show that this yields our desired direct sum decomposition.

The key property for this proof to work is saturation. □ Lemma 1.63

Corollary 1.64. *If X is an affine normal toric variety then X can be written (non-canonically) as $X \cong T \times X'$ for some torus T and $X' = \operatorname{Spec}(k[P'])$ where P' is sharp.*

Definition 1.65. If $X = \operatorname{Spec}(k[P])$ with P sharp, we say X is *pointed*.

2 Convex geometry

Fix a finite-dimensional vector space V over $\mathbb{R}$.

Definition 2.1. We say a subset $Z \subseteq V$ is *convex* if whenever $x, y \in Z$ and $t \in [0, 1]$ we have $tx + (1-t)y \in Z$. We say Z is a *cone* if for all $z \in Z$ and $\lambda \in \mathbb{R}_{\geq 0}$ we have $\lambda z \in Z$.

Exercise 2.2. $Z \subseteq V$ is a convex cone if and only if it is a cone and for all $z, z' \in Z$ we have $z + z' \in Z$.

Definition 2.3. Given a subset $Z \subseteq V$ we define the *convex hull* of Z to be

$$\operatorname{Conv}(Z) = \bigcap_{\substack{Z \subseteq Y \\ Y \text{convex}}} Y$$

Lemma 2.4. *If $Z \subseteq V$ then*

$$\operatorname{Conv}(Z) = \left\{ \sum_{i=1}^{r} \lambda_i v_i : \lambda_i \geq 0, v_i \in Z, \sum_{i=1}^{r} \lambda_i = 1, r \in \mathbb{N} \right\}$$

Proof. The right-hand side is clearly convex. If $Y \supseteq Z$ and Y is convex then by induction on r we have $Y \supseteq \operatorname{Conv}(Z)$, as required. □ Lemma 2.4

Definition 2.5. The *convex cone* generated by a subset $Z \subseteq V$ is

$$\operatorname{Cone}(Z) = \bigcap_{\substack{Y \supseteq Z \\ Y \text{ a convex cone}}} Y$$

Lemma 2.6. *If $Z \subseteq V$ then*

$$\operatorname{Cone}(Z) = \left\{ \sum_{i=1}^{r} \lambda_i v_i : \lambda_i \geq 0, v_i \in Z, r \in \mathbb{N} \right\}$$

Definition 2.7. A *polytope* is the convex hull of a finite set. A *polyhedral convex cone* is the convex cone generated by some finite set.

Lemma 2.8. *Polyhedral convex cones and polytopes are closed. Polytopes are also compact.*

Proof. If P is a polytope then $P = \operatorname{Conv}(\{ v_1, \ldots, v_n \})$. Then the map

$$[0, 1]^n \to V$$
$$(\lambda_i : i < n) \mapsto \sum_{i=1}^{n} \lambda_i v_i$$

is continuous and has image P; so P is compact (and thus closed).

Claim 2.9. *If σ is a polyhedral convex cone then σ admits a decomposition $\sigma \cong W \times \tau$ where $W \subseteq V$ is a subspace and τ is a polyhedral convex cone containing no lines.*

Proof. Let $W = \sigma \cap -\sigma = \{ v \in \sigma : -v \in \sigma \}$; this is the largest vector subspace contained in σ. Let $\pi \colon V \twoheadrightarrow V/W$ be the quotient. Then $\pi(\sigma)$ is a convex cone in V/Q. Choose a splitting; then $V \cong W \times V/Q$, so $\sigma \cong W \times \pi(\sigma)$. Why is $\pi(\sigma)$ pointed? This is because if v and $-v$ are in $\pi(\sigma)$ then $v, -v \in \sigma + W \subseteq \sigma$, so $v \in W$, and $\pi(v) = 0$. $\square$ Claim 2.9

Note that W is canonical, but the splitting is not.

Hence, since W is closed, we may assume $\sigma = \tau$; i.e. that σ contains no lines. i.e. If $0 \neq v \in \sigma$ then $-v \notin \sigma$.

We know that $\sigma = \mathrm{Cone}(v_1, \ldots, v_n)$, so we can let $P = \mathrm{Conv}(v_1, \ldots, v_n)$. Now $\sigma = \{ \lambda v : v \in P, \lambda \geq 0 \}$. Given a sequence $(u_m : m \in \mathbb{N})$ in σ with $(u_m : m \in \mathbb{N}) \to u$, we wish to show $u \in \sigma$. Writing $u_m = \lambda_m v_m$ with $\lambda_m \geq 0$ and $v_m \in P$ for all m, it follows from compactness of P that a subsequence of the v_m converges to some $v \in P$. Without loss of generality we may assume $0 \notin P$, so $v \neq 0$. Also $(\lambda_m : m \in \mathbb{N})$ is bounded since $v \neq 0$; it thus has a convergent subsequence, converging to some $\lambda \geq 0$. Then $\lambda_m u_m \to \lambda v \in \sigma$, as desired. $\square$ Lemma 2.8

Definition 2.10. An *affine hyperplane* is some subset $H \subseteq V$ with $H = W + v$ for some $v \in V$ and some subspace $W \subseteq V$ of codimension 1.

Definition 2.11. Suppose $K \subseteq V$ is a closed and convex subset. We say an affine hyperplane H is a *supporting hyperplane* of K if $H \cap K \neq \emptyset$ and K is contained in one of $H_{\leq}$ and $H_{\geq}$. A *supporting half-space* of K is a closed half-space containing K that is determined by a supporting hyperplane of K.

Proposition 2.12. *If K is closed and convex, then it is the intersection of its supporting half-spaces.*

Proof. Fix an inner product $\langle \cdot, \cdot \rangle$ on V; this yields a metric. We need to show that if $x \in V$ but $x \notin K$ then there is a supporting half-space $H_{\geq}$ of K such that $x \notin K$; i.e. x is on the wrong side of the hyperplane from K.

Since K is closed, there is $x' \in K$ such that

$$d(x, x') = \min_{y \in K} d(x, y)$$

Note that $x' \neq x$ because $x \notin K$. Let H be a hyperplane perpendicular to the line from x to x'. Then $H = \{ v \in V : \langle v, x - x' \rangle = 0 \}$. By translating, we may assume $x' = 0$. Note that $\langle x, x - 0 \rangle > 0$, so the half-space we're interested in is $H_{\leq} = \{ v : \langle v, x \rangle \leq 0 \}$. We wish to show that $K \subseteq H_{\leq}$.

Suppose $y \in K$; we wish to show that $\langle x, y \rangle \leq 0$. If $y = 0$, then $\langle y, 0 \rangle = 0$; assume then that $y \neq 0$. Since K is convex, it follows that for all $t \in [0, 1]$ we have

$$ty + \underbrace{(1-t)x'}_{0} \in K$$

By definition of $x' = 0$, we have $d(x, 0) \leq d(x, ty)$. Expanding, we find

$$\langle x, x \rangle \leq \langle x - ty, x - ty \rangle = \langle x, x \rangle - 2t \langle x, y \rangle + t^2 \langle y, y \rangle$$

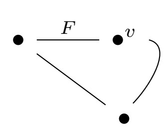
and hence that $2\langle x, y \rangle \leq t \langle y, y \rangle$. As $t \to 0$, we find $\langle x, y \rangle \leq 0$, and $y \in H_{\leq}$. $\square$ Proposition 2.12

Definition 2.13. A subset $F \subseteq K$ is a *face* if $F = K \cap H$ where H is a supporting hyperplane.

Definition 2.14. We define $\dim(K)$ to be the dimension of the affine subspace generated by K.

Definition 2.15. A *vertex* of K is a 0-dimensional face, and a *facet* is a face of codimension 1.

Remark 2.16. If K is not polyhedral, then a face of a face need not be a face. Consider



Note that F is a face of K and v is a face of F, but v is not a face of K.

Lemma 2.17. *If K is convex and closed and $F_1, \ldots, F_n \subseteq K$ are faces then $F = \bigcap_i F_i$ is a face or is empty.*

Proof. Suppose $F \neq \emptyset$. Let $u_i \in V^*$ and $a_i \in \mathbb{R}$ be such that if $H_i = \{ v \in V : \langle u_i, v \rangle = a_i \}$ then $F_i = H_i \cap K$ and $K \subseteq \{ v : \langle u_i, v \rangle \geq a_i \}$. Let $u = \sum u_i$. If $u = 0$, replace u_1 by $2u_1$ and a_1 by $2a_1$; we may thus assume that $u \neq 0$.

Then $\langle u, v \rangle \geq \sum a_i$ for all $v \in K$. Moreover, we have equality if and only if $\langle u_i, v \rangle = a_i$ for all i. So $\{ v \in V : \langle u, v \rangle \geq \sum a_i \} \supseteq K$ and $\{ v \in V : \langle u, v \rangle = \sum a_i \} \cap K = \bigcap_i F_i$. $\square$ Lemma 2.17

Suppose σ is a closed convex cone and H is a supporting hyperplane of σ; suppose $\sigma \subseteq H_{\geq} = \{ v \in V : \langle v, u \rangle \geq a \}$ where $u \in V^*$ and $a \in \mathbb{R}$. Then $H \cap \sigma \neq \emptyset$, so there is $v \in \sigma$ such that $\langle v, u \rangle = a$. But σ is a cone; so $tv \in \sigma \subseteq H_{\geq}$ for all $t \geq 0$. So $ta = \langle tv, u \rangle \geq a$ for all $t \geq 0$; so $a = 0$.

Hence all supporting hyperplanes are in correspondence with certain $u \in V^*$; i.e. we can ignore a (since $a = 0$).

Definition 2.18. Given a cone σ we define the dual cone of σ to be $\sigma^\vee = \{ u \in V^* : \langle v, u \rangle \geq 0 \text{ for all } v \in \sigma \}$; i.e. the set of $u \in V^*$ such that H_u is a supporting hyperplane of σ.

Remark 2.19. $\sigma^\vee$ is in fact a convex cone. One checks for all $u, u' \in \sigma^\vee$ that $u + u' \in \sigma^\vee$ and for all $u \in \sigma^\vee$ and $t \geq 0$ that $tu \in \sigma^\vee$. For illustration, for the second property, note that for all all $v \in \sigma$ we have $\langle v, tu \rangle = t\langle v, u \rangle \geq 0$, and hence that $tu \in \sigma^\vee$.

Example 2.20. $\sigma = \mathrm{Cone}((1,0),(1,2)) \subseteq \mathbb{R}^2 = V$. Then

$$\sigma^\vee = \{ u \in V^* \cong \mathbb{R}^2 : \langle u, v \rangle \geq 0 \text{ for all } v \in \sigma \}$$

Now, $v \in \sigma$ implies $v = a(1,0) + b(1,2)$ with $a, b \geq 0$; then $\langle u, v \rangle \geq 0$ for all $a, b \geq 0$ if and only if $\langle u, (1,0) \rangle \geq 0$ and $\langle u, (1,2) \rangle \geq 0$. Hence if $u = (c)$ then $0 \leq c$ and $0 \leq c + 2d$; solving, we find $\sigma^\vee = \mathrm{Cone}((0,1),(2,-1))$.

Proposition 2.21. *If σ is a closed convex cone then $(\sigma^\vee)^\vee = \sigma$.*

Proof. $v \in (\sigma^\vee)^\vee$ if for all $u \in \sigma^\vee$ we have $\langle v, u \rangle \geq 0$; i.e. if $v \in (H_u)_{\geq}$. Hence $v \in (\sigma^\vee)^\vee$ if

$$v \in \bigcap_{u \in \sigma^\vee} (H_u)_{\geq}$$

But the latter is the intersection of the supporting hyperplanes of σ, which by last time is just σ. Hence $(\sigma^\vee)^\vee = \sigma$. $\square$ Proposition 2.21

From now on "cone" will always mean a convex polyhedral cone.

Exercise 2.22. If σ is a cone and $\tau \subseteq \sigma$ is a face and $\sigma = \mathrm{Cone}(v_1, \ldots, v_n)$ then $\tau = \mathrm{Cone}(v_j : v_j \in \tau)$.

Corollary 2.23. *Every cone has finitely many faces.*

Recall from last time that in general faces of faces aren't faces.

Lemma 2.24. *If σ is a cone and $\tau \subseteq \sigma$ is a face and $\varepsilon \subseteq \tau$ is a face, then $\varepsilon \subseteq \sigma$ is a face.*

Proof. Since τ is a face of σ we get that $\tau = \sigma \cap u^\perp$ for some supporting hyperplane $u^\perp$; likewise we get $\varepsilon = \tau \cap w^\perp$. Then $u \in \sigma^\vee$ and $w \in \tau^\vee$. Write $\sigma = \mathrm{Cone}(v_1, \ldots, v_n)$ and $\tau = \mathrm{Cone}(v_1, \ldots, v_s)$ (with $v_i \notin \tau$ for $i > s$); hence $\langle u, v_i \rangle > 0$ for $i > s$. Then $\langle w + tu, v_i \rangle > 0$ for t sufficiently large and $i > s$; hence $\sigma \cap (w + tu)^\perp = \tau \cap (w + tu)^\perp = \tau \cap w^\perp = \varepsilon$ (since $u^\perp \supseteq \tau$). $\square$ Lemma 2.24

Exercise 2.25. Every face of σ is an intersection of facets.

Corollary 2.26. *$\sigma^\vee$ is a polyhedral cone.*

So far, we've only talked about cones in a real vector space. We now give them rational structure.

Definition 2.27. If N is a lattice (i.e. finite rank free abelian grape), let $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ (considered as a real vector space). We say a cone $\sigma \subseteq N_{\mathbb{R}}$ is *rational* if it is generated by elements of $N_{\mathbb{Q}}$ (or equivalently if it's generated by elements of N).

Example 2.28. Consider $N = \mathbb{Z}^2$; then $N_{\mathbb{R}} = \mathbb{R}^2$. Let $\sigma = \operatorname{Cone}((1,0),(1,\pi))$. Then σ is not rational.

We dislike non-rational σ since $\sigma \cap N$ will not be a finitely generated monoid.
The point of convex geometry:

Proposition 2.29. *If σ is a rational (polyhedral) cone over some lattice N, then $\sigma \cap N$ is a finitely generated, integral, saturated monoid.*

Proof. Let $P = \sigma \cap N$.

(P a monoid) Suppose $p, q \in P$. Then since $p, q \in N$ we get $p + q \in N$; since σ is a cone, we get $p + q \in \sigma$. So $p + q \in P$. It's also clear that $0 \in P$.

(P integral) Simply because $P \subseteq N$ is a submonoid and N is an abelian grape.

(P saturated) Suppose $v \in P^{\mathrm{gp}} \subseteq N$ and $nv \in P$ where $n \in \mathbb{Z}^+$; we wish to show that $v \in P$. Then $nv \in \sigma$ and σ is a cone; so $v = \frac{1}{n}(nv) \in \sigma$. So $v \in P$.

(P finitely generated) Let $\sigma = \operatorname{Cone}(v_1, \ldots, v_n)$; let

$$K = \left\{ \sum \lambda_i v_i : \lambda_i \in [0,1] \right\}$$

so K is compact. Then $K \cap N$ is finite, since N is discrete. Let $w_1, \ldots, w_\ell$ be the lattice points; we will show that the w_j generate P. (Note that the w_j contain the v_i.)

Suppose $p \in P$; so

$$p = \sum \lambda_i v_i$$

for $\lambda_i \geq 0$. Then

$$v - \sum \lfloor \lambda_i \rfloor v_i \in K \cap N$$

So

$$v = \sum \lfloor \lambda_i \rfloor v_i + w_j$$

for some w_j. Hence v is in the monoid generated by the w_j (which include the v_i). $\square$ Proposition 2.29

Fact 2.30. *In fact, if $\dim(\sigma) = \dim(N_{\mathbb{R}})$ then $(\sigma \cap N)^{\mathrm{gp}} = N$.*

If σ not full-dimensional, then $(\sigma \cap N)^{\mathrm{gp}}$

TODO 2. *Missing some words?*

Notation 2.31. From now on a "cone" will mean a rational, polyhedral, convex cone.

From the above work we get:

Theorem 2.32. *We have an inclusion-preserving bijection between finitely generated, integral, saturated submonoids of N and rational polyhedral cones on N given by $P \mapsto \operatorname{Cone}(P)$ and $\sigma \mapsto \sigma \cap N$.*

We thus get a correspondence between cones on N and normal affine toric varieties with $T = \operatorname{Spec}(k[N])$.

Notation 2.33. In the future we will sometimes drop "normal" from "affine normal toric variety".

Definition 2.34. If $P \ni 0$ is a polytope, then its *polar dual* is $P^0 = \{ u \in V^* : \langle u, v \rangle \geq -1 \text{ for all } v \in P \}$.

Lemma 2.35. *If P is a polytope and $\sigma = \operatorname{Cone}(P \times \{1\})$ in $V \times \mathbb{R}$ then $\sigma^\vee = \operatorname{Cone}(P^0 \times \{1\})$.*

Proof. Well,

$$\begin{aligned} \sigma^\vee &= \{ (u,s) \in V^* \times \mathbb{R} : \langle (u,x),(v,t) \rangle \geq 0 \text{ for all } (v,t) \in \sigma \} \\ &= \{ (u,s) : \langle (u,x),(v,1) \rangle \geq 0 \text{ for all } v \in P \} \end{aligned}$$

. Note, however, that $\langle (u,s),(v,1)\rangle = s + \langle u,v\rangle$; furthermore, since $0 \in P$, taking $v = 0$ we find $s \geq 0$. On the other hand,

$$P^0 = \{\, u \in V^* : kangsu, v \geq -1 \text{ for all } v \in P \,\}$$

and

$$\operatorname{Cone}(P^0 \times \{\,1\,\}) = \left\{\, (u,s) \in V^* \times \mathbb{R} : s \geq 0, \frac{u}{s} \in P^0 \,\right\}$$

i.e. we require that $\left\langle \frac{u}{s}, v \right\rangle \geq -1$ for all $v \in P$; i.e. that for all $v \in P$ we have $\langle u,v\rangle \geq -s$, which is the same condition as $(u,s) \in \sigma^\vee$. $\square$ Lemma 2.35

Corollary 2.36. *$(P^0)^0 = P$.*

3 Fans and toric varieties

Definition 3.1. A *fan* Σ on a lattice N is a finite set of pointed rational cones on N such that

TODO 3. *Pointed is not containing a line?*

1. For all $\sigma \in \Sigma$ and all faces τ of σ we have $\tau \in \Sigma$.
2. For all $\sigma, \sigma' \in \Sigma$ we have $\sigma \cap \sigma'$ is a face of σ and a face of σ'.

Example 3.2. In $N = \mathbb{Z}^2$, let $\sigma = \operatorname{Cone}((1,0),(1,1))$ and $\sigma' = \operatorname{Cone}((1,1),(0,1))$.

$$\begin{aligned}
\sigma &= \operatorname{Cone}((1,0),(1,1)) \\
\sigma' &= \operatorname{Cone}((1,1),(0,1)) \\
\rho_1 &= \operatorname{Cone}((1,0)) \\
\rho_2 &= \operatorname{Cone}((1,1)) \\
\rho_3 &= \operatorname{Cone}((0,1))
\end{aligned}$$

Then $\Sigma = \{\, \{\,0\,\}, \rho_1, \rho_2, \rho_3, \sigma, \sigma' \,\}$ is a fan.

Editor's note 3.3. I had to transcribe the above example from a diagram, so transcription errors may have occurred.

Definition 3.4. The *support* of a fan Σ is

$$|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma$$

Remark 3.5. If Σ is a fan then each $\sigma \in \Sigma$ yields an affine toric variety via $U_\sigma = \operatorname{Spec}(k[\sigma^\vee \cap M])$ where $M = N^* = \hom_{\mathbb{Z}}(N, \mathbb{Z})$.

We could look at $\operatorname{Spec}(K[\sigma \cap N])$ but we don't; one reason is that $\sigma^\vee$ is full-dimensional in M even if σ isn't.

Editor's note 3.6. I think this requires that σ be pointed, which is indeed true for all σ in a fan Σ.

Example 3.7. Consider $\sigma = \operatorname{Cone}((1,0))$ in $\mathbb{Z}^2$. Then σ is not full-dimensional, but $\sigma^\vee$ is the right half of the plane, and is full-dimensional.

Since $\sigma^\vee$ is full-dimensional, we have $(\sigma^\vee \cap M)^{\mathrm{gp}} = M$, so the torus of the affine toric variety U_σ is $T = \operatorname{Spec}(k[M])$. Hence all U_σ for $\sigma \in \Sigma$ have the same torus.

Notice if $\tau < \sigma$ is a face then $\tau^\vee \supseteq \sigma^\vee$; so $\tau^\vee \cap M \supseteq \sigma^\vee \cap M$, and we get a map $U_\tau \to U_\sigma$. As it turns out, U_τ sits in U_σ as a principal affine open subset. If $\tau = \sigma \cap u^\perp$ for $u \in \sigma^\vee \cap M$, then U_τ is affine open in U_σ where we've inverted x^u.

TODO 4. *What?*

Property 2 tells us that if $\sigma_1, \sigma_2 \in \Sigma$ then we have embeddings of $U_{\sigma_1 \cap \sigma_2}$ into U_{σ_1} and U_{σ_2}. Gluing together all of the U_σ, we get a variety which we denote $X(\Sigma)$ or X_Σ; it turns out this is a *toric variety.* (In fact it's a normal toric variety; one can check normality on an affine cover and each U_σ is normal.)

Example 3.8. In $\mathbb{Z}$, let

$$\begin{aligned} \sigma &= \operatorname{Cone}(1) \\ \sigma' &= \operatorname{Cone}(-1) \\ \Sigma &= \{ \{ 0 \}, \sigma, \sigma' \} \end{aligned}$$

Then

$$\begin{aligned} \sigma^\vee \cap M &= \mathbb{N} \\ (\sigma')^\vee \cap M &= -\mathbb{N} \\ 0^\vee \cap M &= \mathbb{Z} \end{aligned}$$

We thus get a diagram:

$$\begin{tikzcd} & \mathbb{P}^1 = X(\Sigma) & \\ \operatorname{Spec}(k[x^{-1}]) = U_{\sigma'} \arrow[ur, hook] & & \operatorname{Spec}(k[x]) = U_\sigma \arrow[ul, hook'] \\ & \operatorname{Spec}(k[x^\pm]) = U_0 \arrow[ul, hook'] \arrow[ur, hook] & \end{tikzcd}$$

Example 3.9. In $\mathbb{Z}^2$, let

$$\begin{aligned} \tau &= \operatorname{Cone}((1,0)) \\ \tau' &= \operatorname{Cone}((0,1)) \\ \Sigma &= \{ \{ 0 \}, \tau, \tau' \} \end{aligned}$$

Then

$$\begin{aligned} \tau^\vee &= \mathbb{N} \times \mathbb{Z} \\ (\tau')^\vee &= \mathbb{Z} \times \mathbb{N} \\ 0^\vee &= \mathbb{Z}^2 \end{aligned}$$

We then get another diagram:

$$\begin{tikzcd} & X(\Sigma) = \mathbb{Z}^2 \setminus \{ 0 \} & \\ \mathbb{A}^1 \times \mathbb{G}_{\mathrm{m}} = U_{\tau'} \arrow[ur, hook] & & \mathbb{A}^1 \times \mathbb{G}_{\mathrm{m}} = U_\tau \arrow[ul, hook'] \\ & T = U_0 = \mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}} \arrow[ul, hook'] \arrow[ur, hook] & \end{tikzcd}$$

Definition 3.10. A *morphism* of fans $(\Sigma' N') \to (\Sigma, N)$ is a morphism of lattices $\varphi\colon N' \to N$ such that for all $\sigma' \in \Sigma'$ there is $\sigma \in \Sigma$ such that $\varphi(\sigma') \subseteq \sigma$.

Example 3.11. Let $N = N' = \mathbb{Z}^2$. Let

$$\begin{aligned} \sigma_1 &= \operatorname{Cone}((0,1),(1,0)) \\ \rho_1 &= \operatorname{Cone}((0,1)) \\ \rho_1' &= \operatorname{Cone}((1,0)) \\ \Sigma &= \{ 0, \rho_1, \rho_1', \sigma_1 \} \\ \sigma_2 &= \operatorname{Cone}((1,0),(1,2)) \\ \rho_2 &= \operatorname{Cone}((0,1)) \\ \rho_2' &= \operatorname{Cone}((1,2)) \\ \Sigma' &= \{ 0, \rho_2, \rho_2', \sigma_2 \} \end{aligned}$$

Consider $\varphi \colon \mathbb{Z}^2 \to \mathbb{Z}^2$ given by

$$\varphi = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

Then every cone of Σ is sent to a cone of Σ'.

If $\varphi \colon (\Sigma', N') \to (\Sigma, N)$ satisfies $\varphi(\sigma') \subseteq \sigma$ for $\sigma' \in \Sigma'$ and $\sigma \in \Sigma$, then this yields maps $U_{\sigma'} \to U_\sigma$; hence we get a morphism of toric varieties $X(\Sigma') \to X(\Sigma)$.

Recall if U_σ is an affine toric variety then the T-orbits correspond to faces of σ: τ is a face of σ if and only if $O_\tau = \operatorname{Spec}(k[\tau^\perp \cap M])$. The corresponding irreducible T-invariant closed subvariety $V(\tau) = \overline{O_\tau}$ is $\operatorname{Spec}(k[\sigma^\vee \cap M \cap \tau^\perp])$, and

$$U_\sigma = \coprod_{\tau < \sigma} O_\tau$$

TODO 5. *$\tau < \sigma$ means τ a face of σ.*

Then $O_{\tau_1} \subseteq \overline{O_{\tau_2}} = V(\tau_2)$ if and only if $V(\tau_1) \subseteq V(\tau_2)$, which occurs by a bijection done previously if and only if $\tau_1 \subseteq \tau_2$.

Remark 3.12. O_σ is closed if and only if

$$O_\sigma = V(\sigma) = \coprod_{\sigma < \tau} O_\tau$$

which occurs if and only if σ is not a face of any τ; i.e. σ is maximal.

Proposition 3.13. *Suppose $\varphi \colon (\Sigma', N') \to (\Sigma, N)$ is a morphism of fans; let $\varphi_* \colon X(\Sigma') \to X(\Sigma)$ be the induced morphism of toric varieties. Suppose $\tau' \in \Sigma'$; let $\tau \in \Sigma$ be the smallest cone such that $\varphi(\tau') \subseteq \tau$. Then $\varphi_*(O_{\tau'}) \subseteq O_\tau$ and $\varphi_*(V(\tau')) \subseteq V(\tau)$.*

Proof. That $\varphi_*(V(\tau')) \subseteq V(\tau)$ follows from $\varphi_*(O_{\tau'}) \subseteq O_\tau$ by taking closures. To show $\varphi_*(O_{\tau'}) \subseteq O_\tau$, we'll show the following: that in general O_τ has an identity element and it looks like $x_\tau \colon M \to k$ given by

$$u \mapsto \begin{cases} 0 & \text{if } u \notin \tau^\perp \\ 1 & \text{if } u \in \tau^\perp \end{cases}$$

We'll show that $\varphi_*(x_{\tau'}) = x_\tau$; this implies the desired statement since $O_\tau = T \cdot x_\tau$. i.e. we need to show that the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "\varphi^*"] \arrow[dr, "x_\tau"'] & M' \arrow[d, "x_{\tau'}"] \\ & k \end{tikzcd}$$

We thus need to show that $u \in M$ satisfies $u \in \tau^\perp$ if and only if $\varphi^*(u) \in (\tau')^\perp$.

If $u \in \tau^\perp$, then $\varphi^*(u) \in (\tau')^\perp$ since for all $v \in \tau'$ we have $\langle \varphi^* u, v\rangle = \langle u, \varphi_* v\rangle = 0$ (since $\varphi_* v \in \varphi_* \tau' \subseteq \tau$). Conversely, if $\varphi^* u \in (\tau')^\perp$, then for all $v \in \tau'$ we have $0 = \langle \varphi^* u, v\rangle = \langle u, \varphi_* v\rangle$; hence $\varphi_* \tau') \subseteq u^\perp$. But $\varphi_*(\tau') \subseteq \tau$; so $\varphi_*(\tau') \subseteq \tau \cap u^\perp$. But τ is the smallest cone containing $\varphi_*(\tau')$; so $\tau \cap u^\perp$ is not a proper face; i.e. $\tau \cap u^\perp = \tau$. So $u \in \tau^\perp$, as desired. ☐ Proposition 3.13

Exercise 3.14 (For the bored). Show that the fan generated by $(-1, a)$, $(0, 1)$, $(1, 0)$, and $(0, -1)$ corresponds to the a^{th} Hirzebruch surface.

Proposition 3.15. *Suppose $\varphi \colon N' \to N$ is a morphism of lattices; let $T_{N'} \to T_N$ be the corresponding morphism of tori. Let $T' = T_{N'}$ and $T = T_N$.*

1. *If φ is surjective then there is T'' such that the following diagram commutes:*

$$\begin{tikzcd} T' \arrow[r, "\cong"] \arrow[dr, "f"] & T \times T'' \arrow[d, two heads] \\ & T \end{tikzcd}$$

In particular, we get that f is smooth, surjective, and has connected fibres.

2. *If φ is injective and* $|\operatorname{coker}\varphi| < \infty$ *then f is finite and surjective with* $\deg(f) = |\operatorname{coker}\varphi|$.

3. *If φ is injective and* $\operatorname{coker}\varphi$ *is free then f is a closed immersion.*

4. *f always admits a decomposition:*

$$T' \twoheadrightarrow T_1 \twoheadrightarrow T_2 \hookrightarrow T$$

where the first map is smooth, surjective, and has connected fibres, the second map is a finite surjection, and the third is a closed immersion.

Proof.

1. Suppose φ is surjective. We get a short exact sequence

$$0 \to N'' \to N' \xrightarrow{\varphi} N \to 0$$

But N' is free, and $N'' \subseteq N''$; so N'' is free. The exact sequence splits because N is free; so the following diagram commutes:

$$\begin{tikzcd} N' \arrow[r, "\cong"] \arrow[dr, "\varphi"'] & N \times N'' \arrow[d, two heads] \\ & N \end{tikzcd}$$

This is true at the level of lattices, and hence is also true of tori: so the following diagram commutes:

$$\begin{tikzcd} T' \arrow[r, "\cong"] \arrow[dr, "f"'] & T \times T'' \arrow[d, two heads] \\ & T \end{tikzcd}$$

Since f is a projection map, we get that it is smooth with connected fibres that are isomorphic to T'.

2. We have a short exact sequence

$$0 \to N' \xrightarrow{\varphi} N \to N/N' \to 0$$

with N/N' finite. Dualize: apply $\hom_{\mathbb{Z}}(-, \mathbb{Z}) = (-)^*$. We get

$$(N/N') = 0 \to M \xrightarrow{\varphi^*} M' \to \underbrace{\operatorname{Ext}'_{\mathbb{Z}}(N/N', \mathbb{Z})}_{\text{finite order } R=|\operatorname{coker}\varphi|} \to \operatorname{Ext}^1(N, \mathbb{Z} = 0$$

(with the last equality because N is free). Choose a basis $e_1, \ldots, e_n$ for M' such that M has basis $a_1e_1, \ldots, a_ne_n$. Then $f\colon T' \to T$ given by $(t_i) \mapsto (t_i^{a_i})$ is finite and surjective with degree $\prod a_i = |\operatorname{coker}\varphi|$.

3. As before we get short exact sequences

$$0 \to N' \xrightarrow{\varphi} N \to N/N' \to 0$$

and

$$0 \to (N/N')^* \to M \xrightarrow{\varphi^*} M' \to 0$$

So the map $k[M] \to k[M']$ is surjective; so $T' \hookrightarrow T$ is a closed immersion.

4. We can factor φ as follows:

$$\begin{tikzcd} N' \arrow[r, "\varphi"] \arrow[d, two heads] & N \\ N_1 \arrow[ur, hook] & \end{tikzcd}$$

where $N_1 = \operatorname{im}(\varphi)$. Choose N_2 such that $N_2/N_1 = (N/N_1)_{\text{torsion}}$. We thus get that the following diagram commutes:

$$\begin{tikzcd} N' \arrow[r, "\varphi"] \arrow[d, two heads] & N \\ N_1 \arrow[r, hook] & N_2 \arrow[u, hook] \end{tikzcd}$$

where the map $N' \twoheadrightarrow N_1$ is surjective, the map $N_1 \hookrightarrow N_2$ is injective with finite cokernel, and the map $N_2 \hookrightarrow N$ is inective with free cokernel. The result then follows from the previous parts.

TODO 6. *Right?*

□ Proposition 3.15

Corollary 3.16. *Suppose $\varphi\colon N' \to N$; let $f\colon T' \to T$ be the corresponding morphism of tori. Then:*

1. *f is a closed immersion if and only if φ is a split injection.*
2. *f is surjective if and only if f is* dominant *(i.e. has dense image), which occurs if and only if $|\operatorname{coker}(\varphi)| < \infty$.*

Proof. We do the first claim.

Suppose φ is a split injection. The

$$0 \to N' \xrightarrow{\varphi} N \to N'/N \to 0$$

splits, and $N'/N \subseteq N$. But N is free; so N/N' is free. So, since φ is injective, we get that $\operatorname{coker}(\varphi)$ is free.

Hence φ is a split injection if and only if φ is injective and has free cokernel, which occurs if and only if f is a closed immersion. □ Corollary 3.16

In understanding maps between toric varieties, our first step is to understand maps between tori; analogously, in understanding fan maps, our first step is to understand maps between lattices. (Hence the above.)

If $\tau \in \Sigma$ then $\overline{O_\tau} = V(\tau)$ is a toric variety, and $O_\tau = \operatorname{Spec}(k[\tau^\perp \cap M]$. What is the lattice for $V(\tau)$? Well, $N(\tau) = (\tau^\perp \cap M)^* = N/N_\tau$ where $N_\tau = \tau \cap N$.

Example 3.17. Consider the fan generated by $\operatorname{Cone}((0,1),(1,1))$ and $\operatorname{Cone}((1,0),(1,1))$. Let $\tau = \operatorname{Cone}((1,0),(1,1))$. Then $\tau^\vee \cap M \cong \mathbb{Z} \times \mathbb{N}$, and $\operatorname{Spec}(k[\tau^\vee \cap M] = \mathbb{A}^1 \times \mathbb{G}_\mathrm{m}$. Further computation yields

$$\begin{aligned} O_\tau &= \operatorname{Spec}(k[\underbrace{\tau^\perp \cap M}_{\mathbb{Z}\langle(1,-1)\rangle}]) \\ &\cong \mathbb{G}_\mathrm{m} \\ N_\tau &= \tau \cap N = \mathbb{Z}\langle(1,1)\rangle \\ N(\tau) &= \mathbb{Z}^2/\mathbb{Z}\langle(1,1)\rangle \end{aligned}$$

and $V(\tau) \cong \mathbb{P}^1$.

Corollary 3.18. *Suppose $\varphi\colon N' \to N$ is a morphism of lattices. Suppose $\operatorname{coker}(\varphi)$ is finite and $\tau' \in \Sigma'$; let $\tau \in \Sigma$ be the smallest cone with $\varphi(\tau') \subseteq \tau$. Then $\varphi_*(O_{\tau'}) = O_\tau$. So $\overline{\varphi_*(V(\tau'))} = V(\tau)$.*

In particular, if φ_* is a proper *map (essentially, φ_* is a closed map), then $\varphi_*(V(\tau')) = V(\tau)$.*

Proof. We already know $\varphi_*(O_{\tau'}) \subseteq O_\tau$; i.e. we have a map of tori $O_{\tau'} \xrightarrow{\varphi_*} O_\tau$. This must then come from a map of lattices $N'(\tau') \to N(\tau)$ such that the following diagram commutes:

$$\begin{tikzcd} N' \arrow[r, "\varphi"] \arrow[d, two heads] & N \arrow[d, two heads] \\ N'(\tau') = N'/N'_{\tau'} \arrow[r, dashed, "\psi"] & N(\tau) \end{tikzcd}$$

Since $|\operatorname{coker}(\varphi)| < \infty$, we then get that $|\operatorname{coker}(\psi)| < \infty$. Hence the map $T_{N'(\tau')} \to T_{N(\tau)}$ is surjective. □ Corollary 3.18

Corollary 3.19. *Suppose $\varphi\colon (\Sigma', N') \to (\Sigma, N)$ is a map of fans; let $f\colon X(\Sigma') \to X(\Sigma)$ be the corresponding map of toric varieties.*

1. *If f is proper and $|\operatorname{coker}(\varphi)| < \infty$ then f is surjective.*
2. *If φ is surjective, then $f^{-1}(T) \cong T \times X(\Sigma'')$ where Σ'' is a fan on $\ker(\varphi)$.*
3. *If φ is surjective and f is proper then $f_*\mathcal{O}_{X(\Sigma')} = \mathcal{O}_{X(\Sigma)}$.*

Proof.

1. Let $\tau' = 0 \in \Sigma'$. Let $\tau \in \Sigma$ be the smallest cone with $\tau \supseteq \varphi(\tau')$; so $\tau = 0$. By the previous corollary, we then get that $f(O_0) = O_0$ and $f(T') = T$). But $'X' = \overline{T'}$ and $X = \overline{T}$; so $f(X) = X$.
2. If φ is surjective then $f^{-1}(T)$ consits of $X(\Sigma'')$ where $\Sigma'' = \{\, \sigma \in \Sigma : \sigma \subseteq \ker(\varphi) \,\}$. Now, we have the following picture on the level of lattices:

$$\begin{tikzcd} N' \arrow[r, "\cong"] \arrow[dr] & N \times \ker(\varphi) \arrow[d, two heads] \\ & N \end{tikzcd}$$

   Hence $f^{-1}(T) \cong T \times X(\Sigma'')$.

3. $f\colon X' \to X$ factors

$$\begin{tikzcd} X' \arrow[rr, "f"] \arrow[d, "g"] & & X \\ \operatorname{Spec}_X(f_*(\mathcal{O}_X)) \arrow[urr, "h"] & & \end{tikzcd}$$

   That f is proper will imply that h is finite. (We call h the *universal affine map*.)

   Editor's note 3.20. I think this is called "Stein factorization".

   Fact 3.21 (Zariski's main theorem)**.** *g has connected fibres.*

   To be continued? ☐ Corollary 3.19

3.1 An aside on one-parameter subgrapes

Recall:

Definition 3.22. A 1-*parameter subgrape* of a torus T is a grape homomorphism $\mathbb{G}_\mathrm{m} \to T$.

Say $T = T_N$ where N is the corresponding lattice. Then $G_\mathrm{m} \to T = \operatorname{Spec}(k[M])$ are maps $M \to \mathbb{Z}$, which are just elements of $M^* = N$.

We say $v \in N$ corresponds to the 1-parameter subgrape $\lambda_v\colon \mathbb{G}_\mathrm{m} \to T$.

Proposition 3.23. *Suppose Σ is a fan on N. A 1-parameter subgrape $\lambda_v\colon \mathbb{G}_\mathrm{m} \to T = T_N$ extends to a map $\mathbb{A}^1 \to X(\Sigma)$ if and only if $v \in |\Sigma|$ (the support of Σ). Moreover, if λ_v extends to $\widetilde{\lambda_v}\colon \mathbb{A}^1 \to X(\Sigma)$ and σ is the smallest cone containing v, then $\widetilde{\lambda_v(0)} = x_\sigma$.*

Proof. We know

$$X(\Sigma) = \bigcup_\sigma U_\sigma$$

Hence λ_v extends if and only if there is $\sigma \in \Sigma$ with $\mathbb{A}^1 \xrightarrow{\widetilde{\lambda_v}} U_\sigma$; i.e. if and only if there is a k-algebra map β such that the following diagram commutes:

$$\begin{tikzcd} k[\sigma^\vee \cap M] \arrow[r, "\beta"] \arrow[d, hook] & k[\mathbb{N}] = k[t] \arrow[d, hook] \\ k[M] \arrow[r, "\alpha \equiv \lambda_v"] & k[\mathbb{Z}] = k[t^\pm] \end{tikzcd}$$

where the map $k[M] \to k[\mathbb{Z}]$ is given by $u \mapsto \langle u, v\rangle$, i.e. $x^u \mapsto t^{\langle u,v\rangle}$. But this occurs if and only if $\alpha(k[\sigma^\vee \cap M]) \subseteq k[t]$; i.e. for all $x^u \in k[\sigma^\vee \cap M]$ we have that $t^{\langle u,v\rangle}$ must be a positive power of t. i.e. for all $u \in \sigma^\vee$ we have $\langle u, v\rangle \geq 0$, i.e. $v \in (\sigma^\vee)^\vee = \sigma$. Hence λ_v extends if and only if $\sigma \in \Sigma$ with $v \in \sigma$.

If σ is the smallest cone in Σ containing v, then $v \in \operatorname{rel\,int}(\sigma)$. Let I be the ideal defined by $\widetilde{\lambda_v}(0)$. If $u \in \sigma^\vee \setminus \sigma^\perp$ then $\langle u, v\rangle > 0$ since $v \in \operatorname{rel\,int}(\sigma)$; hence $x^u \in K$. If on the other hand $u \in \sigma^\perp$ then $\langle u, v\rangle = 0$; so $x^u - 1 = x^u - t^{\langle u,v\rangle} \in I$. So I is generated by x^u for $u \in \sigma^\vee \setminus \sigma^\perp$ and $x^u - 1$ for $u \in \sigma^\perp$. This is the definition of the point x_σ. □ Proposition 3.23

Example 3.24. Let $N = \mathbb{Z}$ and consider Σ generated by the cone of non-negative reals; so $X(\Sigma) = \mathbb{A}^1$. Consider $v = -1$. The induced map is then $\lambda_v \colon \mathbb{G}_\mathrm{m} \twoheadrightarrow \mathbb{G}_\mathrm{m} = \operatorname{Spec}(k[t, t^{-1}])$ given by $1 \mapsto t^{-1}$. This does not extend to $\mathbb{A}^1$ since if it extended to $\widetilde{\lambda_v} \colon \mathbb{A}^1 \to \mathbb{A}^1$ then

$$\widetilde{\lambda_v}(0) = \lim_{t\to 0} \lambda(t) = \lim_{t\to 0} t^{-1} = \infty \notin \mathbb{A}^1$$

The reason we care: the proposition lets us recover the fan Σ just by knowing $X(\Sigma)$.

Example 3.25. Consider $\mathbb{P}^2 \supseteq \mathbb{G}_\mathrm{m}^2 = \{\, (x : y : 1) : x, y \in k^* \,\}$. We then have $(x, y)\cdot(x' : y' : z') = (xx' : yy' : z')$ for $(x, y) \in \mathbb{G}_\mathrm{m}^2$. What are the one-parameter subgrapes? They are $\lambda_{(a,b)} \colon \mathbb{G}_\mathrm{m} \to \mathbb{G}_\mathrm{m}^2 \subseteq \mathbb{P}^2$ given by $t \mapsto (t^a : t^b : 1)$. Does this extend to $\widetilde{\lambda_{(a,b)}} \colon \mathbb{A}^1 \to \mathbb{P}^2$? It always does, since $\mathbb{P}^2$ is projective (and thus has all limit points). What's thae limit? It is

$$\widetilde{\lambda_{(a,b)}}(0) = \lim_{t\to 0} \lambda_{(a,b)}(t)$$

This depends on (a, b); for example, $(a, b) = (0, 0)$ has limit $(1 : 1 : 1)$. If $a, b > 0$ then $(t^a : t^b : 1) \mapsto (0 : 0 : 1)$. If $b < 0$ and $a > b$ then $(t^a : t^b : 1) = (t^{a-b} : 1 : t^{-b}) \mapsto (0 : 1 : 0)$. These form the cones of the original fan.

Proposition 3.26. *Suppose $\varphi\colon (\Sigma', N') \to (\Sigma, N)$ is a map of fans inducing $f\colon X(\Sigma') \to X(\Sigma)$. Then f is finite and surjective if and only if $\varphi\colon N' \hookrightarrow N$ and $|\operatorname{coker}\varphi| < \infty$ (so φ induces an isomorphism $N'_\mathbb{R} \to N_\mathbb{R}$) and $\Sigma' = \Sigma$.*

Example 3.27. Consider

$$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} : \mathbb{Z}^2 \hookrightarrow \mathbb{Z}^2$$

Consider Σ' a fan of $\mathbb{A}^2$ generated by $\operatorname{Cone}((0, 1), (1, 0))$; consider Σ a fan of $\mathbb{A}^2$ generated by $\operatorname{Cone}((1, 0), (1, 2))$. (Note that we can think of the lattice of integer points of Σ as a sublattice of the lattice of integer points of Σ'.) On tori, we have a map $\mathbb{G}_\mathrm{m}^2 \to \mathbb{G}_\mathrm{m}^2$ of degree $2 = |\operatorname{coker}(\varphi)|$. It turns out this map is $\mathbb{A}^2 \to X(\Sigma) = V(xy - z^2)$. The proposition tells us that this is fiinite and surjective; it turns out to be a grape quotient.

When is a toric variety smooth? It's enough to check when its affine pieces are smooth.

Theorem 3.28. *Let Σ be a full-dimensional fan in a lattice N. Then $X(\Sigma)$ is smooth if and only if for all maximal cones $\sigma \in \Sigma$ we have that the first lattice points on the rays of σ are a basis for N.*

Proof. We may assume X is affine; say $X = \operatorname{Spec}(k[P])$. If X is not normal, then X is not smooth and P does not satisfy our criterion for smoothness.

Assume then that P is integral, finitely generated, and saturated.

Claim 3.29. *X is smooth if and only if P is isomorphic to $\mathbb{Z}^r \oplus \mathbb{N}^t$.*

Proof. By homework we have $P \cong P^* \oplus \overline{P}$ where P^* is the units of P and $\overline{P} = P/P^*$. Hence $X \cong X(P^*) \times X(\overline{P}) \cong \mathbb{G}_\mathrm{m}^r \times X(\overline{P})$. Thus X is smooth if and only if $X(\overline{P})$ is smooth; so we can assume P is sharp. So (0) is a face of P, which corresponds to a point of X, fixed by the torus action and defined by $k[P] \mapsto k$ with $x^0 \mapsto 1$ and $x^u \mapsto 0$ for all other $u \in P$. The maximal ideal M associated to this point is the one that's generated by $x^{u_1}, \dots, x^{u_n}$ for $u_1, \dots, u_n$ a generating set of P. Now, $M/M^2 = (x^{u_i+u_j} : i, j \in \{\, 1, \dots, n \,\})$. Hence M/M^2 has basis $x^{u_1}, \dots, x^{u_n}$. So $\dim(M/M^2) = n$, and so the point is smooth if and only if $n = \dim(X) = \operatorname{rank}(N) = t$, which occurs if and only if $\{\, u_1, \dots, u_n \,\}$ is a basis for N. So X is smooth if and only if $\overline{P} \cong \mathbb{N}^t$. □ Claim 3.29

The desired result folows from the claim because P is sharp. (Using the reduction noted in the proof of the claim.) Indeed, since P is saturated, the criterion in the theorem is equivalent to requiring that $P \cong \mathbb{N}^t$.
□ Theorem 3.28

(One can check that a set is a basis for $\mathbb{Z}^n$ if and only if its determinant is ± 1.)

Example 3.30. Consider $\mathbb{P}^2$ with the fan generated by cones between $(1,0)$, $(0,1)$, and $(-1,-1)$. One can use the above criteria to check that $\mathbb{P}^2$ is smooth.

Example 3.31. $\mathbb{A}^n$ is smooth because its associated fan is a single cone spanned by the standard unit basis vectors; this too is smooth.

Example 3.32. Fan generated by cones between $(1,0)$, $(1,1)$, $(0,1)$, and $(-1,-1)$. The associated variety is $\mathbb{P}^2$ blown up at a point; it is smooth.

Example 3.33. Fan generated by cones between $(1,0)$ and $(1,2)$. The associated toric variety is the cone over a smooth quadric in $\mathbb{P}^2$; it is not smooth.

Example 3.34 (Weighted projective spaces). Let $X = \mathbb{P}(d_0, \dots, d_n)$ for $d_j \in \mathbb{Z}_{\geq 1} = \mathbb{P}^n / \mu_{d_0} \times \cdots \times \mu_{d_n}$, where the μ_j are the roots of $x^j = 1$. The action is given by $(g_0, \dots, g_n) \cdot [z_0 : \cdots : z_n] = [g_0 z_0 : \cdots : g_n z_n]$. $\mathbb{P}(d_0, \dots {}_n)$ is a toric variety associated to the fan $\{ \frac{1}{d_i} \overrightarrow{e_i} \}$ where $\overrightarrow{e_i}$ is the i^{th} standard unit basis vector.

We now introduce some analogues of classical topological concepts: *separated* will be the analogue of Hausdorff, and *proper* will be the analogue of compact.

In algebraic geometry, the Zariski topology is not nice (i.e. not Hausdorff). Under some conditions in point-set topology, we get that X is Hausdorff if and only if $\Delta \colon X \to X \times X$ given by $\Delta(x) = (x,x)$ has $\Delta(X) \subseteq X \times X$ is closed. Given $(x,y) \in X$ with $x \neq y$ we want a neighbourhood separating them; that is the same as giving a neighbourhood of (x,y) separating it from $\Delta(X)$.

Definition 3.35. We say a scheme X is *separated* if $\Delta_x \colon X \to X \times X$ is a closed immersion.

Fact 3.36. *Δ_X is always an immersion.*

One reason we care is that if X is a separated scheme, then for all open affine U, V we have $U \cap V \subseteq X$ is also affine.

Example 3.37. Consider $\mathbb{A}^1$ glued to $\mathbb{A}^1$ along $\mathbb{G}_{\mathrm{m}}$; then X looks like the affine line with two points at the origin. This is called the non-separated line.

Example 3.38. We have a similar definition of the non-separated plane. Now $\mathbb{A}^2 \cup \mathbb{A}^2$ is a covering, but $\mathbb{A}^2 \cap \mathbb{A}^2 = \mathbb{A}^2 \setminus \{ 0 \}$ is not affine.

Another way of thinking about separatedness is that any limit that exists is unique.

Example 3.39. In the non-separated line,

$$\lim_{t \to 0} t$$

has two possible values.

Properness can be viewed as “all limits exist and are unique”. (A combination of complete and separated.)

Definition 3.40. A morphism $f \colon X \to Y$ is *proper* if f is separated, of finite type, and universally closed. Universally closed means $f \colon X \to Y$ is closed and for any morphism $Z \to Y$

$$\begin{tikzcd} X \arrow[d, "f"] & X \times_Y Z \arrow[l] \arrow[d] \\ Y & Z \arrow[l] \end{tikzcd}$$

Now f is of finite type if for all $U = \operatorname{Spec}(A) \subseteq Y$ we have

$$f^{-1}(U) = \bigcup_i V_i$$

where $V_i = \operatorname{Spec}(B_i)$ and B_i is a finitely generated A-algebra.

A criterion for being proper is the *valuative criterion.* Essentially, given a small punctured curve $C \setminus \{ c \} \subseteq X$, we want there to be only one way to fill in the hole. What's a small curve in algebraic geometry (over some field k)? We take a look at $\operatorname{Spec}(k[[t]])$ around the origin. As a space, this has two points: (0) and (t). ($k[[t]]$ is a local ring.)

The closed point (t) corresponds to $\operatorname{Spec}(k)$, mapping $k[[t]] \twoheadrightarrow k$ by $t \mapsto 0$. The generic point corresponds to the inclusion $k[[t]] \subseteq k((t))$. The analogue of a small curve is the spectrum of a discrete valuation ring.

Definition 3.41. A *valuation ring* is a ring $R \subseteq k$ (a field) such that for all $\alpha \neq 0 \in k$ we have $\alpha \in k$ or $\alpha^{-1} \in k$. A *discrete valuation ring* is an integral domain R plus a function $v \colon R \twoheadrightarrow \mathbb{Z} \cup \{ \infty \}$ such that

1. $v(x) = \infty$ if and only if $x = 0$.
2. $v(fg) = v(f) + v(g)$.
3. $v(f + g) \geq \min(v(f), v(g))$.

i.e. a valued field with value grape isomorphic to $\mathbb{Z}$.

Given a small curve $\operatorname{Spec}(R)$ and a small punctured curve $\operatorname{Spec}(k) \hookrightarrow \operatorname{Spec}(R)$ and a commuting diagram

$$\begin{tikzcd}
\operatorname{Spec}(R) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & X \\
\operatorname{Spec}(k) \arrow[u, hook] \arrow[ur] &
\end{tikzcd}$$

we can ask whether $f = g$; if this always holds, then X is separated. This corresponds to our intuition of "if the limit exists, then it is unique".

Theorem 3.42 (Valuative criterion for morphisms)**.**

1. *A morphism $f \colon X \to Y$ is separated if and only if for all discrete valuation rings R with $k = \operatorname{Frac}(R)$ and all commutative diagrams*

$$\begin{tikzcd}
\operatorname{Spec}(k) \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\
\operatorname{Spec}(R) \arrow[r] \arrow[ur, shift left, "g"] \arrow[ur, shift right, "h"'] & Y
\end{tikzcd}$$

   we have $g = h$.

2. *A morphism $f \colon X \to Y$ is proper if and only if the above holds and there is a unique arrow*

$$\begin{tikzcd}
\operatorname{Spec}(k) \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\
\operatorname{Spec}(R) \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

We'll show that if $X = X(\Sigma)$, then X is proper if and only $|\Sigma| = N_{\mathbb{R}}$.

Aside 3.43. Consider moduli of genus 1 curves plus a point (elliptic curves). Elliptic curves are classified by the j-invariant, so the space of elliptic curves is $\mathbb{A}^1$, given by the j-invariant. This is not proper (non-compact), so we have $\mathbb{A}^1 \to \mathbb{P}^1$ producing non-smooth genus 1 curves. (Not elliptic anymore.)

The point of last time: $f \colon X \to Y$ is proper if and only if given a discrete valuation ring R with $K = \operatorname{Frac}(R)$ we have a unique map with the following diagram commutes:

$$\begin{tikzcd}
\operatorname{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\operatorname{Spec}(R) \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

This is the *valuative criterion.*

Proposition 3.44. *Suppose $\varphi\colon (\Sigma', N') \to (\Sigma, N)$ is a morphism of fans. Then $\varphi_*\colon X(\Sigma') \to X(\Sigma)$ is proper if and only if $\varphi^{-1}(|\Sigma|) = |\Sigma'|$.*

Corollary 3.45. *$X(\Sigma)$ is proper if and only if $|\Sigma| = N_{\mathbb{R}}$.*

Proof. We have a map from $X(\Sigma)$ to a point; this is induced from the fan map $(\Sigma, N) \to (\{0\}, 0)$. Then $X(\Sigma)$ is proper if and only if $N_{\mathbb{R}} = \varphi^{-1}(0) = |\Sigma|$. $\square$ Corollary 3.45

Example 3.46. If Σ is the $\mathbb{P}^2$ fan as previously

TODO 7. *where*

but with one of the maximal cones missing, then $|\Sigma| \neq \mathbb{R}^2$, and indeed $X(\Sigma)$ is $\mathbb{P}^2$ minus a point, which is not proper.

Recall:

Proposition 3.47. *Given a 1-parameter subgrape $v \in N$ we have that $\lambda_v\colon \mathbb{G}_{\mathrm{m}} \to T \subseteq X(\Sigma)$ extends to $\mathbb{A}^1 \to X(\Sigma)$ if and only if $v \in |\Sigma|$.*

Proof of Proposition 3.44.

($\implies$) Suppose φ_* is proper. We need to show that $\varphi^{-1}(|\Sigma|) = |\Sigma'|$. Pick $v' \in N'$ such that $\varphi(v') \in |\Sigma|$; let $v = \varphi(v')$. So $v \in \sigma$ for some $\sigma \in \Sigma$. Now, since $v \in \sigma$ we have $\widetilde{\lambda_v}\colon \mathbb{A}^1 \to X(\Sigma)$ extending $\lambda_v\colon \mathbb{G}_{\mathrm{m}} \to T \subseteq X(\Sigma)$. Then since φ_* is proper, the valuative criterion yields:

$$\begin{tikzcd} \mathbb{G}_{\mathrm{m}} \arrow[r, "\lambda_{v'}"] \arrow[d] & X(\Sigma') \arrow[d, "\varphi_*"] \\ \mathbb{A}^1 \arrow[r, "\widetilde{\lambda_v}"] \arrow[ur, dashed] & X(\Sigma) \end{tikzcd}$$

So $\mathbb{A}^1 \to X(\Sigma')$ extends $\lambda_{v'}\colon \mathbb{G}_{\mathrm{m}} \to X(\Sigma')$. So, again by the 1-parameter subgrape proposition, we know that $v' \in |\Sigma'|$. Hence $\varphi^{-1}(|\Sigma|) \subseteq |\Sigma'|$.

The other containment holds because φ is a map of fans.

($\impliedby$) Suppose $\varphi^{-1}(|\Sigma|) = |\Sigma'|$; we wish to show that φ_* is proper. We use the valuative criterion: suppose we have a commuting diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "\alpha"] \arrow[d] & X(\Sigma') \arrow[d, "\varphi_*"] \\ \mathrm{Spec}(R) \arrow[r, "\beta"] & X(\Sigma) \end{tikzcd}$$

It turns out it's enough to check the valuative criterion when $\alpha\colon \mathrm{Spec}(K) \to T' \subseteq X(\Sigma')$. Say $\beta(\mathrm{Spec}(R)) \subseteq U_\sigma$ for some $\sigma \in \Sigma$. Our diagram is thus

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "\alpha"] \arrow[d] & T \arrow[d, "\varphi_*"] \\ \mathrm{Spec}(R) \arrow[r, "\beta"] & U_\sigma \end{tikzcd}$$

We want a map $\mathrm{Spec}(R) \to X(\Sigma')$. On the level of rings, we have

$$\begin{tikzcd} K & M' \arrow[l, "\nu"'] \\ R \arrow[u] & \sigma^\vee \cap M \arrow[l] \arrow[u, "\varphi^*"] \end{tikzcd}$$

Now, R is a discrete valuation ring, so there is a valuation $\mathrm{ord}\colon K \to \mathbb{Z} \cup \{\infty\}$ with

$$R = \{\gamma \in K : \mathrm{ord}(\gamma) \geq 0\}$$

Now, $v = \operatorname{ord} \circ \nu \in (M')^* = N'$. Since the above diagram commutes, we get that $\operatorname{ord} \circ v \circ \varphi^*$ factors through R, so it's non-negative on $\sigma^\vee \cap M$; i.e. $\varphi(v) \in \sigma$, and $v \in \varphi^{-1}(\sigma)$. Now, by hypothesis we have $\varphi^{-1}(|\Sigma|) = |\Sigma'|$. We know that there is $\sigma' \in \Sigma'$ with $v \in \sigma'$. So $\nu((\sigma')^\vee \cap M') \subseteq R$; so we have our arrow

$$\begin{tikzcd} K & M' \arrow[l, "v"'] & (\sigma')^\vee \cap M' \arrow[l, "\supseteq"'] \arrow[dll, dashed] \\ R \arrow[u] & & \sigma^\vee \cap M \arrow[ll] \arrow[u] \end{tikzcd}$$

To show uniqueness, one shows that every toric variety $X(\Sigma)$ is separated, and that $X(\Sigma') \to X(\Sigma)$ is separated. □ Proposition 3.44

An objection to the forward direction of the above proof: the valuative criterion is stated for discrete valuation rings, not $\mathbb{A}^1$. To get around this, look at $R = \mathcal{O}_{\mathbb{A}^1,0} = k[x]_{(x)}$, which is a discrete valuation ring; let $k = \operatorname{Frac}(R) = k(x)$, the generic point of $\mathbb{G}_\mathrm{m}$. We now have the following diagram:

$$\begin{tikzcd} \operatorname{Spec}(k) \arrow[r] \arrow[d] & \mathbb{G}_\mathrm{m} \arrow[r] \arrow[d, "\varphi"] & X(\Sigma') \arrow[d] \\ \operatorname{Spec}(R) \arrow[r] \arrow[urr, dashed] & \mathbb{A}^1 \arrow[r] & X(\Sigma) \end{tikzcd}$$

But then $\varphi\colon \operatorname{Spec}(R) \to X(\Sigma')$ extends to a map $U \to X(\Sigma')$ for some open $0 \in U \subseteq \mathbb{A}^1$. Now we have $\mathbb{G}_\mathrm{m} \to X(\Sigma')$ and $0 \in U \to X(\Sigma')$; we can glue these morphisms together to get $\mathbb{A}^1 \to X(\Sigma')$.

Our basic example of a proper map is a blow-up. The topogical picture: $\operatorname{Bl}_{(0,0)} \mathbb{A}^2$ is obtained by replacing the origin with a projective space $\mathbb{P}^1$ such that different lines through the origin no longer cross.

Let $X = X(\Sigma)$ be a smooth toric variety; let $X' = \operatorname{Bl}_Y X$ where $Y \subseteq X$ is a closed T-invariant subvariety. It turns out that X' is again a toric variety. What is its fan? Let $Y = V(\tau)$ where $\tau \in \Sigma$. Now, X is smooth, so we can choose a basis $v_1, \dots, v_n$ of N so that $\tau = \operatorname{Cone}(v_1, \dots, v_r)$. Let

$$v = \sum_{i=1}^{r} v_i$$

If $\sigma \in \Sigma$ has $v \in \sigma$ then replace σ with new cones $\sigma_1, \dots, \sigma_r$ where σ_i is generated by the same rays as σ but replace v_i by v. This yields a new fan Σ'; we set $X' = X(\Sigma')$.

Example 3.48. $\operatorname{Bl}_0 \mathbb{A}^2$. Well, 0 corresponds to τ the maximal 2-dimensional cone (in the fan in $\mathbb{R}^2$ generated by $\operatorname{Cone}((0,1),(1,0))$). Running through the above, we find

$$\begin{aligned} v_1 &= e_1 \\ v_2 &= e_2 \\ v &= e_1 + e_2 \end{aligned}$$

Going through all $\sigma \supseteq \tau$, which in this case is just $\sigma = \tau$, we replace τ by τ_1 and τ_2 where

$$\begin{aligned} \tau_1 &= \operatorname{Cone}(v, v_2) \\ \tau_2 &= \operatorname{Cone}(v_1, v) \end{aligned}$$

Example 3.49. Consider blowing up a line in $\mathbb{A}^3$. Choose the natural basis v_1, v_2, v_3; consider $\tau = \operatorname{Cone}(v_1, v_2)$. Let $v = v_1 + v_2$. If σ is the maximal cone then we end up with

$$\begin{aligned} \sigma_1 &= \operatorname{Cone}(e_2, e_1 + e_2, e_3) \\ \sigma_2 &= \operatorname{Cone}(e_1, e_1 + e_2, e_3) \end{aligned}$$

Why is $X' = X(\Sigma')$? Well, $\operatorname{Bl}_Y X = X$ on $X \setminus Y$. So far, our fan is correct, meaning the only change comes from $\sigma \supseteq \tau$.

Let's just look at maximal cones $\sigma \supseteq \tau$. Since σ is smooth, we get that its rays form a basis for N; so $U_\sigma \cong \mathbb{A}^n = \operatorname{Spec}(k[t_1, \dots, t_n])$. Write $\tau = \operatorname{Cone}(v_1, \dots, v_r)$; then

$$\sigma_i^\vee = \{ (u_1, \dots, u_n) : u_j \geq 0 \text{ for all } j \neq i, u_1 + \cdots + u_r \geq 0 \}$$

We've replaced the condition $u_{\geq} 0$ in $\sigma^\vee$ by

$$\sum_{\ell=1}^{r} u_\ell \geq 0$$

in $\sigma_i^\vee$.

Example 3.50. Consider again the blowup of the origin in $\mathbb{A}^2$. Then $\sigma_1 = \operatorname{Cone}((1,0),(-1,1))$. On coordinate rings, we see $k[\sigma_1^\vee \cap \mathbb{Z}^2] = k\left[t, \frac{t_2}{t_1}\right]$.

In general:

$$k[\sigma_i^\vee \cap \mathbb{Z}^n] = k\left[\frac{t_1}{t_i}, \dots, \frac{t_r}{t_i}, t_i, t_{r+1}, \dots, t_n\right]$$

This is the usual affine patch of the blowup. (Recall that

$$\operatorname{Bl}_{(t_1,\dots,t_r)} \mathbb{A}^n = \operatorname{Proj} \frac{k[t_1, \dots, t_n][Y_1, \dots, Y_r]}{(t_i Y_j - t_j Y_i : 1 \leq, i, j \leq r}$$

which are covered by the affine patches $Y_i \neq 0$. On the patch when $Y_i \neq 0$, we note that $t_i Y_j = t_j Y_i$ implies $t_i \frac{Y_j}{Y_i} = t_j$ for $1 \leq j \leq r$; hence the $t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_r$ are irrelevant variables.)

If X is singular, what's the fan Σ' of $\operatorname{Bl}_Y X$? It's not always just inserting a ray.

Lemma 3.51. *If $\varphi^{-1}(|\Sigma|) = |\Sigma|$ (i.e. φ_* is proper) then for all $\sigma \in \Sigma$ we have*

$$\varphi^{-1}(\sigma) = \bigcup \tau_i$$

where the τ_i are some subset of Σ'.

Proof. Suppose $v \in \varphi^{-1}(\sigma)$. Then $v \in |\Sigma'|$, so there is $\tau' \in \Sigma'$ such that $v \in \tau'$. We need $\tau' \in \Sigma'$ with $v \in \tau'$ and $\tau' \subseteq \varphi^{-1}(\sigma)$; let τ' be the smallest $\tau' \in \Sigma'$ such that $v \in \tau'$. Now let τ be the smallest cone in σ such that $\varphi(\tau') \subseteq \tau$; then $\varphi(v) \in \operatorname{rel\,int}(\tau)$. But $v \in \sigma$. So $\tau \subseteq \sigma$. Hence $\tau' \subseteq \varphi^{-1}(\tau) \subseteq \varphi^{-1}(\sigma)$. ☐ Lemma 3.51

Definition 3.52. We say $f \colon X \to Y$ is *birational* if there is $U \subseteq X$ and $V \subseteq Y$ open such that $f \restriction U \colon U \to V$ is an isomorphism.

Blow-ups are examples of these.

The lemma implies that φ_* is proper and birational if and only if $\varphi \colon N' \to N$ is an isomorphism and all Σ-cones are unions of Σ' cones. (For this we need that every birational map of toric varieties is an isomorphism on the tori.) In this case we say that Σ' is a *refinement* of Σ; i.e. same lattice and replace some cones σ by new ones.

Definition 3.53. We say a cone σ is *simplicial* if the rays of σ form a basis for $(N_\sigma)_\mathbb{R}$. We say Σ or $X(\Sigma)$ is simplicial if all $\sigma \in \Sigma$ are simplicial.

Example 3.54. A cone over a non-triangular polygon is not simplicial.

We have seen that $X(\Sigma)$ is smooth if and only if for all maximal $\sigma \in \Sigma$ we have that the rays of σ are a basis for N. So simplicial is "rationally smooth".

We'll later see that every toric variety is a global quotient $X(\Sigma) = U/G$ where $U \subseteq \mathbb{A}^n$ is open and $G = T \times A$ for some finite abelian grape A. The simplicial $X(\Sigma)$ are those where each affine piece U_σ is a quotient $\mathbb{A}^n/G$ where G is a finite grape.

Recall that X is a *toric variety* if X is a separated normal scheme with a dense open torus $T \subseteq X$ such that the action of T on T extends to an action on X.

Theorem 3.55 (Characterization of toric varieties)**.** *There is an equivalence of categories between fans and toric varieties given by $\Sigma \mapsto X(\Sigma)$.*

Proof.

Essential surjectivity Suppose X is a toric variety; we must show there is Σ such that $X = X(\Sigma)$.

> **Fact 3.56** (Sumihiro's theorem)**.** *If X is a normal variety and we have an action of T on X (where T is a torus), then we can write $X = \bigcup U_i$ with the U_i affine, open, and T-invariant.*

Apply this to X to get U_i as above. Now, since X is irreducible, the U_i are open, and $T \subseteq X$ is dense, we then get that $T \cap U_i \neq \emptyset$. We also know that U_i is T-invariant; hence $T \subseteq U_i$.

Let N be the lattice of 1-parameter subgrapes of T; let $M = N^*$. By our characterization theorem for affine toric varieties, we get that $U_i = \operatorname{Spec}(k[\sigma_i^m v \cap M])$ where the σ_i are rational polyhedral cones on N. But X is separated; so $U_i \cap U_j$ is also affine and T-invariant, and we may write $U_i \cap U_j = \operatorname{Spec}(k[\tau_{ij}^\vee \cap M])$. Since $U_i \cap U_j$ is contained in both U_i and U_j, we get that $\tau_{ij} \subseteq \sigma_i \cap \sigma_j$. For $v \in \sigma_i \cap \sigma_j$, let $\lambda_v \colon \mathbb{G}_\mathrm{m} \to T$ be the corresponding 1-parameter subgrape. We know it extends to $\lambda \colon \mathbb{A}^1 \to U_i \subseteq X$ since $v \in \sigma_i$; likewise it extends to $\lambda' \colon \mathbb{A}^1 \to U_j \subseteq X$. By the valuative criterion, these two extensions must be equal on $\mathbb{A}^1$; ones uses the following diagram:

$$\begin{tikzcd}
\mathbb{G}_\mathrm{m} \arrow[r, "\lambda_v"] \arrow[d, hook] & X \arrow[d] \\
\mathbb{A}^1 \arrow[r] \arrow[ur, shift left, "\lambda"] \arrow[ur, shift right, "\lambda'"'] & \text{point}
\end{tikzcd}$$

Now $U_i \supseteq \lambda(\mathbb{A}^1) = \lambda'(\mathbb{A}^1) \subseteq U_j$; so $\lambda = \lambda' \colon \mathbb{A}^1 \to U_i \cap U_j$ is an extension of $\lambda_v \colon \mathbb{G}_\mathrm{m} \to T$. So $v \in \tau_{ij}$, and $\sigma_i \cap \sigma_j \subseteq \tau_{ij}$; so $\sigma_i \cap \sigma_j \subseteq \tau_{ij}$.

Finally, we must check that τ_{ij} is a face of σ_i and σ_j. We then take Σ to be the set of σ corresponding to affine open T-invariant $U \subseteq X$, which is then a fan.

Full faithfulness If $f \colon X(\Sigma') \to X(\Sigma)$ then there is a unique $\varphi \colon \Sigma' \to \Sigma$ inducing f; since $f \restriction T' \colon T' \to T$ is a grape homomorphism we get that $\varphi \colon N' \to N$. This is our unique morphism provided φ sends cones into cones. Let $\sigma' \in \Sigma'$; then there is $\sigma \in \Sigma$ such that $f(O_{\sigma'}) \subseteq O_\sigma$ since we can choose the identity of $O_{\sigma'}$, and then f is equivariant, so $O_\sigma = T \cdot f(1_{O_{\sigma'}})$. We wish to show that $\varphi(\sigma') \subseteq \sigma$.

Suppose $v' \in \sigma'$. We get that the following diagram commutes:

$$\begin{tikzcd}
 & \mathbb{A}^1 \arrow[r] & U_{\sigma'} \arrow[d, "\subseteq"] \\
\mathbb{G}_\mathrm{m} \arrow[ur, hook] \arrow[r, "\lambda_{v'}"] \arrow[dr, "\lambda_{\varphi(v')}"] & T' \arrow[ur, "\subseteq"] \arrow[d, "f \restriction T'"] & X(\Sigma') \arrow[d, "f"] \\
 & T \arrow[r, "\subseteq"] & X(\Sigma)
\end{tikzcd}$$

This diagram shows that $\lambda_{\varphi(v')}$ does enxtend to $\mathbb{A}^1 \to X(\Sigma)$. So there is some $\sigma_{v'} \in \Sigma$ such that $\varphi(v') \in \sigma_{v'}$.

We need to show that $\sigma_{v'}$ is independent of v', and in fact is just σ; we will then have $\varphi(\sigma') \subseteq \sigma$. But the following diagram commutes:

$$\begin{tikzcd}
\mathbb{A}^1 \arrow[r] \arrow[dr] & X' \arrow[d, "f"] \\
 & X
\end{tikzcd}$$

Following $0 \in \mathbb{A}^1$, we find that $f(x_{\sigma'}) = x_{\sigma'_v}$; so $\sigma_{v'} = \sigma$. $\square$ Theorem 3.55

We will see that whereas fans correspond to toric varieties, polytopes correspond to projective toric varieties.

4 Polytopes

Let P be a rational polytope on M; so P is the convex hull of some $u_1, \ldots, u_n \in M$, with $\dim(P) = \operatorname{rank}(M)$. If $Q \subseteq P$ is a face, we let $\sigma_Q = \{ v \in N_{\mathbb{R}} : \langle q, v\rangle \leq \langle p, v\rangle \text{ for all } q \in Q, p \in P \}$.

Example 4.1. Let P be the convex hull of $\{ 0, (0,1), (1,0) \}$; let $Q = \{ (0,1) \}$. Then $\sigma_Q = \{ v \in N_{\mathbb{R}} : \langle (1,0), v\rangle \leq \langle p, v\rangle \text{ for all } p \in P \}$. Squinting a bit, we find that this is just the dual cone of $P - (1,0)$; i.e. we have recentred the polytope so that $(1,0)$ is the origin.

Hence $\sigma_{(1,0)} = \operatorname{Cone}((0,0) - (1,0), (0,1) - (1,0))^\vee$. Piecing together $\sigma_{(1,0)}$, $\sigma_{(0,1)}$, and $\sigma_{(0,0)}$, we get the maximal cones of the fan associated with $\mathbb{P}^2$.

What of $Q = \operatorname{Conv}((0,1), (1,0))$? Well, $v \in \sigma_Q$ if and only if $\langle q, v\rangle \leq \langle p, v\rangle$ for all $q \in Q$ and $p \in P$. But $q \in Q$ takes the form $q = \lambda e_1 + (1-\lambda) e_2$; if we stare at this for a bit, we see that $\sigma_Q = \sigma_{(1,0)} \cap \sigma_{(0,1)}$. Filling in the rest of the σ_Q, we get the rest of the fan associated with $\mathbb{P}^2$.

Proposition 4.2. *If we let*

$$\Sigma_P = \{ \sigma_Q : Q \leq P \text{ is a face} \}$$

then Σ_P is a fan, called the normal fan *of P. Furthermore, $\dim(\sigma_Q) = \operatorname{codim}(Q)$ and $|\Sigma_P| = N_{\mathbb{R}}$.*

In particular, $X(\Sigma_P)$ is proper. (We'll later show that $X(\Sigma)$ is projective if and only if $\Sigma = \Sigma_P$ for some P.)

Proof. We can translate to assume $0 \in \operatorname{int}(P)$. If $Q = P$ then $v \in \sigma_Q$ implies $\langle q, v\rangle \leq \langle p, v\rangle$ for all $q \in P$ and $p \in P$. Choose $p = 0$: so $\langle q, v\rangle \leq 0$ for all $q \in P$, and $v = 0$.

Now suppose $Q \subsetneq P$ is a face. Recall that $P^\circ = \{ v : \langle uv\rangle \geq -1 \text{ for all } u \in P \}$. Let $Q^* = \{ v \in P^\circ : \langle u, v\rangle = -1 \text{ for all } u \in Q \}$. Then $Q^{\subseteq}\sigma_Q$ since for all $p \in P$ and $q \in Q$ and for all $v \in Q^*$ we have $\langle q, v\rangle = -1$. Now, $\langle p, v\rangle \geq -1$ by definition of P°; so $\langle p, v\rangle \geq \langle q, v\rangle$, and $v \in \sigma_Q$.

Let's show that $\sigma_Q = \operatorname{Cone}(Q^*)$. Suppose $v \in \sigma_Q \setminus \{ 0 \}$, so $\langle q, v\rangle \leq \langle p, v\rangle$ for all $q \in Q$ and $p \in P$. But now the map $q \mapsto \langle q, v\rangle$ is constant, since if $q, q' \in Q$ have $\langle q, v\rangle > \langle q', v\rangle$ then we can let $p = q' \in Q \leq P$; this then violates $\langle q, v\rangle \leq \langle p, v\rangle$. Say $\langle q, v\rangle = c$ for all $q \in Q$. Then $0 \in P$, so $c = \langle q, v\rangle \leq \langle 0, v\rangle = 0$. So $-\frac{1}{c} v \in Q^*$ since $c = \langle q, v\rangle$ implies $\langle q, c^{-1} v\rangle = -1$. We also know that $\langle q, v\rangle \leq \langle p, v\rangle$; so $-1 = \langle q, c^{-1}v\rangle \leq \langle p, c^{-1}v\rangle$ (since $-c^{-1} > 0$), and $-\frac{1}{c} v \in Q^*$.

The point is that if $0 \neq v \in \sigma_Q$, we found a positive $-c^{-1}$ such that $-c^{-1}v \in Q^*$; so $\sigma_Q = \operatorname{Cone}(Q^*)$. At this point the result is clear. ☐ Proposition 4.2

Definition 4.3. Suppose $Z \subseteq X$ is a subvariety. Let I be the ideal of Z; so

$$\bigoplus_{i=0}^{\infty} I^i$$

is a graded $k[\sigma^\vee \cap M]$-algebra; we then set

$$\operatorname{Bl}_Z X = \operatorname{Proj} \bigoplus_{i=0}^{\infty} I^i$$

A construction: take a cone σ on N, dualize to get the dual cone in M for coordinates on the toric variety, compute the blow-up by hand to get a new variety, and dualize again to get a Σ for the new variety.

We can generalize this: suppose σ is a pointed cone in N; suppose $Z \subseteq U_\sigma$ is closed, irreducible, and T-invariant. Let $\mathcal{I} \subseteq k[\sigma^\vee \cap M]$ be the corresponding ideal; say $\mathcal{I} = (x^{u_1}, \ldots, x^{u_r})$. Let $P = \operatorname{Conv}(u_1, \ldots, u_r) + \sigma^\vee$; this is called the *Newton polyhedron* of $\mathcal{I}$. Let Σ_P be the normal fan.

Proposition 4.4. *$X(\Sigma_P) \to U_\sigma$ is the normalization of the blowup $\widetilde{\operatorname{Bl}_Z(U_\sigma)}$.*

Proof. Since $P \subseteq \sigma^\vee$, it's not hard to see that Σ_p is a refinement of σ. We thus get a proper birational map $X(\Sigma_P) \to X$. Let $Y = \widetilde{\operatorname{Bl}_Z(X)}$ be the normalization of the blowup; so we have a proper birational map $Y \to X$.

From th definition, we have an action of T on $\mathrm{Bl}_Z X$. We also have an action of T on Yk since normalization is functorial. Also, $\mathrm{Bl}_Z X \to X$ is an isomorphism over T; so $Y \to X$ is a map of toric varieties. If $Y = X(\Sigma')$, then we have proper birational maps

$$\begin{tikzcd} X(\Sigma_p) \arrow[dr] & & X(\Sigma') \arrow[dl] \\ & X & \end{tikzcd}$$

So Σ' is a refinement of σ.

Now, blow-ups are universal with respect to $Y \xrightarrow{g} X$ proper, birational, and $g^{-1}(I)$ locally principal. The maximal cones of $X(\Sigma_P)$ correspond to the vertices of P. Look at the patch corresponding to the vertex $u_i \in P$. Let τ be the corresponding cone; so $u - u_i \in \tau^\vee$ for all $x^u \in I$ (since τ is the dual cone of $P - u_i$). So $x^u \in I$ if and only if $u \in P$. So $u - u_i \in P - u_i$. So $u - u_i \in \tau^\vee$.

As a result, we get that $x^{u-u_i} \in k[\tau^\vee \cap M]$; i.e. $\frac{x^u}{x^{u_i}}$ is a regular function on the patch corresponding to vertex u_i. So $I = (x^{u_i})$.

We've thus covered X by U_i such that $I \restriction U_i$ is principal; so, by the universal property, we get a map

$$\begin{tikzcd} X(\Sigma_P) \arrow[rr] \arrow[dr] & & Y = X(\Sigma') \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

We lastly need that the map $X(\Sigma_P) \to Y$ is an isomorphism. But $g^{-1}(I)$ is locally principal; so for all $\tau \in \Sigma'$ maximal we have $g^{-1}(I) \restriction U_\tau$ principal. Hence one of the u_i generates $g^{-1}(I) \restriction U_\tau/$ Sp, for all $x^u \in I$, we have that x^u is some multiple of x^{u_i}. In fact we must have $x^u = x^{u-u_i}x^{u_i}$; i.e. x^{u-u_i} is a regular function, and $u - u_i \in k[\tau^\vee \cap M]$ for all $u \in P$. So $P - u_i \subseteq \tau^\vee$, and $P - u_i = \tau^\vee$. So our two refinements Σ_P and Σ' are the same. $\square$ Proposition 4.4

Example 4.5. Let $\sigma = \mathrm{Cone}((0,1),(1,0))$, and consider $\mathrm{Bl}_0 \mathbb{A}^2$ with $\mathcal{I} = (x,y) \subseteq k[x,y]$. So $P = \mathrm{Conv}(e_1,e_2) + \sigma^\vee$.

Example 4.6. Let $\sigma^\vee = \mathrm{Cone}((1,0),(1,3))$ in M; we call this the A_3-singularity. Then $\sigma = \mathrm{Cone}((0,1),(3,-1))$ in N. The ideal of the singular point is given by $(1,0)$, $(1,1)$, $(1,2)$, and $(1,3)$. Taking the convex hull and adding $\sigma^\vee$ we get Σ_p, and then the blowup at the singular point of X.

4.1 Resolution of toric surfaces

Suppose X is an affine toric surface.

Fact 4.7 (Proven later). *σ can be written as* $\mathrm{Cone}((0,1),(m,-\ell))$ *where $m \geq 0$ and $0 \leq \ell < m$. (This comes from writing $X = \mathbb{A}^2/(\mathbb{Z}/n\mathbb{Z})$.) In fact σ is smooth if and only if $\ell = 0$ and $m = 1$; this is because*

$$\begin{vmatrix} 0 & m \\ 1 & -\ell \end{vmatrix} = m$$

The first step in our resolution is to add the ray e_1. We get a smooth cone and $\sigma' = \mathrm{Cone}(e_1,(m,-\ell))$. Let's put σ' in "standard form" by change of basis. We want $e_1 \mapsto e_2$ and we want

$$\begin{pmatrix} 0 & a \\ 1 & b \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$$

so $a = \pm 1$. Also

$$\begin{pmatrix} 0 & a \\ 1 & b \end{pmatrix} \begin{pmatrix} m \\ -\ell \end{pmatrix} = \begin{pmatrix} -a\ell \\ m - b\ell \end{pmatrix}$$

We require that $m' = -a\ell \geq 0$; so $a = -1$ and $m = \ell$. We also require that $0 \leq \ell' = b\ell - m < m'$; so $0 \leq b\ell - m < \ell$, and

$$\frac{m}{\ell} \leq b < 1 + \frac{m}{\ell}$$

and

$$b = \left\lceil \frac{m}{\ell} \right\rceil$$

is uniquely determined. Now

$$\begin{aligned}
\frac{m}{\ell} - b &= \frac{m - b\ell}{\ell} \\
&= -\frac{b\ell - m}{\ell} \\
&= -\frac{1}{\frac{\ell}{b\ell - m}} \\
&= -\frac{1}{\frac{m'}{\ell'}}
\end{aligned}$$

Hence

$$\frac{m}{\ell} = b - \frac{1}{\frac{m'}{\ell'}}$$

This is continued fractions using ceilings instead of floors; this is called Hirzebruch-Jung continued fractions.

Given a toric surface singularity, we can put it in standard form, take the Hirzebruch-Jung continued fraction; the b_i we get will be exactly the new rays we need to insert. If the Hirzebruch-Jung continued fraction has r steps, then the toric surface is resolved after r steps.

How to find m and ℓ? Express $X = \mathbb{A}^2/(\mathbb{Z}/m\mathbb{Z})$. Given an action $\mathbb{Z}/m\mathbb{Z}$ on $\mathbb{A}^2$, it must be

$$\zeta_m(x, y) = (\zeta_m^a x, \zeta_m^b y)$$

where $\mathbb{Z}/m\mathbb{Z} = \langle \zeta_m \rangle$. We may assume $a = 1$. The action is then completely described by b; so $\ell = b$.