Course notes for PMATH 945

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1 Preliminaries

Can collaborate with classmates on homework problems, and can looks things up on the internet. *Not* permitted to ask profs or *post* questions on the internet.

Classes vs. sets: classes are sets or proper classes. Any reasonably defined collection of objects should form a class.

2 Category theory

2.1 Categories

Definition 2.1. A *category* $\mathcal{C}$ has two parts:

- $\mathrm{Ob}(\mathcal{C})$, a class of objects
- for each $A, B \in \mathrm{Ob}(\mathcal{C})$ a *set* of morphisms $\hom_{\mathcal{C}}(A, B)$.

We also require a composition law $\circ\colon \hom_{\mathcal{C}}(B,C) \times \hom_{\mathcal{C}}(A,B) \to \hom_{\mathcal{C}}(A,C)$ for all $A, B, C \in \mathrm{Ob}(\mathcal{C})$ such

- Composition is associative, when defined: $f \circ (g \circ h) = (f \circ g) \circ h$.
- For all $A \in \mathrm{Ob}(\mathcal{C})$ there is $\mathrm{id}_A \in \hom_{\mathcal{C}}(A, A)$ such that $\mathrm{id}_A \circ f = f$ and $g \circ \mathrm{id}_A = g$ when defined.

Example 2.2.

1. **Grp**, the category of all grapes: $\mathrm{Ob}(\mathbf{Grp})$ is the class of all groups and $\hom_{\mathbf{Grp}}(G, H)$ the set of grape homomorphisms $G \to H$. Notice we have composition and $\mathrm{id}_G\colon G \to G$.
2. **Set**, the category of all sets: $\mathrm{Ob}(\mathbf{Set})$ is the class of all sets and $\hom_{\mathbf{Set}}(X, Y)$ is the set of functions $X \to Y$.
3. **Top**, the category of topological spaces: $\mathrm{Ob}(\mathbf{Top})$ is the class of all topological spaces and $\hom_{\mathbf{Top}}(X, Y)$ is the set of continuous maps $X \to Y$.
4. **Ab**, the category of abelian grapes.
5. $\mathbf{Top}^*$, the category of pointed topological spaces (topological spaces with an identified point); morphisms will be continuous maps sending the identified point of the domain to the identified point of the codomain.

An important example for sheaves:

Example 2.3. Suppose X is a topological space. We define the category $\mathbf{Top}_X$ by

- $\mathrm{Ob}(\mathbf{Top}_X)$ is the set of open subsets of X
- If U, V are open subsets of X, then we set

$$\hom_{\mathbf{Top}_X}(U, V) = \begin{cases} \emptyset & U \not\subseteq V \\ \{\, i\colon U \to V \,\} & \text{else} \end{cases}$$

Why are we interested in category theory? Categories can provide a unification tool.

2.2 Functors

Definition 2.4. Suppose $\mathcal{C}$ and $\mathcal{D}$ are categories. A *functor* $F\colon \mathcal{C} \to \mathcal{D}$ consists of

- $F\colon\ \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$
- $F\colon\ \hom_{\mathcal{C}}(A, B) \to \hom_{\mathcal{D}}(F(A), F(B))$ for any $A, B \in \mathrm{Ob}(\mathcal{C})$

such that

- $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ for all $A \in \mathrm{Ob}(\mathcal{C})$
- $F(f \circ g) = F(f) \circ F(g)$

Example 2.5.

1. $F\colon \mathbf{Ab} \to \mathbf{Grp}$ given by $F(A) = A$ and $F(f) = f$.

2. $T\colon \mathbf{Grp} \to \mathbf{Ab}$ by $T(G) = G/G'$ (where G' is the commutator subgrape of G) and if $f\colon G \to H$ then $T(f)\colon G/G' \to H/H'$ is given by $T(f)(gG') = f(g)H'$.

3. $\pi_1\colon \mathbf{Top}^* \to \mathbf{Grp}$ that sends a pointed topological space to its fundamental grape; i.e. the grape of loops based at the identified point modulo homotopy equivalence. (Recall that h_0 is homotopic to h_1 if there are h_t for all $t \in (0,1)$ such that the map $[0,1]^2 \to X$ given by $(x,t) \to h_t(x)$ is continuous.) Given $f\colon (X,x_0) \to (Y,y_0)$, we define $\pi_1(f)\colon \pi_1(X,x_0) \to \pi_1(Y,y_0)$ by $\pi_1(f)(g) = f \circ g\colon [0,1] \to Y$.

   Apparently the composition $T \circ \pi_1$ is the first homology grape of a path-connected topological space.

4. The *forgetful functor* $F\colon \mathbf{Grp} \to \mathbf{Set}$.

2.3 Natural transformations

Definition 2.6. Suppose $\mathcal{C}$ and $\mathcal{D}$ are categories; suppose $F, G\colon \mathcal{C} \to \mathcal{D}$ are functors. A *natural transformation* $\alpha\colon F \to G$ consists of a morphism $\alpha_A\colon F(A) \to G(A)$ (i.e. $\alpha_A \in \hom_{\mathcal{D}}(F(A), G(A))$) for all $A \in \mathrm{Ob}(\mathcal{C})$ such that for all $f\colon A \to B$ (where $A, B \in \mathrm{Ob}(\mathcal{C})$), we have that the following diagram commutes:

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"] & F(B) \arrow[d, "\alpha_B"] \\ G(A) \arrow[r, "G(f)"] & G(B) \end{tikzcd}$$

Definition 2.7. If there are natural transformations $\alpha\colon F \to G$ and $\beta\colon G \to F$ such that $\alpha \circ \beta\colon G \to G$ and $\beta \circ \alpha\colon F \to F$ are the respective identity maps, then we say the functors F and G are *isomorphic*.

Example 2.8. Suppose $F\colon \mathcal{C} \to \mathcal{D}$ is a functor. Then $\alpha = \mathrm{id}\colon F \to F$ given by $\alpha_A = \mathrm{id}_A\colon F(A) \to F(A)$

Definition 2.9. Functors $F, G\colon \mathcal{C} \to \mathcal{D}$ are *isomorphic* if there is $\alpha\colon F \to G$ and $\beta\colon G \to F$ such that $\beta \circ \alpha = \mathrm{id}\colon F \to F$ and $\alpha \circ \beta = \mathrm{id}\colon G \to G$.

Example 2.10 (Double duals). Let $\mathcal{C}$ be the category of finite-dimensional vector spaces over $\mathbb{C}$. We define $F\colon \mathcal{C} \to \mathcal{C}$ to be the identity functor; i.e. $F(V) = V$ for $V \in \mathrm{Ob}(\mathcal{C})$ and $F(T) = T$ for $T\colon V \to W$. We define $G\colon \mathcal{C} \to \mathcal{C}$ by $G(V) = V^{**}$ and for $T\colon V \to W$ we let $G(T)\colon V^{**} \to W^{**}$ be $G(T) = T^{**}$. We define a natural transformation $\alpha\colon F \to G$ by $\alpha_V\colon V \to V^{**}$ is $\alpha_V(\overrightarrow{v}) = e_{\overrightarrow{v}}$ (where $e_{\overrightarrow{v}} \in V^{**} = \hom_{\mathbb{C}}(V^*, \mathbb{C})$ is $e_{\overrightarrow{v}}(f) = f(\overrightarrow{v})$ for $f \in V^*$).

Then for $T\colon V \to W$ we have the following diagram commutes:

$$\begin{tikzcd} F(V) \arrow[r, "\alpha_V"] \arrow[d, "F(T)"] & G(V) \arrow[d, "G(T)"] \\ F(W) \arrow[r, "\alpha_W"] & G(W) \end{tikzcd}$$

So $\alpha\colon F \to G$ is indeed a natural transformation.

2.4 Opposite category

Definition 2.11. Suppose $\mathcal{C}$ is a category. We define the *opposite category* $\mathcal{C}^{\mathrm{op}}$ by $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C})$ and for $A, B \in \mathrm{Ob}(\mathcal{C})$ we let $\hom_{\mathcal{C}^{\mathrm{op}}}(A, B) = \hom_{\mathcal{C}}(B, A)$; composition is then given by $\widetilde{f} \circ \widetilde{g} = \widetilde{g \circ f}$ for $\widetilde{f} \in \hom_{\mathcal{C}^{\mathrm{op}}}(B, A)$ and $\widetilde{g} \in \hom_{\mathcal{C}^{\mathrm{op}}}(C, B)$ (i.e. $f \in \hom_{\mathcal{C}}(A, B)$ and $g \in \hom_{\mathcal{C}}(B, C)$). The identity morphisms are then the same.

Example 2.12. If $\mathcal{C}$ is the category of finite-dimensional vector spaces over $\mathbb{C}$ then $F\colon \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ given by $F(V) = V^*$ and $F(T) = T^*\colon W^* \to V^*$ for $T\colon V \to W$ is a functor. Also $G\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ given by $G(V) = V^*$ and $G(T) = T^*\colon W^* \to V^*$ for $T\colon V \to W$ is also a functor. Then $G \circ F\colon \mathcal{C} \to \mathcal{C}$ sends $V \mapsto V^{**}$ and $T \mapsto T^{**}\colon V^{**} \to W^{**}$ for $T\colon V \to W$. Likewise $F \circ G\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ sends $V \mapsto V^{**}$.

Exercise 2.13. Show that $G \circ F$ is naturally isomorphic to the identity functor $\mathcal{C} \to \mathcal{C}$; i.e. there are natural transformations $\alpha\colon G \circ F \to \mathrm{id}$ and $\beta\, \mathrm{id} \to G \circ F$ such that $\beta \circ \alpha = \mathrm{id}\colon G \circ F \to G \circ F$ and $\alpha \circ \beta = \mathrm{id}\colon F \circ G \to F \circ G$.

Definition 2.14. Suppose $\mathcal{C}$ and $\mathcal{D}$ are categories and $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ are functors such that $F \circ G\colon \mathcal{D} \to \mathcal{D}$ and $G \circ F\colon \mathcal{C} \to \mathcal{C}$ are isomorphic to the respective identity functors. Then we say $\mathcal{C} \cong \mathcal{D}$ are *equivalent*.

Example 2.15. If $\mathcal{C}$ is the category of finite-dimensional vector spaces over $\mathbb{C}$, then $\mathcal{C} \cong \mathcal{C}^{\mathrm{op}}$.

Example 2.16 (Algebraic geometry).

Definition 2.17. Let k be a field. A *k-algebra* B is a commutative ring with an injective homomorphism $\varphi\colon k \to B$ such that $\varphi(1_k) = 1_B$.

Remark 2.18. Then $B \supseteq \varphi(k) \cong k$; so B is a vector space over k.

Example 2.19. $B = \mathbb{C}[x, y]$ is a $\mathbb{C}$-algebra with $\varphi\colon \mathbb{C} \to B$ given by $\varphi(\lambda) = \lambda$.

Definition 2.20. B is *finitely generated as a k-algebra* if there are $a_1, \dots, a_d \in B$ such that every $b \in B$ can be written as a polynomial $p(a_1, \dots, a_d)$ for some $p \in k[x_1, \dots, x_d]$. B is *reduced* if whenever $b \in B$ satisfies $b^n = 0$ for some $n \geq 1$ we have $b = 0$.

Example 2.21. $\mathbb{C}[x]/(x)$ is not reduced; $\mathbb{C}[x_1, x_2, x_3, \dots]$ is not finitely generated.

We can then form the category $\mathcal{C}$ of finitely generated, reduced $\mathbb{C}$-algebras. We can also form the category $\mathcal{D}$ of complex affine varieties, whose objects are $Y \subseteq \mathbb{C}^n$ for some $n \geq 1$ such that Y is the zero set of a finite set of polynomials $p_1(x_1, \dots, x_n), \dots, p_d(x_1, \dots, x_n)$. (Note that we don't require irreducibility here.)

Example 2.22. $Y = \{\, (a, b) \in \mathbb{C}^2 : b^2 = a^3 + 1 \,\} \subseteq \mathbb{C}^2$ is the zero set of $x_2^2 - x_1^3 - 1$.

Then algebraic geometry tells us that $\mathcal{C} \cong \mathcal{D}^{\mathrm{op}}$. The nullstellensatz gives us that for $B \in \mathcal{C}$, say $B \cong \mathbb{C}[x_1, \dots, x_n]/(p_1(x_1, \dots, x_n), \dots, p_d(x_1, \dots, x_n))$, that we can set $F(B)$ to Y the zero set of $p_1, \dots, p_n$ in $\mathbb{C}^n$. Also $G\colon \mathcal{D}^{\mathrm{op}} \to \mathcal{C}$ sends $Y \mapsto \mathbb{C}[x_1, \dots, x_n]/(p_1, \dots, p_d)$ where Y is the zero set of $p_1, \dots, p_d \in \mathbb{C}[x_1, \dots, x_n]$.

2.5 Adjoints

Definition 2.23. Suppose $\mathcal{A}, \mathcal{B}$ are categories. We say $F\colon \mathcal{A} \to \mathcal{B}$ is *left adjoint to* $G\colon \mathcal{B} \to \mathcal{A}$ if, intuitively, we have

$$\hom_{\mathcal{A}}(A, G(B)) \cong \hom_{\mathcal{B}}(F(A), B)$$

for all $A \in \mathrm{Ob}(\mathcal{A})$ and $B \in \mathrm{Ob}(\mathcal{B})$. More formally, we require that for all $A \in \mathrm{Ob}(\mathcal{A})$ and $B \in \mathrm{Ob}(\mathcal{B})$ there be a bijection $\alpha_{A,B}\colon \hom_{\mathcal{A}}(A, G(B)) \to \hom_{\mathcal{B}}(F(A), B)$ such that whenever $A, A' \in \mathrm{Ob}(\mathcal{A})$, $B, B' \in \mathrm{Ob}(\mathcal{B})$, $\varphi \in \hom_{\mathcal{A}}(A, A')$ and $\psi \in \hom_{\mathcal{B}}(B, B')$, we have the following diagram commutes:

$$\begin{tikzcd}
\hom_{\mathcal{A}}(A', G(B)) \arrow[r, "\alpha_{A',B}"] \arrow[d, "\hom_{\mathcal{A}}(\varphi, G(\psi))"] & \hom_{\mathcal{B}}(F(A'), B) \arrow[d, "\hom_{\mathcal{B}}(F(\varphi), \psi)"] \\
\hom_{\mathcal{A}}(A, G(B')) \arrow[r, "\alpha_{A,B'}"] & \hom_{\mathcal{B}}(F(A), B')
\end{tikzcd}$$

where $\hom_{\mathcal{A}}(\varphi, G(\psi))\colon \hom_{\mathcal{A}}(A', G(B)) \to \hom_{\mathcal{A}}(A, G(B'))$ is given by $f \mapsto G(\psi) \circ f \circ \varphi$. We then write $F \rightleftarrows G$.

Example 2.24. If $\mathcal{G}$ is the category of grapes and $\mathcal{A}$ is the category of abelian grapes, then we have an inclusion functor $I\colon \mathcal{A} \to \mathcal{G}$ (given by $I(A) = A$ and $I(f) = f$ for $f \in \hom_{\mathcal{A}}(A, B)$) and a reduction functor $R\colon \mathcal{G} \to \mathcal{A}$ (given by $R(G) = G/G'$ and $R(f)$ is the descent of f to $G/G' \to H/H'$ for $f\colon G \to H$). Then these are adjoint; which is left adjoint and which is right adjoint?

Example 2.25. If $\mathcal{A}$ is the category of abelian grapes and **Set** is the category of sets then we have a forgetful functor $G\colon \mathcal{A} \to \mathbf{Set}$ (given by $G(A) = A$ and $G(f) = f$). Consider $F\colon \mathbf{Set} \to \mathcal{A}$ given by

$$F(X) = \mathbb{Z}^X = \bigoplus_{x \in X} \mathbb{Z} = \left\{ \sum_{x \in X} n_x e_x : n_x = 0 \text{ for all but finitely many } x \in X \right\}$$

where e_x are formal "basis vectors". Then $F \rightleftarrows G$; if X is a set and A is an abelian grape, then

$$\hom_{\mathbf{Set}}(X, G(A)) \cong \hom_{\mathcal{A}}(F(X), A)$$

with $f\colon X \to A$ being sent to $\widetilde{f}\colon \mathbb{Z}^X \to A$ given by $e_x \mapsto f(x)$. Furthermore, if $\varphi \in \hom_{\mathbf{Set}}(X, X'$ and $\psi \in \hom -\mathcal{A}(A, A')$, then the following diagram commutes:

$$\begin{tikzcd} \hom_{\mathbf{Set}}(X', G(A)) \arrow[r, "\alpha_{X',A}"] \arrow[d, "\hom_{\mathbf{Set}}(\varphi, G(\psi))"] & \hom_{\mathcal{A}}(F(X'), A) \arrow[d, "\hom_{\mathcal{A}}(F(\varphi),\psi)"] \\ \hom_{\mathbf{Set}}(X, G(A')) \arrow[r, "\alpha_{X,A'}"] & \hom_{\mathcal{A}}(F(X), A) \end{tikzcd}$$

Exercise 2.26 (Stone-Čech compactification). Idea: we have **CHaus**, the category whose objects are compact Hausdorff spaces and whose morphisms are continuous maps, and we have **Top**, the category of topological spaces. We have an inclusion functor $G\colon \mathbf{CHaus} \to \mathbf{Top}$ (given by $G(X) = X$ and $G(f) = f$). In other words, **CHaus** is a subcategory of **Top**; i.e. $\mathrm{Ob}(\mathbf{CHaus}) \subseteq \mathrm{Ob}(\mathbf{Top})$, $\hom_{\mathbf{CHaus}}(X,Y) \subseteq \hom_{\mathbf{Top}}(X,Y)$ for all $X, Y \in \mathrm{Ob}(\mathbf{CHaus})$, $f \circ_{\mathbf{CHaus}} g = f \circ_{\mathbf{Top}} g$ when it makes sense, and id_X in **CHaus** equals id_X in **Top** whenever $X \in \mathrm{Ob}(\mathbf{CHaus})$.

What would a left adjoint do? We would have $F\colon \mathbf{Top} \to \mathbf{CHaus}$ and bijective $\alpha_{X,F(X)}\colon \hom_{\mathbf{Top}}(X, F(X)) \to \hom_{\mathbf{CHaus}}(F(X), F(X))$. Let $\beta = \alpha^{-1}_{X,F(X)}(\mathrm{id}_{F(X)})$; then $\beta\colon X \to F(X)$. Moreover, the adjoint property shows that if $f\colon X \to K$ is continuous (where $K \in \mathrm{Ob}(\mathbf{CHaus})$) then there is a unique $\widetilde{f}\colon F(X) \to K$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\beta"] \arrow[d, "f"'] & F(X) \arrow[dl, dashed, "\widetilde{f}"] \\ K & \end{tikzcd}$$

Example 2.27. Recall we have $\mathbf{Top}^*$, the category of pointed topological spaces, and **Grp**, the category of grapes. Recall we also have $\pi_1\colon \mathbf{Top}^* \to \mathbf{Grp}$ given by $(X, x_0) \mapsto \pi_1(X, x_0)$. For example, if $(X, x_0) = (\mathbb{C}, 1)$, then $\pi_1(X, x_0) = \{\,\mathrm{id}\,\}$, since if $g\colon [0,1] \to \mathbb{C}$ is continuous, then we can define $g_t(x) = g(x)t + 1\cdot(1-t)$; then $g_1 = g$ and $g_0 = 1$. Now, consider $(Y, y_0) = (S^1, 1)$; let $H = \mathbb{Z} \in \mathrm{Ob}(\mathbf{Grp})$. Suppose π_1 had a left adjoint $F\colon \mathbf{Grp} \to \mathbf{Top}^*$. Then $\hom_{\mathbf{Grp}}(H, \pi_1(X, x_0)) \cong \hom_{\mathbf{Top}^*}(F(H), (X, x_0))$; so $|\hom_{\mathbf{Top}^*}(F(H), (X, x_0))| = 1$. On the other hand, we also have $\hom_{\mathbf{Grp}}(H, \pi_1(Y, y_0)) \cong \hom_{\mathbf{Top}^*}(F(H), (Y, y_0))$, and $\hom_{\mathbf{Top}^*}(F(H), (Y, y_0))$ is infinite. But $\hom_{\mathbf{Top}^*}(F(H), (Y, y_0))$ embeds into $\hom_{\mathbf{Top}^*}(F(H), (X, x_0))$, a contradiction. So π_1 does not have a left adjoint.

As a general principle, forgetful functors (like $\mathcal{A} \to \mathbf{Set}$) are right adjoint to "free" functors (like $F\colon \mathbf{Set} \to \mathcal{A}$).

Definition 2.28. Given a category $\mathcal{A}$ and a set X, we say $F(X)$ is the *free object* in X in $\mathcal{A}$ if there is a set map $f\colon X \to F(X)$ such that if $g\colon X \to A$ is a set map to some $A \in \mathrm{Ob}(\mathcal{A})$, then there is a unique $\widetilde{g} \in \hom_{\mathcal{A}}(F(X), A)$ such that the following diagram commutes:

$$\begin{tikzcd} F(X) \arrow[r, dashed, "\widetilde{g}"] & A \\ X \arrow[u, "f"] \arrow[ur, "g"'] & \end{tikzcd}$$

Exercise 2.29. If free objects exist, then $F \rightleftarrows G$ (where G is the forgetful functor).

Exercise 2.30. Free objects don't exist in the category of fields.

The most important example will be tensor-hom adjunction, which we will see later.

Theorem 2.31. *Right adjoints are unique up to natural isomorphism; i.e. if $F\colon \mathcal{A} \to \mathcal{B}$ and $G, G'\colon \mathcal{B} \to \mathcal{A}$ are right adjoints for F then there are natural transformations $\eta\colon G \to G'$ and $\mu\colon G' \to G$ such that $\mu \circ \eta = \mathrm{id}_G\colon G \to G$ and $\eta \circ \mu = \mathrm{id}_{G'}\colon G' \to G'$.*

(A similar proof will show that left adjoints are also unique up to natural isomorphism.)

Proof. Suppose $F\colon \mathcal{A} \to \mathcal{B}$; suppose $G, G'\colon \mathcal{B} \to \mathcal{A}$ are right adjoints for F. We wish to find a natural isomorphism $\eta\colon G \to G'$. Suppose $A \in \mathrm{Ob}(\mathcal{A})$ and $D \in \mathrm{Ob}(\mathcal{B})$. Then we are given

$$\hom_{\mathcal{A}}(A, GD) \xrightarrow{\alpha_{A,D}} \hom_{\mathcal{B}}(FA, D) \xleftarrow{\alpha'_{A,D}} \hom_{\mathcal{A}}(A, G'D)$$

Taking $A = GD$, we have

$$\hom_{\mathcal{A}}(GD, GD) \xrightarrow{\alpha_{GD,D}} \hom_{\mathcal{B}}(FGD, D) \xleftarrow{\alpha'_{GD,D}} \hom_{\mathcal{A}}(GD, G'D)$$

In particular, we have

$$\mathrm{id}_{GD} \mapsto \alpha_{GD,D}(\mathrm{id}_{GD}) \mapsto (\alpha'_{GD,D})^{-1}(\alpha_{GD,D}(\mathrm{id}_{GD}))\colon GD \to G'D$$

Define $\eta_D\colon GD \to G'D$ to be $(\alpha'_{GD,D})^{-1}(\alpha_{GD,D}(\mathrm{id}_{GD}))$; we must show that for $f\colon D \to D'$, the following diagram commutes:

$$\begin{tikzcd}
GD \arrow[r, "\eta_D"] \arrow[d, "Gf"] & G'D \arrow[d, "G'f"] \\
GD' \arrow[r, "\eta_{D'}"] & G'D'
\end{tikzcd}$$

We apply the naturality of the adjoint map twice. The first time we use $A = A' = GD$, $B = D$, $B' = D'$, $\varphi = \mathrm{id}_{GD}\colon A \to A'$, and $\psi, f\colon D \to D'$. Then the following diagram commutes:

$$\begin{tikzcd}
\hom(GD, GD) \arrow[r, "\alpha_{GD,D}"] \arrow[d] & \hom(FGD, D) \arrow[d] & \hom(GD, G'D) \arrow[l, "\alpha'_{GD,D}"'] \arrow[d] \\
\hom(GD, GD') \arrow[r, "\alpha_{GD,D'}"] & \hom(FGD, D') & \hom(GD, G'D') \arrow[l, "\alpha'_{GD,D'}"']
\end{tikzcd}$$

Starting with id_{GD} in the top left corner, we get

$$\mathrm{id}_{GD} \mapsto \eta_D \mapsto G'(f) \circ \eta_D$$

and

$$\mathrm{id}_{GD} \mapsto \Psi(G(f))$$

(where $\Psi = (\alpha'_{GD,D'})^{-1} \circ \alpha_{GD,D'}$). Applying naturality again, this time with $A = GD$, $A' = GD'$, $\varphi = GF\colon GD \to GD'$, $B = B' = D'$ and $\psi = \mathrm{id}_{D'}$, we find the following diagram commutes:

$$\begin{tikzcd}
\hom(GD', GD') \arrow[r] \arrow[d] & \hom(FGD', D') \arrow[d] & \hom(GD', G'D') \arrow[l] \arrow[d] \\
\hom(GD, GD') \arrow[r] & \hom(FGD, D') & \hom(GD, G'D') \arrow[l]
\end{tikzcd}$$

Chasing $\mathrm{id}_{GD'}$, we find

$$\mathrm{id}_{GD'} \mapsto \eta_{D'} \mapsto \eta_{D'} \circ G(f)$$

and

$$\mathrm{id}_{GD'} \mapsto \mathrm{id}_{GD'} \circ G(f) \mapsto \Psi \circ G(f)$$

So the first square yields

$$\Psi \circ G(f) = G'(f) \circ \eta_D$$

and the second yields

$$\Psi \circ G(f) = \eta_{D'} \circ G(f)$$

So the following diagram commutes:

$$\begin{array}{ccc} GD & \xrightarrow{\eta_D} & G'D \\ \downarrow{\scriptstyle Gf} & & \downarrow{\scriptstyle G'f} \\ GD' & \xrightarrow{\eta_{D'}} & G'D' \end{array}$$

And η is a natural transformation; one checks that it is a natural isomorphism. □ Theorem 2.31

Remark 2.32*.* If G is naturally isomorphic to G' and G' is a right adjoint for F, then G is also a right adjoint for F.

Proof. Suppose$\varphi\colon A \to A'$ and $\psi\colon B \to B'$. Then since $F \rightleftarrows G'$, we have the following diagram commutes:

$$\begin{array}{ccc} \hom(A', G'B) & \xrightarrow{\alpha_{A',B}} & \hom(FA', B) \\ \downarrow & & \downarrow \\ \hom(A, G'B') & \xrightarrow{\alpha_{A,B'}} & \hom(FA, B') \end{array}$$

Suppose $\eta\colon G \to G'$ is a natural isomorphism; then the following diagram commutes:

$$\begin{array}{ccc} \hom(A', GB) & \xrightarrow{(\eta_B \circ)} & \hom(A', G'B) \\ \downarrow & & \downarrow \\ \hom(A, GB') & \xrightarrow{(\eta_{B'} \circ)} & \hom(A, G'B') \end{array}$$

(where $(\eta_B\circ)$ maps $f \mapsto \eta_B \circ f$) since

$$\eta_{B'} \circ G(\psi) \circ f \circ \varphi = G'(\psi) \circ \eta_B \circ \circ f \circ \varphi$$

So if $\beta_{A,B} = \alpha_{A,B} \circ (\eta_B \circ)$, then $\beta_{A,B}$ are bijections $\hom(A, GB) \to \hom(FA, B)$ such that the following diagram commutes:

$$\begin{array}{ccc} \hom(A', GB) & \xrightarrow{\beta_{A',B}} & \hom(FA', B) \\ \downarrow & & \downarrow \\ \hom(A, GB') & \xrightarrow{\beta_{A,B'}} & \hom(FA, B') \end{array}$$

So $F \rightleftarrows G$. □ Remark 2.32

2.6 Tensor-Hom adjunction

Let R be a commutative ring, and consider R-**Mod**, the category of R-modules with $\hom_R(M, N) = \hom_{R\text{-}\mathbf{Mod}}(M, N)$ the set of R-module homomorphisms $M \to N$. Fix an R-module M, and consider $F\colon R\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod}$ given by $N \mapsto M \otimes_R N$. Then we have the universal property that if P is an R-module and $f\colon M \times N \to P$ is bilinear, then there is a unique homomorphism of R-modules $\widetilde{f}\colon M \otimes_R N \to P$ such that the following diagram commutes:

$$\begin{tikzcd} M \times N \arrow[r, "f"] \arrow[d, "i"] & P \\ M \otimes_R N \arrow[ur, dashed, "\widetilde{f}"'] & \end{tikzcd}$$

(Given $f\colon N \to N'$, we get $\mathrm{id} \otimes f = F(f)\colon M \otimes_R N \to M \otimes_R N'$ by $m \otimes n \mapsto m \otimes f(n)$.) We also have $G\colon R\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod}$ given by $G(N) = \hom_R(M, N)$ and if $f\colon N \to N'$ then $G(f)\colon \hom_R(M, N) \to \hom_R(M, N')$ is given by $\psi \mapsto f \circ \psi$.

Theorem 2.33 (Tensor-Hom adjunction)**.** *$F \rightleftarrows G$.*

Proof. Given $A, B \in R\text{-}\mathbf{Mod}$, we need $\alpha_{A,B}\colon \hom_R(A, GB) \to \hom_R(FA, B)$; that is, $\hom_R(A, \hom_R(M, B)) \to \hom_R(M \otimes_R A, B)$. Suppose we have $\psi \in \hom_R(A, \hom_R(M, B))$. Then for $a \in A$ we have $\psi(a)\colon M \to B$; in particular, for $m \in M$ we have $\psi(a)(m) \in B$. We then define $\psi_0\colon M \times A \to B$ by $\psi_0(m, a) = \psi(a)(m)$. Then ψ_0 is bilinear:

$$\begin{aligned}\psi_0(rm + m', a) &= \psi(a)(rm + m') \\ &= r\psi(a)(m) + \psi(a)(m') \\ &= r\psi_0(m, a) + \psi_0(m', a)\end{aligned}$$

and

$$\begin{aligned}\psi_0(m, ra + a') &= \psi(ra + a')(m) \\ &= (r\psi(a) + \psi(a'))(m) \\ &= r\psi_0(m, a) + \psi_0(m, a')\end{aligned}$$

So by the universal property for tensor products, we get a unique homomorphism of R-modules $\widehat{\psi_0}\colon M \otimes_R A \to B$ such that the following diagram commutes:

$$\begin{tikzcd} M \times A \arrow[r, "\psi_0"] \arrow[d] & B \\ M \otimes_R A \arrow[ur, dashed, "\widehat{\psi_0}"'] & \end{tikzcd}$$

We then set $\alpha_{A,B}(\psi) = \widehat{\psi_0}$. This is reversible: if $\varphi\colon M \otimes_R A \to B$, then $\widetilde{\varphi}\colon M \times A \to B$ given by $(m, a) \mapsto \varphi(m \otimes a)$ is bilinear:

$$\begin{aligned}\widetilde{\varphi}(rm_1 + m_2, a) &= \varphi((rm_1 + m_2) \otimes a) \\ &= \varphi(r(m_1 \otimes a) + m_2 \otimes a) \\ &= r\varphi(m_1 \otimes a) + \varphi(m_2 \otimes a) \\ &= r\widetilde{\varphi}(m_1, a) + \widetilde{\varphi}(m_2, a)\end{aligned}$$

and likewise with the other side. We can then think of $\widetilde{\varphi}$ as morphism $A \to \hom_R(M, B)$ by $a \mapsto \widetilde{\varphi}(a)$ (where $\widetilde{\varphi}(a)(m) = \widetilde{\varphi}(m, a)$); so $\widetilde{\varphi} \in \hom_R(A, \hom_R(M, B))$.

So $\alpha_{A,B}$ is an isomorphism (i.e. bijection); it remains to check the compatibility condition. Suppose $\varphi\colon A \to A'$, $\psi\colon B \to B'$. We wish to check that the following diagram commutes:

$$\begin{tikzcd} \hom(A', GB) \arrow[r, "\alpha_{A',B}"] \arrow[d] & \hom(FA', B) \arrow[d] \\ \hom(A, GB') \arrow[r, "\alpha_{A,B'}"] & \hom(FA, B') \end{tikzcd}$$

Suppose $h \in \hom(A', \hom(M, B))$; then, going one way, we get

$$h \mapsto \widehat{h_0} \mapsto \psi \circ g \circ F(\varphi) = \psi \circ g \circ (\mathrm{id} \otimes \varphi)$$

Going the other way, we get

$$h \mapsto G(\psi) \circ h \circ \varphi = \psi \circ h \circ \varphi (\widehat{\psi \circ h \circ \varphi})_0$$

One checks that $(\widehat{\psi \circ h \circ \varphi})_0 = \psi \circ \widehat{h_0} \circ (\mathrm{id} \otimes \varphi)$. (Hint: look at what they do to $m \otimes a$.) $\square$ Theorem 2.33

2.7 Yoneda's lemma

Example 2.34. Let $\mathbf{Ab}_{\text{fin}}$ be the category of finite abelian grapes. Suppose $A \in \mathrm{Ob}(\mathbf{Ab}_{\text{fin}})$; suppose for all finite abelian grapes B we know $|\hom_{\mathbf{Ab}}(A, B)|$. Can we recover A? Equivalently, if $A_1 \not\cong A_2$, is there necessarily a B such that $|\hom(A_1, B)| \neq |\hom(A_2, B)|$.

For example, consider

$$\begin{aligned} A_1 &= \mathbb{Z}_2^3 \oplus \mathbb{Z}_4^3 \oplus \mathbb{Z}_5 \\ A_2 &= \mathbb{Z}_2^4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_5 \end{aligned}$$

Then

$$\begin{aligned} |\hom(A_1, \mathbb{Z}_5)| &= 5 \\ |\hom(A_1, \mathbb{Z}_5)| &= 5 \\ |\hom(A_1, \mathbb{Z}_2)| &= 2^6 \\ |\hom(A_1, \mathbb{Z}_2)| &= 2^6 \\ |\hom(A_1, \mathbb{Z}_4)| &= 2^3 \cdot 4^3 \\ |\hom(A_1, \mathbb{Z}_4)| &= 2^4 \cdot 4^2 \end{aligned}$$

The answer turns out to be "yes" for $\mathbf{Ab}_{\text{fin}}$, but not in general.

Yoneda's lemma says roughly that we can understand $A \in \mathrm{Ob}(\mathcal{A})$ by understanding $\hom_{\mathcal{A}}(A, B)$ for all $B \in \mathrm{Ob}(\mathcal{A})$.

Definition 2.35. Suppose $\mathcal{A}$ is a category; suppose $A \in \mathrm{Ob}(\mathcal{A})$. We can make a functor $h_A \colon \mathcal{A} \to \mathbf{Set}$ by $h_A(B) = \hom_{\mathcal{A}}(A, B)$ and $h_A(f) \colon \hom_{\mathcal{A}}(A, B) \to \hom_{\mathcal{A}}(A, B')$ is $h_A(f)(\psi) = f \circ \psi$ whenever $f \in \hom_{\mathcal{A}}(B, B')$. Such an h_A is called a *representable functor*. (We also give this name to a functor that is naturally isomorphic to a representable functor.)

On the assignment, we define a category $\mathbf{Funct}(\mathcal{A}, \mathbf{Set})$ whose objects are functors $\mathcal{A} \to \mathbf{Set}$ and whose morphisms $F \to G$ are natural transformations $\eta \colon F \to G$. Let $\mathcal{F}$ be the (full) subcategory of $\mathbf{Funct}(\mathcal{A}, \mathbf{Set})$ whose objects are representable functors; i.e. $\hom_{\mathcal{F}}(h_A, h_B)$ is the class of natural transformations $h_A \to h_B$.

Theorem 2.36 (Yoneda's lemma)**.** $\mathcal{A} \cong \mathcal{F}^{\mathrm{op}}$.

Recall if $\eta \colon h_A \to h_B$ is a natural isomorphism then for each $C \in \mathrm{Ob}(\mathcal{A})$ we get an isomorphism $\eta_C \colon h_A(C) \to h_B(C)$; i.e. $\hom(A, C) \cong \hom(B, C)$. Yoneda's lemma gives a partial converse to this.

Example 2.37. Consider the forgetful functor $G \colon \mathbf{Grp} \to \mathbf{Set}$ given by $G(H) = H$. Then G is a representable functor: note that $\hom_{\mathbf{Grp}}(\mathbb{Z}, H) \cong H$ for all $H \in \mathrm{Ob}(\mathbf{Grp})$. So $G \cong h_{\mathbb{Z}}$.

Another way to view the above: consider $F \colon \mathbf{Set} \to \mathbf{Grp}$ where $F(X)$ is the free grape on X. Then $\mathbb{Z} = F(\{x\})$; so by the adjoint property we have $\hom_{\mathbf{Grp}}(F(X), H) \cong \hom_{\mathbf{Set}}(X, H)$. But in $\mathbf{Set}$, we have $H \cong \hom_{\mathbf{Set}}(\{x\}, H) \cong \hom_{\mathbf{Grp}}(F(\{x\}), H) = \hom_{\mathbf{Grp}}(\mathbb{Z}, H)$.

Example 2.38. Let $\mathcal{C}$ be the category of commutative k-algebras (where k is a field). Given a ring C we can form a category C-$\mathbf{Mod}$. If M is a C-module, a *derivation* $\delta \colon C \to M$ is a k-linear map satisfying $\delta(c_1 c_2) = c_1 \delta(c_2) + c_2 \delta(c_1)$. Consider $\mathrm{Der}_k(C, M)$ the set of derivations $\delta \colon C \to M$; this is a C-module with $(c \cdot f)(a) = c \cdot f(a)$. So we have a functor $\mathrm{Der} \colon C\text{-}\mathbf{Mod} \to C\text{-}\mathbf{Mod}$ given by $M \mapsto \mathrm{Der}_k(C, M)$ and $\mathrm{Der}_k(f)(\delta) = f \circ \delta$. (Note that $f \circ \delta$ is indeed a derivation: $(f \circ \delta)(ab) = f(a\delta(b) + b\delta(a)) = af(\delta(b)) + bf(\delta(a))$.)

Claim 2.39. Der_k *is representable.*

Proof. We use Kähler differentials. Given C a k-algebra, we construct a C-module $\Omega_{C/k}$ which is the free C-module on all symbols of the form dc for $c \in C$ modulo the relations

$$\begin{aligned} d(c_1 + \lambda c_2) &= dc_1 - \lambda dc_2 \\ d(c_1 c_2) &= c_1 dc_2 + c_2 dc_1 \end{aligned}$$

For example, consider $C = k[t]$. Then in $\Omega_{k[t]/k}$, we have

$$d(a_0 + a_t + \cdots + a_s t^s) = a_0 d1 + a_1 dt + \cdots + a_s dt^2 = 0 + a_1 dt + 2a_2 t dt + \cdots + s a_2 t^{s-1} dt = p'(t) dt$$

So $\Omega_{k[t]/t} = k[t]dt$. In general $\mathrm{Der}_k(k[t], M) \cong \hom_{k[t]}(\Omega_{k[t]/k}, M)$ where given $\delta \colon k[t] \to M$ a derivation we associate $f_\delta \colon \Omega_{k[t]/k} \to M$ given by $f_\delta(dt) = \delta(t)$. (In general we want $f_\delta(dc) = \delta(c)$.) Then $f_\delta(p(t)dt) =$

$p(t)\delta(t)$. Conversely, for $f\colon \Omega_{k[t]/k} \to M$ can associate $\delta_f\colon k[t] \to M$ given by $\delta_f(p(t)) = f(dp(t)) = f(p'(t)dt) = p'(t)f(dt)$; then $\delta_f(c) = f(dc)$ and

$$\begin{aligned}\delta_f(p(t)q(t)) &= (p(t)q(t))' f(dt)\\ &= p'(t)q(t)f(dt) + p(t)q'(t)f(dt)\\ &= q \cdot \delta_f(p) + p \cdot \delta_f(q)\end{aligned}$$

So δ_f is indeed a differential. □ Claim 2.39

We digress from Yoneda's lemma for a bit to give an exposition of presheaves.

Definition 2.40 ((Topological) presheaves)**.** Recall that if X is a topological space we defined $\mathbf{Top}_X$ to have open subsets of X as objects and

$$\hom_{\mathbf{Top}_X} = \begin{cases} i & U \overset{i}{\hookrightarrow} V \\ \emptyset & \text{else} \end{cases}$$

Then a *presheaf of* $\mathcal{C}$ (where $\mathcal{C} \in \{\, \mathbf{Ab}, \mathbf{Ring}, \mathbf{Grp}, \mathbf{Set}, \dots \,\}$) is a functor $S\colon \mathbf{Top}_X^{\mathrm{op}} \to \mathcal{C}$ (i.e. a contravariant $S\colon \mathbf{Top}_X \to \mathcal{C}$); then if $i\colon U \hookrightarrow V$, we get $p_{V,U} = S(i)\colon S(V) \to S(U)$, which we think of as "restriction" from V to U.

Example 2.41. Consider $\mathcal{O}\colon \mathbf{Top}_X^{\mathrm{op}} \to \mathbf{Set}$ given by $U \mapsto \{\, f\colon U \to \mathbb{C} \text{ continuous} \,\}$ where given $f \in \mathcal{O}(V)$ we define $p_{V,U}(f) = f \restriction U \in \mathcal{O}(U)$.

Example 2.42. let $X = \mathbb{C}$ with the Euclidean topology, and let $\mathcal{F}\colon \mathbf{Top}_X^{\mathrm{op}} \to \mathbf{Ring}$ be $\mathcal{F}(U) = \{\, f\colon U \to \mathbb{C} \mid f \text{ analytic} \,\}$. If $U \subseteq V$, we get $\mathcal{F}(U) \to \mathcal{F}(V)$ by $f \mapsto f \restriction U$.

Definition 2.43. A presheaf $\mathcal{F}\colon \mathbf{Top}_X^{\mathrm{op}} \to \mathcal{C}$ is a *sheaf* if it satisfies

1. It is *separated*: if $U \subseteq X$ is open and
$$U = \bigcup_{i \in I} U_i$$
then if $f, g \in \mathcal{F}(U)$ satisfy $f \restriction U_i = g \restriction U_i$ for all $i \in I$, we have $f = g$.

2. We should be able to *glue*: if
$$U = \bigcup_{i \in I} U_i$$
and we are given $(f_i : i \in I)$ such that $f_i \restriction (U_i \cap U_j) = f_j \restriction (U_i \cap U_j)$, then there is some $f \in \mathcal{F}(U)$ such that $f \restriction U_i = f_i$ for all $i \in I$.

Example 2.44. For example, $\mathcal{F}\colon \mathbf{Top}_X^{\mathrm{op}} \to \mathbf{Ring}$ given by $\mathcal{F}(U) = \{\, f\colon U \to \mathbb{C} \mid f \text{ continuous} \,\}$ is a sheaf of rings.

Example 2.45. Let $X = \mathbb{R}$ with the Euclidean topology. Let $\mathcal{F}(U)$ be the set of bounded continuous function $U \to \mathbb{R}$, and endow $\mathcal{F}$ with the restriction mapping. This is a presheaf but not a sheaf, since we don't have gluing:

$$\mathbb{R} = \bigcup_{n=1}^{\infty} (-n, n)$$

and we can set $f_n(x) = x \in \mathcal{O}(U_n)$ (where $U_n = (-n, n)$) and $f_n \restriction (U_n \cap U_m) = f_m \restriction (U_n \cap U_m)$ but there is no $f\colon \mathbb{R} \to \mathbb{R}$ bounded such that $f \restriction U_n = f_n$ for all n.

We now bring things back to Yoneda's lemma.

What are the representable presheaves of sets; i.e. representable functors $h\colon \mathbf{Top}_X^{\mathrm{op}} \to \mathbf{Set}$? Well, we fix $U \subseteq X$ open and get $h_U\colon \mathbf{Top}_X^{\mathrm{op}} \to \mathbf{Set}$ given by $h_U(V) = \hom_{\mathbf{Top}_X^{\mathrm{op}}}(U, V) = \hom_{\mathbf{Top}_X}(V, U)$ and $\psi \mapsto \psi \circ i$ for $\psi \in \hom_{\mathbf{Top}_X^{\mathrm{op}}}(V_2, V_1) = \hom_{\mathbf{Top}_X}(V_1, V_2)$. Then $h_U(V)$ is empty if $V \not\subseteq U$ and is $\{\, i\colon V \hookrightarrow U \,\}$ otherwise.

Now, if h_U and $\mathcal{F}$ are two presheaves $\mathbf{Top}_X^{\mathrm{op}} \to \mathbf{Set}$, what is a natural transformation $\eta\colon h_U \to \mathcal{F}$? Well, if $V_1 \hookrightarrow V_2$ then we get the following diagram commutes:

$$\begin{tikzcd} h_U(V_2) \arrow[r, "\eta_U"] \arrow[d] & \mathcal{F}(V_2) \arrow[d] \\ h_U(V_1) \arrow[r, "\eta_V"] & \mathcal{F}(V_1) \end{tikzcd}$$

If $\mathcal{F} = h_V$, then the $\eta\colon h_U \to h_V$ are in bijection with $h_V(U) = \hom_{\mathbf{Top}_X^{\mathrm{op}}}(V, U) = \hom_{\mathbf{Top}_X}(U, V)$.

Claim 2.46. *Any $\eta\colon h_U \to \mathcal{F}$ is completely determined by η_U.*

Proof. If $V_1 \hookrightarrow V_2$ then we get the following diagram commutes:

Case 1. Suppose $V \subseteq U$ is open; so we have $V \overset{i}{\hookrightarrow} U$, and hence $U \to V$ in $\mathbf{Top}_X^{\mathrm{op}}$. We get

$$\begin{tikzcd} h_U(U) \arrow[r, "\eta_U"] \arrow[d] & \mathcal{F}(U) \arrow[d] \\ h_U(V) = \hom(U, V) \arrow[r, "\eta_V"] & \mathcal{F}(V) \end{tikzcd}$$

So η_V is determined by η_U.

Case 2. Suppose $V \not\subseteq U$; then $h_U(V) = \emptyset$. □ Claim 2.46

We now prove Yoneda's lemma.

Proof of Theorem 2.36. We have a category $\mathcal{A}$ with objects $A, B, C, \dots$ and morphisms $A \xrightarrow{f} B$; we have a category $\mathcal{F} \subseteq \mathbf{Funct}(\mathcal{A}, \mathbf{Set})$ with objects $h_A, h_B, h_C, \dots$ and morphisms $\eta\colon h_A \to h_B$. We claim that $A \cong \mathcal{F}^{\mathrm{op}}$. We need to construct $F\colon \mathcal{A} \to \mathcal{F}^{\mathrm{op}}$ and $G\colon \mathcal{F}^{\mathrm{op}} \to \mathcal{A}$. We define $F(A) = h_A$; given $A \xrightarrow{f} B$ we define $\eta_f = F(f)\colon h_B \to h_A$ by, for $C \in \mathrm{Ob}(\mathcal{A})$, setting $(\eta_f)_C\colon h_B(C) \to h_A(C)$ (i.e. $\hom(B, C) \to \hom(A, C)$) to be $\psi \mapsto \psi \circ f$ for $\psi \in \hom(B, C)$.

To check that $\eta_f\colon h_B \to h_A$ is a natural transformation, suppose $g\colon C \to C'$ for $C, C' \in \mathrm{Ob}(\mathcal{A})$. We wish to check that the following diagram commutes:

$$\begin{tikzcd} h_B(C) \arrow[r, "(\eta_f)_C"] \arrow[d] & h_A(C) \arrow[d, "h_A(g)"] \\ h_B(C') \arrow[r, "(\eta_f)_{C'}"] & h_A(C') \end{tikzcd}$$

But going one way, we get

$$\psi \mapsto g \circ \psi \mapsto g \circ \psi \circ f$$

and going the other way, we get

$$\psi \mapsto \psi \circ f \mapsto g \circ \psi \circ f$$

So the map $A \mapsto h_A$ and $f \mapsto \eta_f$ is a functor $F\colon \mathcal{A} \to \mathcal{F}^{\mathrm{op}}$.

Now, define $G\colon \mathcal{F}^{\mathrm{op}} \to \mathcal{A}$ by $G(h_A) = A$. For $\eta\colon h_A \to h_B$, we wish to define $f(\eta) = G(\eta)\colon B \to A$. But $\eta_A\colon h_A(A) \to h_B(A)$; so we may set $f(\eta) = G(\eta) = \eta_A(\mathrm{id}_A)\colon B \to A$. One checks that G is a functor.

Look at $G \circ F\colon \mathcal{A} \to \mathcal{A}$ and $F \circ G\colon \mathcal{F}^{\mathrm{op}} \to \mathcal{F}^{\mathrm{op}}$. We claim that these are the respective identity functors. Well, note that

$$\begin{aligned} (G \circ F)(A) &= G(h_A) \\ &= A \\ (F \circ G)(h_A) &= F(A) \\ &= h_A \end{aligned}$$

Suppose $A \xrightarrow{f} B$; we get $A \xrightarrow{GF(f)} B$. We need to check that $GF(f) = f$. Well, $F(f) = \eta_f \colon h_B \to h_A$ is given by $(\eta_f)_C \colon h_B(C) \to h_A(C)$ is $\psi \mapsto \psi \circ f$; then $G(\eta_f) = (\eta_f)_B(\mathrm{id}_B) = \mathrm{id}_B \circ f = f$.

Suppose now that $\eta \colon h_B \to h_A$. Then $F(G(\eta)) = F(\eta_B(\mathrm{id}_B))$, and for $C \in \mathrm{Ob}(\mathcal{A})$ we have $(F(\eta_B(\mathrm{id}_B)))_C \colon h_B(C) \to h_A(C)$ is given by $\psi \mapsto \psi \circ \eta_B(\mathrm{id}_B)$. But by naturality of η we have the following diagram commutes:

$$\begin{tikzcd} h_B(B) \arrow[r, "\eta_B"] \arrow[d, "h_B(\psi)"] & h_A(B) \arrow[d, "h_A(\psi)"] \\ h_B(C) \arrow[r, "\eta_C"] & h_A(C) \end{tikzcd}$$

and hence, following $\mathrm{id}_B \in h_B(B)$, we find $\eta_C(\psi) = \psi \circ \eta_B(\mathrm{id}_B)$. So $\eta_C = (F(\eta_B(\mathrm{id}_B)))_C$ for all $C \in \mathrm{Ob}(\mathcal{A})$. So $\eta = F(G(\eta))$.

So $G \circ F = \mathrm{id}_{\mathcal{A}}$ and $F \circ G = \mathrm{id}_{\mathcal{F}^{\mathrm{op}}}$, as desired. So $\mathcal{A} \cong \mathcal{F}^{\mathrm{op}}$. □ Theorem 2.36

Corollary 2.47. *Any small category (i.e. in which* $\mathrm{Ob}(\mathcal{C})$ *is a set and* $\hom(A, B)$ *is a set for all* $A, B \in \mathrm{Ob}(\mathcal{C})$*) is* concretizable*; i.e. is equivalent to a category in which each object is a set.*

Idea of proof. Let $\mathcal{C}$ be a small category. Then by Yoneda's lemma we have $\mathcal{C} \cong \mathcal{F}^{\mathrm{op}} \subseteq \mathbf{Funct}(\mathcal{C}, \mathbf{Set})^{\mathrm{op}}$ via $C \mapsto h_C$. We make a new category $\widetilde{\mathcal{C}}$ whose objects are given as follows: for $B \in \mathrm{Ob}(\mathcal{C})$ we make a set

$$\widehat{B} = \coprod_{C \in \mathrm{Ob}(\mathcal{C})} h_B(C)$$

Given $f \colon B \to B'$ we define a map $\widehat{f} \colon \widehat{B'} \to \widehat{B}$ by $\varphi_C \mapsto \varphi_C \circ f$ where

$$\varphi_C \in \widehat{B'} = \coprod_{C \in \mathrm{Ob}(\mathcal{C})} h_{B'}(C)$$

This gives us a concrete category $\widehat{\mathcal{C}}$ with $\mathcal{C} \cong \mathcal{F}^{\mathrm{op}} \cong \widehat{\mathcal{C}}^{\mathrm{op}}$. □ Corollary 2.47

2.8 Initial and terminal objects

Definition 2.48. We say $I \in \mathrm{Ob}(\mathcal{C})$ is an *initial object* of $\mathcal{C}$ if for all $C \in \mathrm{Ob}(\mathcal{C})$ there is a unique $f \colon I \to C$. We say T is a *terminal object* if for all $C \in \mathrm{Ob}(\mathcal{C})$ there is a unique $g \colon C \to T$.

Example 2.49. Consider **Set**. Then $\emptyset$ is the unique initial object, and the terminal objects are exactly the singletons.

Remark 2.50. If they exist, initial and terminal objects are unique up to unique isomorphism.

Proof. We do the case of initial objects. Suppose I_1 and I_2 is initial. Then there is a unique $i_1 \colon I_1 \to I_2$ and $i_2 \colon I_2 \to I_1$; then $i_2 \circ i_1 \colon I_1 \to I_1$. But there is a unique map $I_1 \to I_1$, and $\mathrm{id}_{I_1} \colon I_1 \to I_1$; so $i_2 \circ i_1 = \mathrm{id}_{I_1}$. Likewise, we get $i_1 \circ i_2 = \mathrm{id}_{I_2}$, and i_1 is an isomorphism. Uniqueness is then immediate. □ Remark 2.50

Example 2.51.

1. In **Ring** (in which we require maps to preserve unity), we have $I = \mathbb{Z}$ is initial and $T = 0_R$ (the zero ring) is terminal.
2. In **Ab** we have (0) is initial and terminal; we call this a *zero object*.
3. In **Field*** (i.e. non-zero fields) there is no initial or terminal object.

2.9 Limits and colimits

We use $\varinjlim$ to denote colimits and $\varprojlim$ to denote limits.

Definition 2.52. Let $\mathcal{C}$ be a category and let $\mathcal{B}$ be a category. (Almost always $\mathcal{B}$ will be small and $\mathcal{B} \subseteq \mathcal{C}$ is not necessarily full.) Then a *diagram* based on $\mathcal{B}$ is a functor $F\colon \mathcal{B} \to \mathcal{C}$ (often the inclusion functor). A diagram is *small* if $\mathcal{B}$ is a small category. A *cone* to F is an object $N \in \mathrm{Ob}(\mathcal{C})$ and a family of morphisms $\varphi_B\colon N \to FB$ for all $B \in \mathrm{Ob}(\mathcal{B})$ such that for all $f\colon B_i \to B_j$ in $\mathcal{B}$ we have the following diagram commutes:

$$\begin{tikzcd} N \arrow[d, "\varphi_{B_i}"] \arrow[dr, "\varphi_{B_j}"] & \\ FB_i \arrow[r, "F(f)"] & FB_j \end{tikzcd}$$

We can make a category of cones in the natural way; we then define a *limit* $\varprojlim F$ of the diagram to be a final (i.e. terminal) object; that is, a cone (L, φ_B) such that every other cone factors uniquely through (L, φ_B).

Remark 2.53. Since terminal objects are unique up to unique isomorphism if they exist, we have that $\varprojlim F$ is unique up to unique isomorphism if it exists.

Definition 2.54. We can dually define a *co-cone* to F to be an object $N \in \mathrm{Ob}(\mathcal{C})$ and a family of morphisms $\varphi_B\colon FB \to N$ for all $B \in \mathrm{Ob}(\mathcal{B})$ such that for all $f\colon B_i \to B_j$ in $\mathcal{B}$ we have the following diagram commutes:

$$\begin{tikzcd} FB_i \arrow[r, "F(f)"] \arrow[d, "\varphi_{B_i}"] & FB_j \arrow[dl, "\varphi_{B_j}"] \\ N & \end{tikzcd}$$

We then define an *inverse limit* of the diagram to be an initial object in the category of co-cones.

Limits	Colimits	Diagrams
$\varprojlim$	$\varinjlim$	
Final object	Initial object	$\emptyset$
Product	Coproduct	Objects in $\mathcal{C}$ with the respective identity morphisms
Equalizer	Coequalizer	$A \rightrightarrows B$
Inverse (projective) limit	Direct limit	Directed set
Pullback	Pushout	$A \longleftarrow B \longrightarrow C$ $E \longrightarrow F \longleftarrow G$

Example 2.55. Recall that a directed set I has a reflexive and transitive (i.e. preorder) $\leq$ such that for all $a, b \in I$ we have an upper bound in I.

Consider $I = \mathbb{N}$ with the usual order. Let **Ring** be the category of rings. Let $\mathcal{B} \subseteq$ **Ring** be the category with objects $\mathbb{Z}/p^n\mathbb{Z}$ for some fixed prime p; for $i \geq 2$, we include a morphism $\varphi_i\colon \mathbb{Z}/p^i\mathbb{Z} \to \mathbb{Z}/p^{i-1}\mathbb{Z}$ given by $[n]_{p^i} \mapsto [n]_{p^{i-1}}$. Take $F\colon \mathcal{B} \to$ **Ring** to be the inclusion functor. Then $L = \varprojlim F = \varprojlim \mathbb{Z}/p^n\mathbb{Z} = \mathbb{Z}_p$ the ring of p-adic integers.

Let's see how to find L. Embed

$$\widetilde{\pi}\colon L \to \prod_{i=1}^{\infty} \mathbb{Z}/p^i\mathbb{Z}$$

by $x \mapsto (\pi_1(x), \pi_2(x), \dots)$. Now, if $\widetilde{\pi}$ is not injective, we can replace L by $L/\ker(\widetilde{\pi})$; so assume $\widetilde{\pi}$ is injective. So

$$L \subseteq \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p^2\mathbb{Z} \times \dots$$

If $(a_1, a_2, \dots) \in L$, then $a_1 = \pi_1((a_1, \dots)) = \varphi_2(\pi_2(a_1, \dots)) = a_2$ in $\mathbb{Z}/p\mathbb{Z}$; likewise we get $a_{n+1} \equiv a_n \pmod{p^n}$. So $L \subseteq \mathbb{Z}_p$. In fact we have equality: $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$.

Example 2.56. Consider the directed set $I = \mathbb{N}$ with $a \leq b \iff a \mid b$ Let $\mathcal{C}$ be the category of fields. Fix a prime p; notice for $n \in \mathbb{N}$ we have $\mathbb{F}_{p^n}$ the splitting field of $x^{p^n} - x$ over $\mathbb{F}_p$. If $\mathbb{F}_{p^i} \subseteq \mathbb{F}_{p^j}$ then we have $\mathbb{F}_{p^j} = \mathbb{F}_{p^i} \cdot 1 \oplus \ldots \oplus \mathbb{F}_{p^i}\alpha_s$ has size $(p^i)^s$; so $j = is$, and $i \mid j$. Conversely, if $i \mid j$, say $j = is$, then we get an embedding $\theta_{ij} \colon \mathbb{F}_{p^i} \hookrightarrow \mathbb{F}_{p^j}$. What is $\varinjlim \mathbb{F}_{p^n}$? The category $\mathcal{B}$ has objects $\mathbb{F}_{p^i}$ for $i \geq 1$ and morphisms venerated by θ_{ij} for $i \mid j$. Then $L = \overline{\mathbb{F}_p}$ is the algebraic closure of $\mathbb{F}_p$.

We have seen that $\mathbb{Z}_p$ is a $\varprojlim$ and $\overline{\mathbb{F}_p}$ is a $\varinjlim$. More generally, if $(I, \leq)$ is a directed set, we define

1. Given category with objects $\{ C_i : i \in I \}$ and morphisms $\varphi_{ij} \colon C_i \to C_j$ for $i \geq j$ such that $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ and $\varphi_{ii} = \mathrm{id}_{C_i}$, we define $\varprojlim C_i$ to be the *inverse limit* of the C_i.
2. Given a category with objects $\{ C_i : i \in I \}$ and morphisms $\theta_{ij} \colon C_i \to C_j$ again satisfying $\theta_{jk} \circ \theta_{ij} = \theta_{ik}$ and $\theta_{ii} = \mathrm{id}_{C_i}$, we define $\varinjlim C_i$ to be the *direct limit* of this system.

Definition 2.57 (Products and coproducts)**.** Suppose $\mathcal{C}$ is a category and $\mathcal{B} \subseteq \mathcal{C}$ is a subcategory whose only morphisms are the identity morphisms; let F be the inclusion functor. We call $\varprojlim F$ the *product*

$$\prod_{C \in \mathrm{Ob}(\mathcal{B})} C$$

and we call $\varinjlim F$ the *coproduct*

$$\coprod_{C \in \mathrm{Ob}(\mathcal{B})} C$$

Mnemonic 2.58. "Colimits are the stalactites of category theory."

$$\begin{tikzcd}
{} \arrow[r] & FB \arrow[rr, "Ff"] \arrow[dr, "\varphi_B"'] & & FB' \arrow[r] \arrow[dl, "\varphi_{B'}"] & {} \\
& & L & &
\end{tikzcd}$$

We can think of the "c" in "colimit" as recalling "**c**eiling". We can also recall the $\varprojlim$ generalizes the inverse/projective limit, and that $\varinjlim$ generalizes the direct limit.

Remark 2.59. When limits/colimits exist, we can regard $\varinjlim$ or $\varprojlim$ as functors. What does this mean? Well, if we fix a category $\mathcal{A}$ and consider all diagrams of type $\mathcal{B}$ into $\mathcal{A}$, we can identify this with $\mathbf{Funct}(\mathcal{B}, \mathcal{A})$. Suppose $F, G \colon \mathcal{B} \to \mathcal{A}$ and $\eta \colon F \to G$ is a natural transformation; consider the colimit case. Then the following diagram commutes:

$$\begin{tikzcd}
FB \arrow[rr, "Ff"] \arrow[dr, "\varphi_B"] \arrow[dd, "\eta_B"] & & FB' \arrow[dl, "\varphi_{B'}"'] \arrow[dd, "\eta_{B'}"] \\
& \varinjlim F & \\
GB \arrow[rr, "Gf"] \arrow[dr, "\psi_B"] & & GB' \arrow[dl, "\psi_{B'}"'] \\
& \varinjlim G &
\end{tikzcd}$$

which then induces a unique morphism $\varinjlim \eta \colon \varinjlim F \to \varinjlim G$ such that the following diagram commutes:

$$\begin{tikzcd}
FB \arrow[rr, "Ff"] \arrow[dr, "\varphi_B"] \arrow[dd, "\eta_B"] & & FB' \arrow[dl, "\varphi_{B'}"'] \arrow[dd, "\eta_{B'}"] \\
& \varinjlim F \arrow[dd, dashed, "\varinjlim \eta"] & \\
GB \arrow[rr, "Gf"] \arrow[dr, "\psi_B"] & & GB' \arrow[dl, "\psi_{B'}"'] \\
& \varinjlim G &
\end{tikzcd}$$

Playing a little more, we get that $\varinjlim$ is indeed a functor $\mathbf{Funct}(\mathcal{B}, \mathcal{A}) \to \mathcal{A}$.

An overview of our coverage of limits and colimits:

1. Examples
2. Left adjoints preserve colimits, right adjoints preserve limits
3. Criteria for (small) colimits and limits to always exists

What does (2) mean? Well, suppose $D\colon \mathcal{D} \to \mathcal{A}$ is a diagram; suppose $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{A}$ have $F \rightleftarrows G$. We do the colimit case.

$$\begin{tikzcd} DX \arrow[rr, "Df"] \arrow[dr, "\varphi_X"'] & & DX' \arrow[dl, "\varphi_{X'}"] \\ & \varinjlim D & \end{tikzcd}$$

Applying F, we get another cone:

$$\begin{tikzcd} FDX \arrow[rr, "FDf"] \arrow[dr, "F\varphi_X"'] & & FDX' \arrow[dl, "F\varphi_{X'}"] \\ & F(\varinjlim D) & \end{tikzcd}$$

A priori, we don't know that it's the universal cone (i.e. colimit).

Theorem 2.60. *$F(\varinjlim D) = \varinjlim(FD)$.*

Mnemonic 2.61. RAPL: "right adjoints preserve limits". Alternatively, left adjoints are right exact.

Example 2.62 (Coproduct in **Grp**). Consider two copies of $\mathbb{Z}/2\mathbb{Z}$: $\langle x \mid x^2 = 1 \rangle$ and $\langle y \mid y^2 = 1 \rangle$.

$$\begin{tikzcd} \langle x \mid x^2 = 1 \rangle \arrow[dr] \arrow[ddr] & & \langle y \mid y^2 = 1 \rangle \arrow[dl] \arrow[ddl] \\ & \mathbb{Z}/2\mathbb{Z} \coprod \mathbb{Z}/2\mathbb{Z} \arrow[d] & \\ & G & \end{tikzcd}$$

where $\mathbb{Z}/2\mathbb{Z} \coprod \mathbb{Z}/2\mathbb{Z}$ is the *free product* of $\mathbb{Z}/2\mathbb{Z}$ with itself. Given maps $\mathbb{Z}/2\mathbb{Z}$ into G as above, we define g to be the image of x and h to be the image of y; this then induces a map $\mathbb{Z}/2\mathbb{Z} \coprod \mathbb{Z}/2\mathbb{Z} \to G$ via $x \mapsto g$ and $y \mapsto h$.

One can check that $\mathbb{Z}/2\mathbb{Z} \coprod \mathbb{Z}/2\mathbb{Z} \cong \langle u, v \mid v^2 = 1, vuv^{-1} = u^{-1} \rangle$, the infinite dihedral grape. In general we have

$$\coprod_{i \in I} G_i$$

is just the free product of the G_i.

Note that the free product of $\mathbb{Z}/2\mathbb{Z}$ with $\mathbb{Z}/2\mathbb{Z}$ in **Ab** is instead the direct sum.

Example 2.63 (Coproduct in **Set**). The coproduct of sets is just the disjoint union.

Example 2.64 (Coproduct in **Ab**). $A \coprod B \cong A \oplus B$. More generally in R-**Mod** we have $M \coprod N \cong M \oplus N$; in fact

$$\coprod_{i \in I} M_i \cong \bigoplus_{i \in I} M_i$$

Example 2.65. Consider $G\colon \mathbf{Grp} \to \mathbf{Set}$ the forgetful functor. We know $F \rightleftarrows G$ where F is the free grape functor; is G a left adjoint? No, as it does not preserve colimits: $G(\mathbb{Z}/2\mathbb{Z} \coprod \mathbb{Z}/2\mathbb{Z})$ is infinite but $G(\mathbb{Z}/2\mathbb{Z}) \coprod G(\mathbb{Z}/2\mathbb{Z}) \cong \{1, 2, 3, 4\}$.

TODO 1. *Get this class.*

Definition 2.66. A category in which all small colimits exist is called *cocomplete*; a category in which all small limits exist is called *complete*. A category that is complete and cocomplete is called *bicomplete*.

Theorem 2.67 (Criterion for existence of small colimits)**.** *Suppose $\mathcal{C}$ is a category in which all small coproducts exist and all coequalizers*

$$\begin{tikzcd} C \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & C' \end{tikzcd}$$

exist. Then $\mathcal{C}$ is cocomplete.

Proof. Suppose that small coproducts exist and all coequalizers exist. Suppose $F\colon \mathcal{B} \to \mathcal{C}$ is a small diagram. We wish to show $\varinjlim F$ exists. Let

$$C' = \coprod_{B \in \mathrm{Ob}(\mathcal{B})} FB \in \mathrm{Ob}(\mathcal{C})$$

Pictorially:

$$\begin{tikzcd} FB \arrow[dr, "i_B"] & FB' \arrow[d, "i_{B'}"] & FB'' \arrow[dl, "i_{B''}"] \\ & C' & \end{tikzcd}$$

Let

$$\mathrm{Mor}(\mathcal{B}) = \bigcup_{B, B' \in \mathrm{Ob}(\mathcal{B})} \hom_{\mathcal{B}}(B, B')$$

Notice that each $\varphi \in \mathrm{Mor}(\mathcal{B})$ has a source and target: if $\varphi\colon B \to B'$, we define $s(\varphi) = B$ and $t(\varphi) = B'$. (A somewhat technical point is that we implicitly require in our definition of a category that these maps be well-defined.) Let

$$C = \coprod_{\varphi \in \mathrm{Mor}(\mathcal{B})} F(s(\varphi)) \in \mathrm{Ob}(\mathcal{C})$$

Pictorially:

$$\begin{tikzcd} FB \arrow[dr, "\alpha_B"] & FB' \arrow[d, "\alpha_{B'}"] & FB'' \arrow[dl, "\alpha_{B''}"] \\ & C & \end{tikzcd}$$

(Note that α_B should really be $\alpha_{B,\varphi}$ where $s(\varphi) = B$; for notational convenience, we instead use α_B.)

We now construct morphisms $\Phi, \Psi\colon C \to C'$ such that we will have $\varinjlim F$ is the coequalizer of Φ and Ψ.

Since each FB has $i_B\colon FB \to C'$, we have that C' together with the i_B is a cocone over $\mathrm{Mor}(\mathcal{B})$; so there is a unique $\Phi\colon C \to C'$ such that the following diagram commutes:

$$\begin{tikzcd} FB \arrow[dr, "\alpha_B"] \arrow[ddr, "i_B"'] & & FB' \arrow[dl, "\alpha_{B'}"'] \arrow[ddl, "i_{B'}"] \\ & C \arrow[d, dashed, "\Phi"] & \\ & C' & \end{tikzcd}$$

It also holds that for each $\varphi \in \hom_{\mathcal{B}}(B, B')$ we have $i_{B'} \circ F(\varphi)\colon FB \to C'$; this yields another cocone to C', and thus we get a unique $\Psi\colon C \to C'$ such that the following diagram commutes:

$$\begin{tikzcd} FB \arrow[dr, "\alpha_B"] \arrow[ddr, "i_{B'} \circ F(\varphi)"'] & \\ & C \arrow[d, dashed, "\Psi"] \\ & C' \end{tikzcd}$$

By assumption, we have that coequalizers exist; so there is an object L and a morphism $v\colon C' \to L$ such that $v \circ \Phi = v \circ \Psi$. We claim that L together with the obvious maps $\gamma_B = v \circ i_B\colon FB \to L$ is a colimit of F.

We first check that (L, γ_B) is a cocone. Suppose $\varphi \in \hom_{\mathcal{B}}(B, B')$. Then

$$\begin{aligned}
\gamma_B &= v \circ i_B \\
&= v \circ \Phi \circ \alpha_B \\
&= v \circ \Psi \circ \alpha_B \\
&= v \circ i_{B'} \circ F\varphi
\end{aligned}$$

So the following diagram commutes:

$$\begin{tikzcd}
FB \arrow[rr, "F\varphi"] \arrow[dr, "\gamma_B"'] & & FB' \arrow[dl, "\gamma_{B'}"] \\
& L &
\end{tikzcd}$$

and (L, γ_B) is indeed a cocone.

Suppose we have another cocone (T, θ_B). Then for each $B \in \mathrm{Ob}(\mathcal{B})$ we have $\theta_B\colon FB \to T$; so, by definition of C', we have a unique $h\colon C' \to T$ such that $h \circ i_B = \theta_B$ for all $B \in \mathrm{Ob}(\mathcal{B})$. We want h to factor through L; i.e. we want a unique $\widetilde{h}\colon L \to T$ such that the following diagram commutes:

$$\begin{tikzcd}
FB \arrow[dr, "i_B"] \arrow[ddr, "\theta_B"'] & FB' \arrow[d, "i_{B'}"] & \\
& C' \arrow[r, "v"] \arrow[d, "h"] & L \arrow[dl, dashed, "\widetilde{h}"] \\
& T &
\end{tikzcd}$$

To get to factor through L we must show that $h \circ \Phi = h \circ \Psi$. But

$$\begin{aligned}
h \circ \Phi \circ \alpha_B &= h \circ i_B \\
&= \theta_B \\
&= \theta_{B'} \circ F(\varphi) \\
&= h \circ i_{B'} \circ F(\varphi) \\
&= h \circ i_{B'} \circ F(\varphi) \\
&= h \circ \Psi \circ \alpha_B
\end{aligned}$$

But by definition of C, we have a *unique* $f\colon C \to T$ such that $\theta_B = f \circ \alpha_B$ for all $B \in \mathrm{Ob}(\mathcal{B})$. So $h \circ \Phi = h \circ \Psi$, and by definition of L as the coequalizer we have our desired $\widetilde{h}$. ☐ Theorem 2.67

Remark 2.68. The exact same argument shows that if $F\colon \mathcal{C} \to \mathcal{D}$ with $\mathcal{C}$ and $\mathcal{D}$ cocomplete satisfies

$$F\left(\coprod_{i \in I} C_i\right) \cong \coprod_{i \in I} FC_i$$

$$F(\mathrm{Coequal}(\ C \overset{f}{\underset{g}{\rightrightarrows}} C'\ )) = \mathrm{Coequal}(\ FC \overset{Ff}{\underset{Fg}{\rightrightarrows}} FC'\ )$$

Then

$$F(\varinjlim D) = \varinjlim FD$$

for all small diagrams $D\colon \mathcal{B} \to \mathcal{C}$.

Corollary 2.69. *The following categories are bicomplete:*

Category	*Product*	*Coproduct*	*Equalizer*	*Coequalizer*
Abelian grapes	$\prod A_i$	$\bigoplus A_i$	$\ker(f-g)$	$\mathrm{coker}(f-g)$
R-modules	$\prod M_i$	$\bigoplus M_i$	$\ker(f-g)$	$\mathrm{coker}(f-g)$
Commutative rings	$\prod R_i$	$\bigotimes_{\mathbb{Z}}^{R} R_i$	$\{\, f(x) = g(x) \,\}$	$R/\langle f(x) - g(x)\rangle$
Grapes	...			

2.10 Govorov-Lazard theorem and filtered subcategories

Recall that an R-module M is *flat* if whenever

$$0 \to N' \xrightarrow{f} N$$

is exact then so is

$$0 \to N' \otimes_R M \to N \otimes_R M$$

Further recall that P is *projective* if $\hom_R(P, -)$ is exact, and I is *injective* if $\hom_R(-, I)$ is exact.

Example 2.70. Free modules are flat.

Theorem 2.71 (Govorov-Lazard)**.** *Let R be a commutative ring and let M be an R-module. Then M is flat if and only if M is a* filtered colimit *of free modules.*

Definition 2.72. Suppose $\mathcal{B}$ is a small category. We say $\mathcal{B}$ is *filtered* if

1. If $B_1, B_2 \in \mathrm{Ob}(\mathcal{B})$ then there is $B \in \mathrm{Ob}(\mathcal{B})$ with $f \in \hom(B_1, B)$ and $g \in \hom(B_2, B)$.
2. If $f \in \hom(B', B_1)$ and $g \in \hom(B', B_2)$ then there are $B'' \in \mathrm{Ob}(\mathcal{B})$ and $u\colon B_1 \to B''$ and $v\colon B_2 \to B''$ such that the following diagram commutes:

$$\begin{tikzcd} & B_1 \arrow[dr, "u"] & \\ B' \arrow[ur, "f"] \arrow[dr, "g"'] & & B'' \\ & B_2 \arrow[ur, "v"'] & \end{tikzcd}$$

If $F\colon \mathcal{B} \to \mathcal{A}$ is a diagram and $\mathcal{B}$ is filtered, we say $\varinjlim F$ is a *filtered colimit*.

Example 2.73 (Filtered limits in R-**Mod**). If $\mathcal{B}$ is a filtered subcategory of R-**Mod**, then what is $\varinjlim \mathcal{B}$? A concrete description is

$$\varinjlim \mathcal{B} = \bigsqcup_{M \in \mathrm{Ob}(\mathcal{B})} M/\sim$$

What is $\sim$? If $x \in M$ and $y \in M'$ then we set $x \sim y$ if and only if $f\colon M \to M''$ and $g\colon M' \to M''$ such that $f(x) = g(y)$. Observe that

1. $\sim$ is an equivalence relation. Reflexivity and symmetry follow immediately; to see transitivity, suppose $x \sim y$ and $y \sim z$, say with $f\colon M \to P$, $g\colon M' \to P$, $h\colon M' \to P'$, and $k\colon M'' \to P'$ such that $f(x) = g(y)$ and $h(y) = k(z)$. Since $\mathcal{B}$ is filtered then we have $Q \in \mathrm{Ob}(\mathcal{B})$ and $u\colon P \to Q$ and $v\colon P' \to Q$ such that the following diagram commutes:

$$\begin{tikzcd} & P \arrow[dr, "u"] & \\ M' \arrow[ur, "g"] \arrow[dr, "h"'] & & Q \\ & P' \arrow[ur, "v"'] & \end{tikzcd}$$

   Then $(u \circ f)(x) = (u \circ g)(y) = (v \circ h)(y) = (v \circ k)(z)$ and $x \sim z$.

2. We have an R-module structure on

$$\bigsqcup_{M \in \mathrm{Ob}(\mathcal{B})} M/\sim$$

In particular, given

$$x, y \in \bigsqcup_{M \in \mathrm{Ob}(\mathcal{B})} M$$

say $x \in M_1$ and $y \in M_2$, we define $x + y$ to be the equivalence class of $f(x) + g(y)$ where we use the fact that $\mathcal{B}$ is filtered to find $N \in \mathrm{Ob}(\mathcal{B})$ and $f\colon M_1 \to N$ and $g\colon M_2 \to N$. One checks that this is well-defined.

3. We have natural maps

$$i_M\colon M \to \bigsqcup_{M \in \mathrm{Ob}(\mathcal{B})} M/\sim$$

Suppose (F, φ_M) is a cocone over $\mathcal{B}$. Suppose $x \sim y$; say $x \in M$, $y \in M'$, $u\colon M \to M''$, $v\colon M' \to M''$ satisfy $u(x) = v(y)$. Then $\varphi_M(x) = \varphi_{M''}(u(x)) = \varphi_{M''}(v(y)) = \varphi_{M'}(y)$. So the φ_M are defined on $\sim$-classes, and thus induce a map

$$\bigsqcup_{M \in \mathrm{Ob}(\mathcal{B})} M/\sim \to F$$

Hence we indeed have

$$\bigsqcup_{M \in \mathrm{Ob}(\mathcal{B})} M/\sim \cong \varinjlim \mathcal{B}$$

Proof of Theorem 2.71. We prove that if the U_i come from is a filtered subcategory $\mathcal{B}$ of R-**Mod** whose objects are free then $\varinjlim U_i$ is flat.

Idea: suppose $0 \to N' \xrightarrow{f} N$ is exact and $M = \varinjlim U_i$. We wish to show that

$$0 \to M \otimes N' \xrightarrow{\mathrm{id} \otimes f} M \otimes N$$

is exact. Let $F\colon \mathcal{B} \to R$-**Mod** be $Q \mapsto Q \otimes N'$ and $G\colon \mathcal{B} \to R$-**Mod** be $Q \mapsto Q \otimes N$. The point is that we get a natural transformation $\alpha\colon F \to G$ given by

$$\begin{array}{ccc} F(U) & \xrightarrow{\alpha_U} & G(U) \\ U \otimes N' & \xrightarrow{\mathrm{id} \otimes f} & U \otimes N \end{array}$$

for $U \in \mathrm{Ob}(\mathcal{B})$. Indeed, if $h\colon U \to U'$ then the following diagram commutes:

$$\begin{tikzcd} F(U) \arrow[r, "\alpha_U"] \arrow[d, "F(h)"] & G(U) \arrow[d, "G(h)"] \\ F(U') \arrow[r, "\alpha_{U'}"] & G(U') \end{tikzcd}$$

since, following $u \otimes n' \in F(U)$ right and down we get

$$u \otimes n' \mapsto u \otimes f(n') \mapsto h(u) \otimes f(n')$$

and following down and right we get

$$u \otimes n' \mapsto h(u) \otimes n' \mapsto h(u) \otimes f(n')$$

The proof is then that, if $M = \varinjlim \mathcal{B}$, then

$$M \otimes N' = (\varinjlim \mathcal{B}) \otimes N' \cong \varinjlim (U_i \otimes N') \xhookrightarrow{\varinjlim f} \varinjlim (U_i \otimes N) \cong (\varinjlim U_i) \otimes N = M \otimes N$$

The isomorphisms follow from the fact that left adjoints preserve colimits and tensor product is a left adjoint; it remains to see that

$$h\colon \varinjlim (U_i \otimes N') \xrightarrow{\varinjlim f} \varinjlim (U_i \otimes N)$$

given by
$$\bigsqcup U_i \otimes N' \xrightarrow{\mathrm{id}\otimes f} \bigsqcup U_i \otimes N/\sim$$
is injective. Suppose
$$x \in \bigsqcup U_i \otimes N'/\sim$$
has $h(x) \sim 0$. Then we have some U_j and $\theta = G(\psi)\colon U_i \otimes N \to U_j \otimes N$ such that $\theta(h(x)) = 0$. But then by naturality of α we have the following diagram commutes:

$$\begin{tikzcd} U_i \otimes N' \arrow[r, "\alpha_{U_i}"] \arrow[d, "F(\psi)"] & U_i \otimes N \arrow[d, "\theta"] \\ U_j \otimes N' \arrow[r, "\alpha_{U_j}"] & U_j \otimes N \end{tikzcd}$$

But α_{U_j} is injective; so $F(\psi)(x) = 0$, and $x \sim 0$. So h is injective as desired. □ Theorem 2.71

3 Abelian categories

Definition 3.1. A *preadditive category* is a category $\mathcal{C}$ is a category in which for all $A, B \in \mathrm{Ob}\,\mathcal{C})$ we have that $\hom_{\mathcal{C}}(A, B)$ has the structure of an abelian grape. (In particlar, there is $0_{A,B}\colon A \to B$ for all $A, B \in \mathrm{Ob}(\mathcal{C})$.) We also require that
$$\circ_{A,B,C}\colon \hom_{\mathcal{C}}(B, C) \times \hom_{\mathcal{C}}(A, B) \to \hom_{\mathcal{C}}(A, C)$$
be bilinear (as a homomorphism of $\mathbb{Z}$-modules) for all $A, B, C \in \mathrm{Ob}(\mathcal{C})$.

Example 3.2. Suppose R is a ring. Define a category with R as the unique object and morphisms $\varphi_r\colon R \to R$ for $r \in R$ given by $\varphi_r(x) = rx$. Then

- $\varphi_0 = 0_{r,r}$
- $(\varphi_r + \varphi_s) \circ \varphi_t = \varphi_{rt} + \varphi_{st} = \varphi_r \circ \varphi_t + \varphi_s \circ \varphi_t$
- $\varphi_r \circ (\varphi_s + \varphi_t) = \varphi_{rs} + \varphi_{rt} = \varphi_r \circ \varphi_s + \varphi_r \circ \varphi_t$

So this category is preadditive.

Definition 3.3. Suppose $\mathcal{C}$ and $\mathcal{D}$ are preadditive categories. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is called *additive* if the map $f \mapsto F(f)$ gives a homomorphism $\hom_{\mathcal{C}}(A, B) \to \hom_{\mathcal{D}}(FA, FB)$ for all $A, B \in \mathrm{Ob}(\mathcal{C})$.

Definition 3.4. A preadditive category is *additive* if all finite (including empty) products and coproducts exist.

Remark 3.5. If $\mathcal{C}$ is additive and $A, B \in \mathrm{Ob}(\mathcal{C})$ then $A \prod B \cong A \coprod B$.

Proof. We are given $p_A\colon A \prod B \to A$, $p_B\colon A \prod B \to B$, $i_A\colon A \to A \coprod B$, and $i_R\colon B \to A \coprod B$. Drawing inspiration from familiar abelian categories, our isomorphism $\theta\colon A \prod B \to A \coprod B$ should be $i_A \circ p_A + i_B \circ p_B$. To get its inverse, note that we have a map $\mu_A\colon A \to A \prod B$ induced by the cone $\mathrm{id}_A\colon A \to A$ and $0_{A,B}\colon A \to B$; likewise we gat a map $\mu_B\colon B \to A \prod B$.

Claim 3.6. *$A \prod B$ is a coproduct.*

Proof. Suppose we have $f\colon A \to C$, $g\colon B \to C$; we wish to find unique $\theta\colon A \prod B \to C$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[dr, "\mu_A"] \arrow[ddr, "f"'] & & B \arrow[dl, "\mu_B"'] \arrow[ddl, "g"] \\ & A \prod B \arrow[d, "\theta"] & \\ & C & \end{tikzcd}$$

What should θ be? It should be $f \circ p_A + g \circ p_B$. We must show $f = \theta \circ \mu_A$ and $g = \theta \circ \mu_B$. But

$$\begin{aligned}
\theta \circ \mu_A &= (f \circ p_A + g \circ p_B) \circ \mu_A \\
&= f \circ (p_A \circ \mu_A) + g \circ (p_B \circ \mu_A) \\
&= f \circ \mathrm{id}_A + g \circ 0 \\
&= f + 0 \\
&= f
\end{aligned}$$

and similarly we get $g = \theta \circ \mu_B$.

It remains to check that θ is unique. Suppose θ and θ' both make the above diagram commute; so $\theta \circ \mu_A = \theta' \circ \mu_A = f$ and $\theta \circ \mu_B = \theta' \circ \mu_B = g$. Let $\psi \colon A \prod B \to C$ be $\psi = \theta - \theta'$; then $\psi \circ \mu_A = \psi \circ \mu_B = 0$.

Subclaim 3.7. $\mu_A \circ p_A + \mu_B \circ p_B = \mathrm{id}_{A \prod B}$.

Proof. Recall that

$$\begin{aligned}
p_A \circ \mu_A &= \mathrm{id}_A \\
p_B \circ \mu_B &= \mathrm{id}_B \\
p_A \circ \mu_B &= 0 \\
p_B \circ \mu_A &= 0
\end{aligned}$$

But then the following diagram commutes:

$$\begin{tikzcd}
& A \prod B \arrow[d, "\mu_A \circ p_A + \mu_B \circ p_B"] \arrow[ddl, "p_A"'] \arrow[ddr, "p_B"] & \\
& A \prod B \arrow[dl, "p_A"] \arrow[dr, "p_B"'] & \\
A & & B
\end{tikzcd}$$

since

$$p_A \circ (\mu_A \circ p_A + \mu_B \circ p_B) = \mathrm{id}_A \circ p_A + 0 = p_A$$

and likewise with p_B. But by the universal property of products we have that $\mathrm{id}_{A \prod B}$ is the unique morphism $A \prod B \to A \prod B$ making the above diagram commute. So $\mathrm{id}_{A \prod B} = \mu_A \circ p_A + \mu_B \circ p_B$, as desired. ☐ Subclaim 3.7

Then

$$\psi = \psi \circ \mathrm{id}_{A \prod B} = \psi \circ (\mu_A \circ p_A + \mu_B \circ p_B) = (\psi \circ \mu_A) \circ p_A + (\psi \circ \mu_B) \circ p_B = 0$$

and $\theta = \theta'$. ☐ Claim 3.6

The isomorphism then follows by uniqueness of coproducts. ☐ Remark 3.5

Remark 3.8. We also have a zero object. Why? The empty coproduct yields an initial object I, and the empty product gives a final object T.

Claim 3.9. $I \cong T$.

Proof. $0_{T,I} \colon T \to I$ and $0_{I,T} \colon I \to T$; the fact that id_T is the unique morphism $T \to T$ and id_I is the unique morphism $I \to I$ yields that $0_{I,T} \circ 0_{T,I} = \mathrm{id}_T$ and $0_{T,I} \circ 0_{I,T} = \mathrm{id}_I$. So $0_{I,T} \colon I \to T$ is an isomorphism. ☐ Claim 3.9

Remark 3.10. Notice if $f \colon A \to B$ then the limit of the diagram:

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "0_{A,B}"'] & B
\end{tikzcd}$$

is the equalizer of f and 0, which we think of as roughly $\{\, x \in A : f(x) = 0 \,\}$.

Definition 3.11. If the equalizer of

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "0_{A,B}"'] & B \end{tikzcd}$$

exists, we call it the *kernel* of f. If the coequalizer exists, we call it the *cokernel*.

Definition 3.12. An additive category in which kernels and cokernels exist is called *pre-abelian*.

Definition 3.13. A map $f\colon A \to B$ is called a *monomorphism* (which we think of as similar to injectivity) if whenever $f \circ h_1 = f \circ h_2$ we also have $h_1 = h_2$. We say f is an *epimorphism* if whenever $h_1 \circ f = h_2 \circ f$ we also have $h_1 = h_2$.

Example 3.14. A morphism can be a monomorphism and an epimorphism without being an isomorphism. Indeed, consider **Ring** with $\mathbb{Z} \overset{i}{\hookrightarrow} \mathbb{Q}$. It is clear that i is a monomorphism.

Claim 3.15. *$h_1 \circ i = h_2 \circ i$ implies $h_1 = h_2$.*

Proof. We are given that $h_1(n) = h_2(n)$ for all $n \in \mathbb{Z}$. Then

$$1 = h_1(1) = h_1(b)h_1(b^{-1}) = h_1(b)h_2(b^{-1}) = 1$$

so $h_1(b^{-1}) = h_2(b^{-1})$; thus

$$h_1(ab^{-1}) = h_1(a)h_1(b^{-1}) = h_2(a)h_2(b^{-1}) = h_2(ab^{-1})$$

So $h_1 = h_2$. □ Claim 3.15

Definition 3.16. A monomorphism $f\colon A \to B$ is *normal* if f is a kernel; i.e. there is $g\colon B \to C$ such that (A, f) is the kernel of g. Dually, an epimorphism $g\colon B \to C$ is *normal* if g is a cokernel.

An *abelian category* is a pre-abelian category in which every monomorphism is *normal* and every epimorphism is *normal*.

Exercise 3.17. This implies that $f\colon A \to B$ admits a factorization

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "u"'] & & B \\ & \mathrm{im}(f) \arrow[ur, "v"'] & \end{tikzcd}$$

where u is an epimorphism and v is a monomorphism.

What is $\mathrm{im}(f)$? It must be $\ker(\mathrm{coker}(f))$.

Example 3.18. Suppose R is a ring with unity (not necessarily commutative). Then R-**Mod**, the category of left R-modules is an abelian category.

Remark 3.19. In R-**Mod**, monomorphisms are exactly injective homomorphisms. Indeed, if $f\colon M \to N$ is a monomorphism and $i\colon \ker(f) \hookrightarrow M$ then $f \circ i = f \circ 0$; so since f is a monomorphism we have $i = 0$, and $\ker(f) = 0$, and f is injective.

Dually, we get that epimorphisms are surjective.

3.1 Mitchell's embedding lemma

We wish to get a notion of exactness. Suppose

$$A \xrightarrow{f} B \xrightarrow{g} C$$

What does it mean to say that this is exact at B?

1. $g \circ f = 0$

2. The canonical map $\widetilde{f}\colon \operatorname{im}(f) \to \ker(g)$ is an isomorphism.

What is the canonical map? Well, let $\pi\colon B \twoheadrightarrow \operatorname{coker}(f)$ and $i\colon \operatorname{im}(f) = \ker(\pi) \hookrightarrow B$ be the canonical maps. Then $\pi \circ f = 0$, so by the universal property of $\ker(\pi)$ we have a unique $\theta\colon A \to \operatorname{im}(f)$ such that the following diagram commutes:

$$\begin{tikzcd}
 & \operatorname{im}(f) \arrow[d, "i"] \\
A \arrow[ur, dashed, "\theta"] \arrow[r, "f"] & B
\end{tikzcd}$$

In fact θ is an epimorphism and i is a monomorphism. But

$$\begin{aligned}
0 = g \circ f &\implies g \circ i \circ \theta = 0 \\
&\implies g \circ i \circ \theta = 0 \circ \theta \\
&\implies g \circ i = 0
\end{aligned}$$

since θ is an epimorphism. So, by the universal property of $\ker(g)$, we have a unique map $\widetilde{f}\colon \operatorname{im}(f) \to \ker(g)$ such that the following diagram commutes:

$$\begin{tikzcd}
\operatorname{im}(f) \arrow[dr, dashed, "\widetilde{f}"] \arrow[dd, "i"] & \\
 & \ker(g) \arrow[dl] \\
B &
\end{tikzcd}$$

Remark 3.20. I think this is equivalent to requiring that the map $\operatorname{im}(f) \to B$ be the kernel of g.

Definition 3.21. Suppose $F\colon \mathcal{C} \to \mathcal{D}$ is a functor. We say F is:

- *full* if $F\colon \hom_{\mathcal{C}}(A, B) \to \hom_{\mathcal{D}}(FA, FB)$ is surjective for all $A, B \in \operatorname{Ob}(\mathcal{C})$.
- *faithful* if $F\colon \hom_{\mathcal{C}}(A, B) \to \hom_{\mathcal{D}}(FA, FB)$ is injective for all $A, B \in \operatorname{Ob}(\mathcal{C})$.
- *exact* if F is additive and if whenever we have
$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$
exact then
$$0 \to FA \xrightarrow{Ff} FB \xrightarrow{Fg} C \to 0$$
is exact.

Lemma 3.22 (Mitchell's theorem)**.** *Suppose $\mathcal{A}$ is a small abelian category. Then there is $F\colon \mathcal{A} \to R$-**Mod** where R is a ring and F is full, faithful, and exact.*

If we start with R-**Mod**, can we recover R?

Remark 3.23. If $\mathcal{A}$ is an abelian category and $A \in \operatorname{Ob}(\mathcal{A})$ then $\hom_{\mathcal{A}}(A, A) \cong \operatorname{End}_{\mathcal{A}}(A)$ is a ring under $\circ$. In R-**Mod**, if we consider R as a left R-module, then $\operatorname{End}_R(R) \cong R^{\mathrm{op}}$ (where R^{op} is R with $r \cdot_{R^{\mathrm{op}}} s = s \cdot_R r$). Indeed, given $\psi \in \operatorname{End}_R(R)$, we have that ψ is determine by $\psi(1)$ since if $\psi(1) = s$ then $\psi(r) = r\psi(1) = rs$. So $\psi = \Phi_s$ for some $s \in R$ where $\Phi_s(x) = xs$. So

$$\operatorname{End}_R(R) \cong \{\, \Phi_s : s \in R \,\} \cong R^{\mathrm{op}}$$

(where the opposite ring comes because $(\Phi_s \circ \Phi_r)(x) = xrs = \Phi_{rs}(x)$).

However, we can have $R \ncong S$ with R-**Mod** $\cong S$-**Mod**.

Example 3.24. R-**Mod** $\cong M_n(R)$-**Mod**.

We might remark, though, that given a free module R^n we have $\mathrm{End}_R(R^n) \cong M_n(R^{\mathrm{op}})$, and thus $\mathrm{End}_R(R^n)^{\mathrm{op}} \cong M_n(R)$; so we might look and the endomorphism ring of free modules. Being a free module, however, is not categorically definable. We instead turn to projective modules:

Definition 3.25. Suppose $\mathcal{A}$ is an abelian category and $M \in \mathrm{Ob}(\mathcal{A})$. We get a functor $\hom(M, -)\colon \mathcal{A} \to \mathbf{Ab}$ by $B \mapsto \hom_{\mathcal{A}}(M, B)$. We say that M is a *projective object* of $\mathcal{A}$ if the functor $\hom(M, -)$ is exact.

What are the projectives in R-**Mod**? Well, one checks that for all P we have $\hom(P, -)$ is left-exact. When is $\hom(P, -)$ right-exact? We need that given exact $M \xrightarrow{g} N \to 0$ that $\hom(P, M) \to \hom(P, N) \to 0$ is exact; i.e. given any $\varphi\colon P \to N$ there is $\psi\colon P \to N$ such that the following diagram commutes:

$$\begin{tikzcd}
M \arrow[r, "g"] & N \\
 & P \arrow[u, "\varphi"] \arrow[ul, dashed, "\psi"]
\end{tikzcd}$$

Remark 3.26. P is projective implies there is Q such that $P \oplus Q \cong R^I$. Indeed, consider $\pi\colon R^I \twoheadrightarrow P$; then since $\mathrm{id}_P\colon P \to P$ we have $s\colon P \to R^I$ such that the following diagram commutes:

$$\begin{tikzcd}
R^I \arrow[r, two heads, "\pi"] & P \\
 & P \arrow[u, "\mathrm{id}_P"'] \arrow[ul, dashed, "s"]
\end{tikzcd}$$

The proof is somewhat involved, so we merely give an overview.

A starting result:

Theorem 3.27. *Suppose $\mathcal{L}$ is a cocomplete abelian category with a projective generator (i.e. $\overline{P}$ such that $\hom(\overline{P}, -)$ is exact and faithful). If $\mathcal{A} \subseteq \mathcal{L}$ (i.e. with $I\colon \mathcal{A} \to \mathcal{L}$ exact) is a small abelian subcategory then there is fully faithful and exact $F\colon \mathcal{A} \to R$-**Mod**.*

Remark 3.28. In R-**Mod**, we have that R is a projective generator.

Our strategy is then to take $\mathcal{A}$, find $\mathcal{B}$ complete, containing $\mathcal{A}$, and having a projective generator, and then apply the theorem.

Remark 3.29. $\hom(\overline{P}, -)$ is an additive functor.

Remark 3.30. Not all projectives are generators. Consider for example $R = \mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/3\mathbb{Z}$; then $P = \mathbb{Z}/2\mathbb{Z}$ is projective and not a generator.

Proof of Theorem 3.27. Suppose $A \in \mathrm{Ob}(\mathcal{A})$; consider

$$\coprod_{g \in \hom(\overline{P}, A)} \overline{P}$$

We get $i_g\colon \overline{P} \to \coprod_g \overline{P}$ for each $g \in \hom(\overline{P}, A)$. Furthermore, since the $g\colon \overline{P} \to A$ form a cocone, we get $p_A\colon \coprod_g \overline{P} \to A$ such that $p_a \circ i_g = g$ for all $g \in \hom(\overline{P}, A)$.

Claim 3.31. *p_A is an epimorphism.*

Proof. In an abelian category, it suffices to verify that if $h \circ p_A = 0$ then $h = 0$ for all $h\colon A \to B$. Suppose then that $h \circ p_A = 0$. Then $h \circ p_A \circ i_g = 0$ for all $g \in \hom(\overline{P}, A)$; so $h \circ g = 0$ for all $g\colon \overline{P} \to A$. So $\hom(\overline{P}, h) = 0\colon \hom(\overline{P}, A) \to \hom(\overline{P}, B)$. But $\hom(\overline{P}, -)$ is faithful since $\overline{P}$ is a generator. So $h = 0$. So p_A is an epimorphism. □ Claim 3.31

Now, let

$$I = \bigsqcup_{A \in \mathrm{Ob}(\mathcal{A})} \hom(\overline{P}, A)$$
$$P = \coprod_I \overline{P}$$

From assignment 3, we will see:

1. P is a projective generator.
2. For all $A \in \mathrm{Ob}(\mathcal{A})$ there is an epimorphism $\theta \colon P \to A$.

Now we can find a ring R:

$$R = \mathrm{End}\left(\coprod_I \overline{P}\right)^{\mathrm{op}} = \mathrm{End}(P)^{\mathrm{op}}$$

Claim 3.32. *There is $F \colon \mathcal{A} \to R$-**Mod** fully faithful and exact given by $M \mapsto \hom(P, M)$ for $M \in \mathrm{Ob}(\mathcal{A})$.*

Proof. We first need to define an R-module structure on $\hom(P, M)$. Well, $R = \mathrm{End}(P)^{\mathrm{op}} = \hom(P, P)^{\mathrm{op}}$. Given $r \in R$ and $\psi \in \hom(P, M)$, we can then set $r \cdot \psi = \psi \circ r \in \hom(P, M)$; bilinearity and associativity of composition yield that this is in fact an R-module structure.

We also need to check that the images of morphisms are morphisms of R-modules. Suppose $f \colon M \to N$ for $M, N \in \mathrm{Ob}(\mathcal{A})$. We must check that $\hom(P, f) \colon \hom(P, M) \to \hom(P, N)$ (given by $g \mapsto f \circ g$) is a homomorphism of R-modules. Additivity follows from bilinearity of composition; for scalar multiplication, note that for $r \in R$ we have

$$r \cdot (\hom(P, f)(g)) = r \cdot (f \circ g) = (f \circ g) \circ r = f \circ (g \circ r) = r \circ (r \cdot g) = \hom(P, f)(r \cdot g)$$

Now we must check that F is fully faithful and exact. Projectivity of P immediately yields exactness; that P is a generator immediately yields faithfulness. It remains to check that F is full.

Suppose then that $\alpha \colon \hom(P, M) \to \hom(P, N)$; we wish to find $f \colon M \to N$ such that $\alpha = \hom(P, f)$. Now we use the second result from the assignment to get epimorphisms $\theta \colon P \to M$ and $\psi \colon P \to N$. Let $K = \ker(\theta)$; then

$$0 \to K \to P \xrightarrow{\theta} M \to 0$$

is a short exact sequence. Since $\hom(P, -)$ is exact, we get

$$0 \to \hom(P, K) \to \hom(P, P) \xrightarrow{\hom(P, \theta)} \hom(P, M) \to 0$$

is exact. But $\hom(P, P) \cong R$ as left R-modules, as one sees by looking at the R-module structure we defined. So

$$\begin{tikzcd}
0 \arrow[r] & \hom(P, K) \arrow[r] & R \arrow[r, "\hom(P,\theta)"] & \hom(P, M)A \arrow[r] \arrow[d, "\alpha"] & 0 \\
& & R \arrow[r, "\hom(P,\psi)"] & \hom(P, N) \arrow[r] & 0
\end{tikzcd}$$

Fact 3.33. *R is projective.*

So there is $\alpha' \colon R \to R$ such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & \hom(P, K) \arrow[r] & R \arrow[r, "\hom(P,\theta)"] \arrow[d, dashed, "\alpha'"] & \hom(P, M)A \arrow[r] \arrow[d, "\alpha"] & 0 \\
& & R \arrow[r, "\hom(P,\psi)"] & \hom(P, N) \arrow[r] & 0
\end{tikzcd}$$

But $\alpha'\colon R \to R$ is a morphism; so $\alpha' = \rho_s$ is right multiplication by some $s \in R$. Now look at the diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] & P \arrow[r] \arrow[d, "s"] & M \arrow[r] & 0 \\ & & P \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

Consider

$$\begin{tikzcd} K \arrow[r] & P \arrow[d, "s"] & & \\ & P \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

We claim that $K \to P \xrightarrow{s} P \to N$ is the 0 morphism. Why? Well,

$$\hom(P, K) \to R \xrightarrow{\rho_s} R \to \hom(P, N)$$

is the 0 map by the preceding commutative diagram and $\hom(P, -)$ is faithful.

Now, $M = \operatorname{coker}(K \to P)$, and $K \to P \xrightarrow{s} P \xrightarrow{\psi} N$ is the 0 map; so there is $h\colon M \to N$; apply $\hom(P, -)$ and use the fact that $\hom(P, \theta)$ is an epimorphism to conclude that $\alpha = \hom(P, h)$. $\square$ Claim 3.32

$\square$ Theorem 3.27

3.2 Projective modules

Definition 3.34. Given a ring R we define R-**Mod** to be the category of left R-modules; we define **Mod**(R) to be the category of right R-modules.

Definition 3.35. Recall that an R-module P is *projective* if $\hom(P, -)\colon R\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod}$ is exact. We know it is left exact; so it is equivalent to requiring that given any surjection $g\colon M \twoheadrightarrow N$ and any $\varphi\colon P \to N$, there is $\psi\colon P \to M$ such that the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, two heads, "g"] & N \arrow[r] & 0 \\ & P \arrow[u, "\varphi"] \arrow[ul, dashed, "\psi"] & \end{tikzcd}$$

Theorem 3.36. *Suppose P is an R-module. Then the following are equivalent:*

1. *We have the condition above; namely that given any surjection $g\colon M \twoheadrightarrow N$ and any $\varphi\colon P \to N$ there is $\psi\colon P \to M$ such that the following diagram commutes:*

$$\begin{tikzcd} M \arrow[r, two heads, "g"] & N \arrow[r] & 0 \\ & P \arrow[u, "\varphi"] \arrow[ul, dashed, "\psi"] & \end{tikzcd}$$

2. *Every short exact sequence*

$$0 \to M \xrightarrow{f} N \xrightarrow{g} P \to 0$$

splits.

3. *There is an R-module Q such that $P \oplus Q$ is free.*

4. *The functor $\hom(P, -)$ is exact.*

Proof.

(1) $\implies$ (2) By (1) we get $s\colon P \to N$ such that the following diagram commutes:

$$\begin{tikzcd} N \arrow[r, "g"] & P \arrow[r] & 0 \\ & P \arrow[u, "\mathrm{id}"] \arrow[ul, dashed, "s"] & \end{tikzcd}$$

So we have s such that $g \circ s = \mathrm{id}_P$. Now define $\psi\colon P \oplus M \to N$ by $(p, m) \to s(p) + f(m)$. One checks that ψ is an isomorphism; so the short exact sequence splits.

(2) $\implies$ (3) Pick a free module F with $F \xrightarrow{g} P \to 0$ exact. Let $Q = \ker(F \xrightarrow{g} P)$. So

$$0 \to Q \to F \to P \to 0$$

is exact. By (2), this splits, and $F \cong P \oplus Q$.

(3) $\implies$ (4) Suppose

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

is exact. We know that $\hom(P, -)$ is left exact; it remains to show that $\hom(P, g)\colon \hom(P, M) \to \hom(P, M'')$ (given by $\psi \mapsto g \circ \psi$) is surjective. Suppose $h\colon P \to M''$; we must show that there is $h'\colon P \to M$ such that $h = g \circ h'$. By (3) we may find an R-module Q such that $F = P \oplus Q$ is free. Define $h_0\colon F \to M''$ by $h_0 \restriction P = h$ and $h_0 \restriction Q = 0$. Then because F is free there is $h_0'\colon F \to M$ such that $g \circ h_0' = h_0$; i.e. the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "g"] & M'' \arrow[r] & 0 \\ & F \arrow[u, "h_0"] \arrow[ul, "h_0'"] & \end{tikzcd}$$

Now let $h' = h_0' \restriction P$. Then

$$g \circ h' = g \circ (h_0' \restriction P) = (g \circ h_0') \restriction P = h_0 \restriction P = h$$

(4) $\implies$ (1) Immediate, since (1) just requires that whenever $M \twoheadrightarrow N \to 0$ is exact then so is $\hom(P, M) \to \hom(P, N) \to 0$. □ Theorem 3.36

Example 3.37. Let $R = \mathbb{Z} \times \mathbb{Z}$; let $P = \mathbb{Z} \times \{ 0 \}$. Then P is not free since $(0,1) \cdot P = (0)$, so $\mathrm{Ann}(P) = \{ 0 \} \times \mathbb{Z}$ is non-trivial. But if $Q = \{ 0 \} \times \mathbb{Z}$ then $P \oplus Q = R$ is free; so P is projective.

We now consider the commutative situation. Suppose $(R, \mathfrak{m})$ is a (commutative) local ring (i.e. $\mathfrak{m}$ is the unique maximal ideal).

Theorem 3.38 (Kaplansky)**.** *If P is a projective R-module then P is free.*

Theorem 3.39. *Suppose $(R, \mathfrak{m})$ is a local ring; suppose P is a finitely generated, projective R-module. Then P is free.*

Proof. Let $p_1, \ldots, p_s$ be a generating set for P with s minimal. Let

$$\begin{array}{rc} g\colon & \underbrace{R \oplus \ldots \oplus R}_{s \text{ times}} \twoheadrightarrow P \\ & (0, 0, \ldots, 0, \underbrace{1}_{i^{\text{th}}}, 0, \ldots, 0) \mapsto p_i \end{array}$$

Let $Q = \ker(g)$. Then

$$0 \to Q \xrightarrow{i} R^s \xrightarrow{g} P \to 0$$

is exact. Since P is projective, we get that $R^s \cong Q \oplus P$. Let $K = R/\mathfrak{m}$; then K is a field. Applying $- \otimes_R K$ to the above isomorphism, we get

$$K^s = (R/\mathfrak{m}R)^s \cong R^s/\mathfrak{m}R^s \cong P/\mathfrak{m}P \oplus Q/\mathfrak{m}Q$$

Claim 3.40. *$P/\mathfrak{m}P \cong K^s$.*

Proof. Suppose not; then, since these are vector spaces over K, we have $P/\mathfrak{m}P \cong K^t$ for some $t < s$ (since $P/\mathfrak{m}P \subseteq K^s$). Pick $a_1, \dots, a_t \in P$ such that $\overline{a_1}, \dots, \overline{a_t} \in P/\mathfrak{m}P$ form a K-basis (i.e. an $R/\mathfrak{m}$-basis). Now let

$$P_0 = Ra_1 + \dots + Ra_t \subsetneq P$$

(The containment is proper because $t < s$ and we chose s to be minimal.) Now let $N = P/P_0 \neq (0)$. Then N is finitely generated since P is finitely generated. What is $\mathfrak{m}N$? Well, notice $P = \mathfrak{m}P + P_0$, since $\overline{P_0} = P/\mathfrak{m}P$. So

$$\mathfrak{m}N = (\mathfrak{m}P + P_0)/P_0 = P/P_0 = N$$

But $\mathfrak{m} = J(R)$ and N is finitely generated; so, by Nakayama's lemma, we get $N = (0)$, a contradiction. □ Claim 3.40

Then since

$$\underbrace{K^s}_{s \text{ dimensional}} = \underbrace{(P/\mathfrak{m}P)}_{s \text{ dimensional}} \oplus (Q/\mathfrak{m}Q)$$

and these are vector spaces over K, we have $Q/\mathfrak{m}Q = 0$. So $Q = \mathfrak{m}Q$. But $\mathfrak{m} = J(R)$, and Q is a direct summand of a finitely generated module, and is thus finitely generated; so, by Nakayama's lemma, we have $Q = (0)$. But $R^s = P \oplus Q$; so $R^s = P$, and P is free. □ Theorem 3.39

Remark 3.41. If R is a PID and P is projective then P is free.

Proof. We prove the case where P is finitely generated. Then by the fundamental theorem for finitely generated modules over a PID, we have $P = R^m \oplus T$, where T is torsion; in particular, we get

$$T = \oplus_I R/I$$

for some collection of ideals I of R. But we say that there is Q finitely generated such that $P \oplus Q \cong R^L$. (In particular, we pick $g \colon R^L \twoheadrightarrow P$, and let $Q = \ker(g)$; then L is the number of generators of P.) Since Q is finitely generated, we have

$$Q \cong R^n \oplus T'$$

where T' is torsion. Then

$$R^L \cong P \oplus Q \cong (R^m \oplus T) \oplus (R^n \oplus T') \cong (R^m \oplus R^n) \oplus (T \oplus T')$$

But R^L is free, and thus has no torsion; so $T = T' = (0)$. So P is free. □ Remark 3.41

Theorem 3.42 (Bass)**.** *Suppose R is a commutative Noetherian ring such that* 0 *and* 1 *are the only idempotents. Suppose P is a projective R-module that is not finitely generated. Then P is free.*

Definition 3.43. Suppose R is a ring. Recall that the *spectrum* of R is $\operatorname{Spec}(R) = \{\, \mathfrak{p} : \mathfrak{p} \text{ a prime ideal of } R \,\}$. We put a topology on $\operatorname{Spec}(R)$ called the *Zariski topology* by declaring the closed sets to be $\{\, \mathfrak{p} : \mathfrak{p} \supseteq I \,\}$ for $I \trianglelefteq R$. We define the *principal open sets* to be $V(f) = \{\, \mathfrak{p} : f \notin \mathfrak{p} \,\}$.

Definition 3.44. Suppose S is a multiplicatively closed subset of R with $0 \notin S$. We set $S^{-1}R = \{\, s^{-1}r : s \in S, r \in R \,\}$ where $s^{-1}r = (r, s)$ and $(r_1, s_1) \sim (r_2, s_2)$ if and only if $s_3(r_1 s_2 - s_1 r_2) = 0$. If M is an R-module, then we define $S^{-1}M = M \otimes_R S^{-1}R$; then elements of $S^{-1}M$ take the form $s^{-1}m = (s, m)$ for $s \in S$ and $m \in M$, where $(s_1, m_1) \sim (s_2, m_2)$ if and only if $s_3(s_1 m_2 - s_2 m_1) = 0$ for some $s_3 \in S$. For $\mathfrak{p} \in \operatorname{Spec}(R)$, we define $M_\mathfrak{p} = R_\mathfrak{p} \otimes_R M$ and $R_\mathfrak{p} = S^{-1}R$ with $S = \{\, x \in R : x \notin \mathfrak{p} \,\}$. If $f \in R$ is not nilpotent, we define $M_f = R_f \otimes_R M$ and $R_f = S^{-1}R$ with $S = \{\, 1, f, f^2, \dots \}$.

Theorem 3.45. *Suppose R is a commutative Noetherian ring; suppose P is a finitely generated R-module. Then the following are equivalent:*

1. *P is projective.*
2. *$P_\mathfrak{p} = P \otimes_R R_\mathfrak{p}$ is free for all $\mathfrak{p} \in \operatorname{Spec}(R)$.*

3. $P_{\mathfrak{m}} = P \otimes_R R_{\mathfrak{m}}$ *is free for all maximal ideals* $\mathfrak{m}$ *of* R.

Proof.

(1) $\implies$ **(2)** If P is finitely generated and projective then we have $n \geq 1$ and a surjection $g\colon R^n \twoheadrightarrow P$. If $Q = \ker(g)$, then
$$0 \to Q \to R^n \to P \to 0$$
is exact. Then, since P is projective, we have $R^n \cong Q \oplus P$. Applying $- \otimes_R R_{\mathfrak{p}}$ we see that
$$\begin{aligned} R_{\mathfrak{p}}^n &\cong (R \otimes_R R_{\mathfrak{p}})^n \\ &\cong R^n \otimes_R R_{\mathfrak{p}} \\ &\cong (P \oplus Q) \otimes_R R_{\mathfrak{p}} \\ &\cong P_{\mathfrak{p}} \oplus Q_{\mathfrak{p}} \end{aligned}$$
So $P_{\mathfrak{p}}$ is a direct summand of a free module. So $P_{\mathfrak{p}}$ is projective. So $P_{\mathfrak{p}}$ is free (since $R_{\mathfrak{p}}$ is a local ring and $P_{\mathfrak{p}}$ is finitely generated).

(2) $\implies$ **(3)** Clear, since $\mathfrak{m}$ maximal implies $\mathfrak{m}$ is prime.

(3) $\implies$ **(1)** Suppose $P_{\mathfrak{m}}$ is free (and of finite rank) for all maximal ideals $\mathfrak{m}$ of R. Recall that P is projective if and only if whenever $M \xrightarrow{g} M' \to 0$ is exact then $\hom(P, M) \to \hom(P, M') \to 0$ (given by $\psi \mapsto g \circ \psi$) is exact. (i.e. $\hom(P, -)$ is exact.)

Our strategy: let $g\colon M \twoheadrightarrow M'$ be epi; we will show that $\hom(P, M) \twoheadrightarrow \hom(P, M')$ is epi. Suppose now that $M \xrightarrow{g} M' \to 0$ is exact. Let $\mathfrak{m}$ be a maximal ideal. Then, by right exactness of $- \otimes_R R_{\mathfrak{m}}$, we have
$$M_{\mathfrak{m}} = M \otimes_R R_{\mathfrak{m}} \xrightarrow{g \otimes \mathrm{id}} M'_{\mathfrak{m}} = M' \otimes_R M_{\mathfrak{m}} \to 0$$
is exact. Since $P_{\mathfrak{m}}$ is projective, we get
$$\hom_{R_{\mathfrak{m}}}(P_{\mathfrak{m}}, M_{\mathfrak{m}}) \xrightarrow{g \otimes -} \hom_{R_{\mathfrak{m}}}(P_{\mathfrak{m}}, M'_{\mathfrak{m}})$$
By assigment 3, since $R_{\mathfrak{m}}$ is a flat R-module and P is finitely presented, we have
$$\hom_{R_{\mathfrak{m}}}(P_{\mathfrak{m}}, M_{\mathfrak{m}}) \cong \hom_R(P, M) \otimes_R R_{\mathfrak{m}} = \hom_R(P, M)_{\mathfrak{m}}$$
(We say P is *finitely presented* if there is an exact sequence $R^m \to R^n \to P \to 0$.)

TODO 2. *Why is P finitely presented?*

So $\hom(P, M)_{\mathfrak{m}} \xrightarrow{g \circ -} \hom(P, M')_{\mathfrak{m}}$ is surjective for all maximal ideals $\mathfrak{m}$.

Claim 3.46. *Suppose* R *is commutative and Noetherian. Suppose* M_1, M_2 *are finitely generated modules with* $g\colon M_1 \to M_2$ *a homomorphism. Suppose* $(M_1)_{\mathfrak{m}} \xrightarrow{g} (M_2)_{\mathfrak{m}}$ *is surjective for all maximal ideals* $\mathfrak{m}$*. Then* g *is surjective.*

Proof. Let $K = \operatorname{coker}(g)$; then
$$M_1 \xrightarrow{g} M_2 \to K \to 0$$
is exact. So, by right exactness of $- \otimes_R R_{\mathfrak{m}}$, we have that
$$(M_1)_{\mathfrak{m}}) \xrightarrow{g} (M_2)_{\mathfrak{m}}) \to K_{\mathfrak{m}} \to 0$$
is exact for all maximal ideals $\mathfrak{m}$. Since
$$(M_1)_{\mathfrak{m}}) \xrightarrow{g} (M_2)_{\mathfrak{m}} \to 0$$
is exact, we have $K_{\mathfrak{m}} = (0)$ for all maximal $\mathfrak{m}$. But for $k \in K$ we have $1^{-1}k \sim 1^{-1}0$ in $K_{\mathfrak{m}}$ if and only if there is $s \notin M$ such that $sk = 0$. Since M_2 is finitely generated, we have $K \cong M_2/\operatorname{im}(g)$ is finitely

generated; let $k_1, \ldots, k_r$ be a set of generators. If $\mathfrak{m}$ is maximal, then the above implies that there are $s_1, \ldots, s_r \notin \mathfrak{m}$ such that $s_i k_i = 0$ for all i. Let $s = s_1 \ldots s_r \notin \mathfrak{m}$; then $sk_i = 0$ for all i. So $sK = 0$ since $k_1, \ldots, k_r$ generate K.

So for all maximal ideals $\mathfrak{m}$ of R there is $s_{\mathfrak{m}} \notin \mathfrak{m}$ such that $s_{\mathfrak{m}} \cdot K = 0$. Now, let $I = \{ s \in R : s \cdot K = 0 \}$. This is an ideal of K (namely $\operatorname{Ann}(K)$), and if I were proper, then it would be contained in a maximal ideal $\mathfrak{m}$; but $s_{\mathfrak{m}} \notin \mathfrak{m}$ is in I, a contradiction. So $I = R$; so $1 \cdot K = (0)$, so $K = (0)$, and g is surjective, as desired. $\square$ Claim 3.46

So if $\hom(P, M)$ and $\hom(P, M')$ are finitely generated and $M \xrightarrow{g} M' \to 0$ is exact then

$$\hom(P, M) \xrightarrow{g} \hom(P, M') \to 0$$

is exact. Notice that if $P = R^n$ and $M = \langle m_1, \ldots, m_s \rangle$ then $\varphi_{r,i} \colon R^n \to M$ given by

$$e_j \mapsto \begin{cases} m_{r(i)} & i = j \\ 0 & \text{else} \end{cases}$$

(where $r(i) \in \{ 1, \ldots, s \}$). Then

$$\begin{aligned} \varphi(e_1) &= a_{11} m_1 + \cdots + a_{1s} m_s \\ &\vdots \\ \varphi(e_n) &= a_{n1} m_1 + \cdots + a_{ns} m_s \end{aligned}$$

Then

$$\varphi = a_{11} \varphi_{1,1} + a_{12} \varphi_{2,1} + \cdots + a_{1s} \varphi_{s,1} + \cdots + a_{ns} \varphi_{s,n}$$

Because P is locally free (and finitely generated) and M, M' are finitely generated, one can show that $\hom(P, M)$ and $\hom(P, M')$ are finitely generated (exercise). So M, M' finitely generated imply $\hom(P, M) \twoheadrightarrow \hom(P, M')$ surjective. Now take $M = R^n$ and $M' = P$. Then there is $s \colon P \to R^n$ such that the following diagram commutes:

$$\begin{tikzcd} R^n \arrow[r, two heads, "g"] & P \arrow[r] & 0 \\ & P \arrow[u, "\mathrm{id}"'] \arrow[ul, dashed, "s"] & \end{tikzcd}$$

So $P \oplus \ker(g) \cong R^n$; so P is projective. $\square$ Theorem 3.45

From here, one notes that given P we have $P_{\mathfrak{p}} \cong R_{\mathfrak{p}}^{d(\mathfrak{p})}$ for $d \geq 1$. Then $\operatorname{Spec}(R) \to \mathbb{Z}$ given by $\mathfrak{p} \mapsto d(\mathfrak{p}) = \operatorname{rank}(P_{\mathfrak{p}})$. By assignment 3, we get that this map is continuous.

Remark 3.47. Suppose P is finitely generated; suppose R is a commutative Noetherian ring. Then if $P_{\mathfrak{m}}$ is free then there is $f \in R \setminus \mathfrak{m}$ such that P_f is free as an R_f-module.

Proof. Since P is finitely generated as an R-module, we can write

$$P = \langle p_1, \ldots, p_m \rangle = R p_1 + \cdots + R p_m$$

By assumption, we have that $P_{\mathfrak{m}} = \{ s^{-1} p : s \notin \mathfrak{m}, p \in P \}$ is free. (Recall that $s_1^{-1} p_1 = s_2^{-1} p_2$ if and only if there is $s_3 \notin \mathfrak{m}$ such that $s_3(s_1 p_2 - s_2 p_1) = 0$.) Pick $s_1^{-1} q_1, \ldots, s_d^{-1} q_d \in P_{\mathfrak{m}}$ such that

$$P_{\mathfrak{m}} = \bigoplus_{i=1}^{d} R_{\mathfrak{m}} s_i^{-1} q_i$$

Then $q_1, \ldots, q_d \in P$ form a basis for $P_{\mathfrak{m}}$; i.e.

$$P_{\mathfrak{m}} = \bigoplus_{i=1}^{d} R_{\mathfrak{m}} q_i$$

Now, for $i \in \{1, \ldots, m\}$ we have $1^{-1}p_i = p_i \in P_{\mathfrak{m}}$; so

$$p_i = (\mu_{i1}^{-1} r_{i1})q_1 + \cdots + (\mu_{id}^{-1} r_{id})q_d$$

where each $\mu_{ij} \in R \setminus \mathfrak{m}$ and each $r_{ij} \in R$. Pick $s \in R \setminus \mathfrak{m}$ such that $s\mu_{ij}^{-1} \in R$ for all i, j; concretely, one could take

$$s = \prod_{i,j} \mu_{ij}$$

Then $sp_i \in Rq_1 + \cdots + Rq_d$ for all i; so $p_i \in R_s q_1 + \cdots + R_s q_d$. So let $Q = Rq_1 + \cdots + Rq_d \subseteq P$; then $Q_s = P_s$. Now consider $R_s^d \twoheadrightarrow Q_s = P_s$ given by $e_i \mapsto q_i$; let K be the kernel of this map. Then

$$0 \to K \to R_s^d \to P_s \to 0$$

is exact; so, localizing to $R_{\mathfrak{m}}$, we find that

$$0 \to K_{\mathfrak{m}} \to R_{\mathfrak{m}}^d \to P_{\mathfrak{m}} \to 0$$

is exact. But the map $R_{\mathfrak{m}}^d \to P_{\mathfrak{m}}$ is an isomorphism; so $K_{\mathfrak{m}} = (0)$. But R is Noetherian; so R_s is Noetherian, and K is finitely generated as an R_s-module.

Exercise 3.48. Since $K_{\mathfrak{m}} = (0)$ there is $s' \notin \mathfrak{m}$ such that $K_{s'} = (0)$.

Now if we invert ss' we get

$$0 \to K_{ss'} = (0) \to R_{ss'}^d \to P_{ss'} \to 0$$

is exact. So $P_{ss'} = R_{ss'}^d$. Taking $f = ss'$, we see $P_f \cong R_f^d$ is a free R_f-module, as desired. □ Remark 3.47

So given $\mathfrak{m}$ a maximal ideal we get $f \notin \mathfrak{m}$ such that $P_f \cong R_f^d$. Note that $\operatorname{Spec}(R_f) \approx \{\mathfrak{p} \in \operatorname{Spec}(R) : f \notin \mathfrak{p}\} = V(f)$ is an open subset of $\operatorname{Spec}(R)$. Notice that for every $\mathfrak{p} \in V(f)$ we have $R_{\mathfrak{p}}$ is a localization of R_f; so $P_f \cong R_f^d$ implies that $P_{\mathfrak{p}} \cong R_{\mathfrak{p}}^d$ (since $P_{\mathfrak{p}} \cong P_f \otimes_{R_f} R_{\mathfrak{p}}$ and $R_{\mathfrak{p}}^d \cong R_f^d \otimes_{R_f} R_{\mathfrak{p}}$).

What does this say? Well, recall that free modules over a commutative ring have a well-defined rank. So we have $\psi\colon \operatorname{Spec}(R) \to \mathbb{Z}$ given by $\mathfrak{p} \mapsto \operatorname{rank}_{R_{\mathfrak{p}}}(P_{\mathfrak{p}})$. Then this says that ψ is constant on $V(f)$; choosing our f judiciously, we get that ψ is locally constant.

Exercise 3.49. ψ is continuous.

Corollary 3.50. *If* $\operatorname{Spec}(R)$ *is connected, then* ψ *is constant. In this case, we can define* $\operatorname{rank}(P)$ *to be the image of* ψ*.*

Exercise 3.51. $\operatorname{Spec}(R)$ is disconnected if and only if $R \cong R_1 \times R_2$ for non-zero R_1, R_2, which holds if and only if R has an idempotent $e^2 = e$ with $e \notin \{0, 1\}$.

Example 3.52. Consider $R = \mathbb{Z} \times \mathbb{Z}$ with $P = \mathbb{Z} \times \{0\}$ and $Q = \{0\} \times \mathbb{Z}$. Then $R = P \oplus Q$ and $\operatorname{Spec}(R) = U \sqcup V$. Furthermore, we have $\operatorname{rank}(P_{\mathfrak{p}}) = 1$ and $\operatorname{rank}(Q_{\mathfrak{p}}) = 0$ for all $\mathfrak{p} \in U$; likewise, we get that $\operatorname{rank}(P_{\mathfrak{p}}) = 0$ and $\operatorname{rank}(Q_{\mathfrak{p}}) = 1$ for all $\mathfrak{p} \in V$. Since rank is additive for free modules, we have that if $\operatorname{Spec}(R)$ is connected, then $\operatorname{rank}(P \oplus Q) = \operatorname{rank}(P) + \operatorname{rank}(Q)$.

We have seen that not all projectives are free.

Definition 3.53. A finitely generated projective module P is *stably free* if there are $m, n \geq 1$ such that $P \oplus R^m \cong R^n$; equivalently such that

$$0 \to R^m \to R^n \to P \to 0$$

is exact.

Example 3.54 (Swan's example). Let $A = \mathbb{R}[x, y, z]/(1 - x^2 - y^2 - z^2)$. We have a surjection $g\colon A^3 \twoheadrightarrow A$ given by $(a, b, c) \mapsto ax + by + cz$; in particular, we have $g(rx, ry, rz) = rx^2 + ry^2 + rz^2 = r$. Let $P = \ker(g)$. So

$$0 \to P \to A^3 \xrightarrow{g} A \to 0$$

is exact, and furthermore is split since $s\colon A \to A^3$ given by $1 \mapsto (x, y, z)$ is a section. So $A^3 \cong P \oplus A$, and P is stably free.

Theorem 3.55 (Swan). *P is not free.*

Proof. Suppose for contradiction that P were free. Then $P \cong A^2$ and $P \subseteq A^3$; so $P = \langle (f_1, f_2, f_3), (g_1, g_2, g_3) \rangle \subseteq A^3$. Now $A^3 = P \oplus s(A) = P \oplus \langle (x, y, z) \rangle$; so $A^3 = \langle (f_1, f_2, f_3), (g_1, g_2, g_3), (x, y, z) \rangle$. So

$$\begin{aligned}(1,0,0) &= a_1(f_1, f_2, f_3) + b_1(g_1, g_2, g_3) + c_1(x, y, z)\\ (0,1,0) &= a_2(f_1, f_2, f_3) + b_2(g_1, g_2, g_3) + c_2(x, y, z)\\ (0,0,1) &= a_3(f_1, f_2, f_3) + b_3(g_1, g_2, g_3) + c_3(x, y, z)\end{aligned}$$

so

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{pmatrix}\begin{pmatrix}f_1 & f_2 & f_3\\ g_1 & g_2 & g_3\\ x & y & z\end{pmatrix}$$

where all entries on the latter two matrices are just functions on S^2. If we plug in any $(\alpha, \beta, \gamma) \in S^2$ (i.e. with $\alpha^2 + \beta^2 + \gamma^2 = 1$), in particular we get that

$$0 \neq \det \begin{pmatrix}f_1(\alpha,\beta,\gamma) & f_2(\alpha,\beta,\gamma) & f_3(\alpha,\beta,\gamma)\\ g_1(\alpha,\beta,\gamma) & g_2(\alpha,\beta,\gamma) & g_3(\alpha,\beta,\gamma)\\ \alpha & \beta & \gamma\end{pmatrix}$$

Now view (f_1, f_2, f_3) as a continuous map $S^2 \to \mathbb{R}^3$.

Claim 3.56. *For any continuous map $\psi\colon S^2 \to \mathbb{R}^3$ there is $p \in S^2$ and $\lambda \in \mathbb{R}$ such that $\psi(p) = \lambda p$.*

Proof. If $0 \in \operatorname{im}(\psi)$, we're done; assume then that $\psi\colon S^2 \to \mathbb{R}^3 \setminus \{0\}$. Without loss of generality, we may then replace $\psi(p)$ by $\frac{\psi(p)}{\|\psi(p)\|}\colon S^2 \to S^2$. One then uses some homotopy and homology to get a contradiction. $\square$ Claim 3.56

But this contradicts the above remark about determinants. $\square$ Theorem 3.55

3.2.1 Vector bundles

Definition 3.57. Suppose S is a connected, compact real manifold. A (real) *vector bundle* over S of rank n is a topological space V with a continuous map $\pi\colon V \to S$ such that

1. For all $x \in S$ we have $\pi^{-1}(x) = \{ v \in V : \pi(v) = x \}$ is a real vector space of dimension n.
2. For all $x \in S$ there is an open neighbourhood U of x in S and a homeomorphism $\varphi\colon U \times \mathbb{R}^n \to \pi^{-1}(U)$ such that $\pi \circ \varphi = p$ (where $p\colon U \times \mathbb{R}^n \to U$ is projection) and for all $y \in U$ we have $\varphi \restriction (\{ y \} \times \mathbb{R}^n)\colon \{ y \} \times \mathbb{R}^n \to \pi^{-1}(\{ y \})$ is a linear isomorphism of vectors spaces.

A vector bundle is *trivial* if $V \cong S \times \mathbb{R}^n$.

There is a correspondence between vector bundles and projective modules as follows: suppose S is a compact, connected real manifold. Then $C(S) = \{ f\colon S \to \mathbb{R} \mid f \text{ is continuous} \}$ has a natural ring structure. Given a vectro bundle $\pi\colon V \to S$ over S of rank n we define a $C(S)$-module $P(V)$ as follows:

Definition 3.58. Let $\pi\colon V \to S$ be as before. A *section* of π is a continuous map $s\colon S \to V$ such that $\pi \circ s = \mathrm{id}_S$. We then set $P(V)$ to be the set of sections.

We put a $C(S)$-module structure on $P(V)$ by

- $(f \cdot s)(x) = f(x)s(x) \in \pi^{-1}(\{ x \})$ for $f \in C(S)$ and $s \in P(V)$.
- $(s + t)(x) = s(x) + t(x)$ for $s, t \in P(V)$.

Theorem 3.59 (Swan)**.** *If V is a vector bundle of rank n then $P(V)$ is a projective $C(S)$-module of rank n. Moreover, the above correspondence gives an equivalence of categories between the category of vector bundles over S and the category of finitely generated projective $C(S)$-modules. In particular, under this equivalence, we have that trivial vector bundles correspond to free modules.*

3.2.2 Loose ends

(Grothendieck grape) Suppose R is a ring. We can make a grape $K_0(R)$ out of the collection of isomorphism classes of finitely generated (left) projective R-modules as follows. Let A be the free abelian grape on the isomorphism classes $[P]$ of finitely generated projective modules P. We then impose the relations $[P_1] + [P_2] = [P_3]$ whenever there is an exact sequence $0 \to P_1 \to P_3 \to P_2 \to 0$.

Example 3.60. If k is a field, then the isomorphism classes of finitely generated projective modules are represented by k^n for $n \in \mathbb{N}$; but we always have an exact sequence $0 \to k^{n-1} \to k^n \to k \to 0$. So $[k^n] = [k^{n-1}] + [k]$ for all $n \in \mathbb{N}$, and $K_0(k) \cong \mathbb{Z}$.

If R is commutative, we can make $K_0(R)$ into a ring via $[P] \cdot [Q] = [P \otimes_R Q]$. One needs to check that $P \otimes_R Q$ is still projective; but if P, Q are finitely generated and projective, then $P \oplus H \cong R^n$ and $Q \oplus E \cong R^m$ for some R-modules H, E. So

$$R^{nm} \cong R^n \otimes_R R^m \cong (P \oplus H) \otimes_R (Q \oplus E) \cong (P \otimes_R Q) \oplus (H \otimes_R Q) \oplus (P \otimes_R E) \oplus (H \otimes_R E)$$

So $P \otimes_R Q$ is a direct summand of a free module, and is thus projective.

(Exterior products) Suppose R is a commutative ring and M is an R-module. We define the i^{th} *exterior product of* M to be

$$\Lambda^i M = \underbrace{M \otimes_R \ldots \otimes_R M}_{i \text{ times}} / N$$

where N is the submodule generated by

$$m_1 \otimes_R \ldots \otimes_R m_i = \operatorname{sgn}(\sigma) m_{\sigma(1)} \otimes_R \ldots \otimes_R m_{\sigma(i)}$$

Then $\Lambda^0 M = R$ and $\Lambda^1 M = M$.

Remark 3.61. $\Lambda^i R^n \cong R^{\binom{n}{i}}$.

Proof. Let $e_1, \ldots, e_n$ be a basis for R^n. Then

$$\underbrace{R^n \otimes_R \ldots \otimes_R R^n}_{i \text{ times}}$$

is spanned by elements of the form $e_{j_1} \otimes_R \ldots \otimes_R e_{j_i}$. But

$$e_{j_1} \otimes_R \ldots \otimes_R e_{j_i} \equiv \pm e_{\ell_1} \otimes_R \ldots \otimes_R e_{\ell_i}$$

where $\ell_1 \leq \ell_2 \leq \cdots \leq \ell_i$. Indeed, one can show that elements of the form $e_{\ell_1} \otimes_R \ldots \otimes_R e_{\ell_i}$ form a basis for $\Lambda^i R^n$. ☐ Remark 3.61

In particular, we get that $\Lambda^n R^n \cong R$. If R is a Noetherian commutative ring with $\operatorname{Spec}(R)$ connected and P is a projective module of rank n then $\Lambda^i P$ is projective of rank $\binom{n}{i}$.

(Picard grape) Now we let $\operatorname{Pic}(R)$ denote the multiplicative subset of $K_0(R)$ generated by projective modules of rank 1; this has a grape structure via $[P] \cdot [Q] = [P \otimes_R Q]$. We call $\operatorname{Pic}(R)$ the *Picard grape of* R. It is indeed a grape: $[P] \otimes_R [\hom(P, R)] = [R]$ is the identity. We have a map $K_0(R)^\times \to \operatorname{Pic}(R)$ given by $[P] \mapsto [\Lambda^{\operatorname{rank}(P)} P]$; this is a homomorphism of semigrapes (under $\otimes_R$).

(A final remark) If R is commutative and $P \oplus R^n \cong R^{n+1}$ then $P \cong R$.

This is left as an exercise.

(Step 1) Check that

$$\Lambda^i (M \oplus N) \cong \bigoplus_{j=1}^{i} \Lambda^j (M) \otimes_R \Lambda^{i-j} N$$

(Step 2) $R^{n+1} \cong R^n \oplus P$, so

$$\begin{aligned} R &= R^{\binom{n+1}{n+1}} \\ &\cong \Lambda^{n+1} R^{n+1} \\ &\cong \Lambda^{n+1}(R^n \oplus P) \\ &\cong \bigoplus_{j=1}^{n+1} R^n \otimes_R \Lambda^{n+1-j} P \end{aligned}$$

(Step 3) Show that since P has rank 1 then $\Lambda^j P = (0)$ for $j > 1$ and $\Lambda^{n+1} R^n = (0)$; then the isomorphism in the previous step shows that

$$R \cong \Lambda^n R^n \otimes_R \Lambda^1 P \cong R \otimes_R P \cong P$$

3.3 Injective modules

We now consider the dual notion of projective modules. Suppose $\mathcal{A}$ is an abelian category. Recall that P is a projective object if and only if $\hom(P, -)$ is exact.

Definition 3.62. We say $I \in \mathrm{Ob}(\mathcal{A})$ is an *injective object* if and only if $\hom(-, I)$ is exact; i.e. whenever

$$0 \to A \to B \to C \to 0$$

is exact, we have that

$$0 \to \hom(C, I) \to \hom(B, I) \to \hom(A, I) \to 0$$

is exact. One checks that this is equivalent to requiring that whenever $0 \to A \xrightarrow{f} B$ is exact then $\hom(B, I) \to \hom(A, I) \to 0$ given by $\psi \mapsto \psi \circ f$ is exact; i.e.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "h"] & B \arrow[dl, dashed, "\exists \widetilde{h}"] \\ & I & \end{tikzcd}$$

Lemma 3.63 (Baer)**.** *Suppose R is a ring; suppose Q is a left R-module. If for every left ideal $I \leq R$ and every homomorphism of R-modules $h\colon I \to Q$ there is a homomorphism of R-modules $\widetilde{h}\colon R \to Q$ such that $\widetilde{h} \restriction I = h$, then Q is injective.*

Proof. Suppose we have

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "f"] \arrow[d, "\beta"] & M \\ & Q & \end{tikzcd}$$

i.e. f is injective; assume without loss of generality we assume f is an inclusion. Consider the set $\mathcal{S}$ of all pairs (N', β') with $N \subseteq N' \subseteq M$ and $\beta'\colon N' \to Q$ such that $\beta' \restriction N = \beta$. We can partially order $\mathcal{S}$ via $(N_1, \beta_1) \leq (N_2, \beta_2)$ if $N_1 \subseteq N_2$ and $\beta_2 \restriction M = \beta_1$. Observe that $(N, \beta) \in \mathcal{S}$, so $\mathcal{S}$ is non-empty. Further observe that $\mathcal{S}$ is closed under unions of chains: given a chain $((N_i, \beta_i) : i \in I)$ in $\mathcal{S}$, we get

$$\left(\bigcup_{i \in I} N_i, \bigcup_{i \in I} \beta_i \right) \in \mathcal{S}$$

So, by Zorn's lemma, there is a maximal such pair (N', β') in $\mathcal{S}$. If $N' = M$ we're done. Assume therefore that there is $m \in M \setminus N'$; look at $N'' = Rm + N'$. Let $I = \{ r \in R : rm \in N' \}$; then I is a left ideal of R. Make a map $\theta\colon I \to Q$ given by $r \mapsto \beta'(rm) \in Q$. By hypothesis we can extend θ to $\delta\colon R \to Q$; i.e. so that $\delta \restriction I = \theta$. Consider $\beta''\colon N'' \to Q$ given by $rm + n' \mapsto \delta(r) + \beta'(n')$. Notice β'' is well-defined: if $r_1 m + n_1 = r_2 m + n_2$ then $(r_1 - r_2)m \in N'$; so $r_1 - r_2 \in I$, and $\delta(r_1 - r_2) = \theta(r_1 - r_2) = \beta'((r_1 - r_2)m)$. So $\beta''(r_1 m + n_1) - \beta''(r_2 m + n_2) = \beta'((r_1 - r_2)m + n_1 - n_2) = 0$. By construction we get $\beta'' \restriction N' = \beta'$, contradicting the maximality of (N', β'). So $N' = M$, and we're done. $\square$ Lemma 3.63

Corollary 3.64. *Let $R = \mathbb{Z}$. Then an R-module M is injective if and only if M is divisible.*

Proof.

($\Longrightarrow$) Assignment 2.

($\Longleftarrow$) Suppose M is divisible; we apply Baer's criterion. Suppose $I \trianglelefteq \mathbb{Z}$; so $I = n\mathbb{Z}$ for some $n \geq 0$. Suppose we are given $\beta\colon I \to M$; we wish to extend β to $\beta'\colon \mathbb{Z} \to M$. If $I = (0)$, we may take $\beta' = 0$. Suppose then that $n \neq 0$; let $m = \beta(n) \in M$. Since M is divisible, there is $x \in M$ such that $nx = m$; define $\beta'\colon \mathbb{Z} \to M$ by $1 \mapsto x$. Then $\beta'(n) = n - x = m = \beta(n)$.

$\square$ Corollary 3.64

Corollary 3.65. *Suppose M is an injective $\mathbb{Z}$-module; suppose $K \leq M$. Then M/K is injective.*

Proof. Suppose $x + K \in M/K$; i.e. suppose $x \in M$. Suppose $n \in \mathbb{Z}$ and $n > 0$; then there is $y \in M$ such that $ny = x$. So $n(y + K) = x + K$; so M/K is divisible. $\square$ Corollary 3.65

Definition 3.66. An abelian category $\mathcal{A}$ has *enough projectives* if for every $A \in \mathrm{Ob}(\mathcal{A})$ there is a projective object P and an epimorphism $f\colon P \twoheadrightarrow A$. It has *enough injectives* if for every $A \in \mathrm{Ob}(\mathcal{A})$ there is an injective object Q and a monomorphism $f\colon A \hookrightarrow Q$.

We'll see that R-**Mod** has enough injectives, where R is a ring. We first verify the case $R = \mathbb{Z}$.

Claim 3.67. **Ab** $= \mathbb{Z}$-**Mod** *has enough injectives.*

Proof. Suppose A is an abelian grape; then there is $\mathbb{Z}^I \twoheadrightarrow A$. So $A \cong \mathbb{Z}^I/K$ where $K \leq \mathbb{Z}^I$ is the kernel. But $\mathbb{Z}^i \hookrightarrow \mathbb{Q}^i$, and Q^i is divisible, and hence injective. So $K \leq \mathbb{Z}^I \leq \mathbb{Q}^I$; so $A \cong \mathbb{Z}^I/K \leq \mathbb{Q}^I/K$ and this last is injective by the corollary. So we have $A \hookrightarrow \mathbb{Q}^I/K$ which is injective. $\square$ Claim 3.67

We lift this result to R-**Mod**. For the setup, suppose S, R are rings. (Ultimately we'll take $S = \mathbb{Z}$.) Suppose F is an (S, R)-bimodule; i.e. suppose F has structure as a left S-module and as a right R-module. We assume that F is a flat right R-module; i.e. if $0 \to M \to N$ is an exact sequence of left R-modules then $0 \to F \otimes_R M \to F \otimes_R N$ is an exact sequence of abelian grapes.

Aside 3.68 (Non-commutative tensor products)*.* Suppose R is a ring, T is a right R-module, and L is a left R-module. Then $T \otimes_R L$ is an abelian grape.

Remark 3.69*.* Suppose M is a left S-module. We define $\widetilde{M} = \hom_s(F, M)$.

Notice that $\widetilde{M}$ is a left R-module via the rule $(r \cdot \varphi)(x) = \varphi(x \cdot r)$. Furthermore, given $r_1, r_2 \in R$ we have $(r_1 \cdot r_2) \cdot \varphi(x) = \varphi(x \cdot r_1 r_2)$. Then

$$r \cdot [(r_2 \cdot \varphi)](x) = \Gamma_2 - \varphi(xr_1) = \varphi(xr, r_2)$$

Lemma 3.70 (Injective production lemma)**.** *Under this setup, if M is an injective left S-module, then $\widetilde{M}$ is an injective left R-module.*

Proof. We check that $\hom_R(-, \widetilde{M})$ is exact. In fact, we know it is enough to show that whenever $0 \to A \xrightarrow{f} B$ is exact (for $A, B \in \mathrm{Ob}(R\text{-}\mathbf{Mod})$), we also have $\hom_R(B, \widetilde{M}) \to \hom_R(A, \widetilde{M}) \to 0$ given by $\psi \mapsto \psi \circ f$ is exact. Suppose then that $0 \to A \xrightarrow{f} B$ is exact. We wish to check that $\hom_R(B, \hom_S(F, M)) \to \hom_R(A, \hom_S(F, M)) \to 0$ given by $\psi \mapsto \psi \circ f$ is exact. From the tensor-hom adjunction, we have an isomorphism of abelian grapes $\hom_R(B, \hom_S(F, M)) \cong \hom_S(F \otimes_R B, M)$ such that given $\psi\colon B \to \hom_S(F, M)$ we have $\psi \mapsto (\theta \otimes_R b \mapsto \psi(b)(\theta))$.

Exercise 3.71*.* We have a map $\hom_S(F \otimes_R B, M) \to \hom_S(F \otimes_R A, M)$ such that given $\psi\colon F \otimes_R B \to M$, we have $\psi \mapsto \widehat{\psi}\colon F \otimes_R A, M)$ given by $\widehat{\psi}(\theta \otimes_R a) = \psi(\theta \otimes_R f(a))$; furthermore, the isomorphisms yield a commuting diagram:

$$\begin{tikzcd}
\hom_R(B, \hom_S(F, M)) \arrow[r] \arrow[d, "\cong"] & \hom_R(A, \hom_S(F, M)) \arrow[d, "\cong"] \\
\hom_S(F \otimes_R B, M) \arrow[r] & \hom_S(F \otimes_R A, M)
\end{tikzcd}$$

So it suffices to show that $\hom_S(F \otimes_R B, M) \to \hom_S(F \otimes_R A, M) \to 0$ is exact. Since M is an injective left module and F is flat as a right R-module, we get

1. $0 \to A \xrightarrow{f} B$ is exact.
2. $0 \to F \otimes_R A \xrightarrow{\mathrm{id} \otimes_R f} F \otimes_R B$ is exact in S-**Mod**.
3. $\hom_S(F \otimes_R B, M) \to \hom_S(F \otimes_R A, M) \to 0$ given by $\psi \mapsto \widehat{\psi}$ is exact.

The result then follows from the commuting diagram above. ☐ Lemma 3.70

For us, we'll take $S = \mathbb{Z}$, $M = \mathbb{Q}/\mathbb{Z}$, and F a free (and hence flat) right R-module; note that M is an injective S-module. In this setup, if F is a right R-module, we define $F^* = \hom_{\mathbb{Z}}(F, \mathbb{Q}/\mathbb{Z})$; this is the *Pontryagin dual of F*. Then F^* is a left R-module.

Remark 3.72. If A is a left or right R-module, we get an embedding $A \hookrightarrow A^{**}$ given by $m \mapsto e_m$ where $e_m \colon A^* \to \mathbb{Q}/\mathbb{Q}$ is given by $e_m(f) = f(m)$. Why is this an injection? Well, suppose we have $m \in A \setminus \{0\}$ such that $e_m = 0$; i.e. suppose $f(m) = 0$ for all $f \in \hom_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$. Let $C = \mathbb{Z}m \subseteq A$.

Claim 3.73. *There is a non-trivial homomorphism $g \colon C \to \mathbb{Q}/\mathbb{Z}$.*

Proof. Well, C is cyclic; so we have two cases.

Case 1. Suppose $C \cong \mathbb{Z}$; then we can just use the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$.

Case 2. Suppose $C \cong \mathbb{Z}/n\mathbb{Z}$; then we can use the map $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ given by $1 + n\mathbb{Z} \mapsto \frac{1}{n} + \mathbb{Z}$. ☐ Claim 3.73

By injectivity of $\mathbb{Q}/\mathbb{Z}$ there is $\widetilde{g} \colon A \to \mathbb{Q}/\mathbb{Z}$ such that the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & C \arrow[r, hook] \arrow[d, "g"] & A \arrow[dl, dashed, "\widetilde{g}"] \\ & \mathbb{Q}/\mathbb{Z} & \end{tikzcd}$$

Then $e_m(\widetilde{g}) = \widetilde{g}(m) = g(m) \neq 0$, and injectivity follows.

Corollary 3.74. *Let R be a ring; then R-**Mod** has enough injectives.*

Proof. If A is a right R-module then there is a free right R-module F and a surjection $F \twoheadrightarrow A$. Since $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module and A, F are $\mathbb{Z}$-modules, we get that $0 \to \hom_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}) \to \hom_{\mathbb{Z}}(F, \mathbb{Q}/\mathbb{Z})$ is exact; i.e. $A^* \hookrightarrow F^*$. By Lemma 3.70, we have that F^* is an injective left R-module. We thus see that any left R-module of the form A^* with A a right R-module embeds in an injective. But every left R-module A has $A \hookrightarrow A^{**} = (A^*)^*$, which we just saw embeds into the injective left R-module F^*. So A embeds into an injective left R-module. So R-**Mod** has enough injectives. ☐ Corollary 3.74

A nice fact:

Fact 3.75. *Any R-module A has a unique minimal injective resolution.*

Definition 3.76. Let R be a ring; let $M \subseteq E$ be left R-modules. We say that M is an *essential submodule of E* (or E is an *essential extension of M*) if $M \cap N \neq (0)$ for all $N \subseteq E$.

Proposition 3.77.

1. *Given a ring R and R-modules $M \subseteq F$ there is a maximal submodule $E \subseteq F$ with M as an essential submodule.*
2. *If F is injective then E is injective.*
3. *There is up to isomorphism a unique essential extension E of M that is an injective R-module. We call this the* injective envelope *of M, denoted $E(M)$.*

Proof.

1. Assignment (up to a small error).
2. Assignment.
3. Since R-**Mod** has enough injective, there is an injective F and an embedding $M \overset{i}{\hookrightarrow} F$; without loss of generality we assume $M \subseteq F$. By (1) and (2) we have that there is an essential extension E of M (with $E \subseteq F$) that is injective. So we at least have existence. To see uniqueness, suppose we have $M \overset{\alpha_1}{\hookrightarrow} E_1$ and $M \overset{\alpha_2}{\hookrightarrow} E_2$ where E_1 and E_2 are injective and essential extensions of M. Then by injectivity of E_2 we get $\beta\colon E_1 \to E_2$ such that the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "\alpha_1"] \arrow[d, "\alpha_2"] & E_1 \arrow[dl, dashed, "\beta"] \\ & E_2 & \end{tikzcd}$$

i.e. $\beta \circ \alpha_1 = \alpha_2$. So, since α_2 is injective, we have that $\ker(\beta \restriction \alpha_1(M)) = (0)$.

Claim 3.78. $\ker(\beta) = (0)$*; i.e.* β *is injective.*

Proof. Well, $\alpha_1(M) \subseteq E_1$ is an essential submodule, and since $\ker(\beta \restriction \alpha_1(M)) = (0)$ we get that $\alpha_1(M) \cap \ker(\beta) = (0)$; so $\ker(\beta) = (0)$. □ Claim 3.78

So β is injective; so $\beta(E_1)$ is an injective submodule of E_2.

TODO 3. *Why an injective submodule?*

So there is E_2' such that $\beta(E_1) \oplus E_2' = E_2$. But now we get $\alpha_2(M) = (\beta \circ \alpha_1)(M) \subseteq \beta(E_1)$ and $\alpha_2(M) \subseteq E_2$ is essential. So if $E_2' \neq (0)$ then $\alpha_2(M) \cap E_2' \neq (0)$, and $\beta(E_1) \cap E_2' \neq (0)$, a contradiction. So $E_2' = (0)$, and $\beta(E_1) = E_2$. So β is bijective, and $E_1 \cong E_2$. □ Proposition 3.77

In particular, then the exact sequence

$$0 \to E \overset{i}{\hookrightarrow} F \to \operatorname{coker}(i) \to 0$$

splits, and $F \cong E \oplus \operatorname{coker}(i)$.

Given an R-module M, we have an embedding $0 \to M \to E(M)$; let $Q_1 = \operatorname{coker}(M \to E(M))$. Continuing, we can extend the sequence

$$0 \to M \to E(M) \to E(Q_1) \to E(Q_2) \to \dots$$

where $Q_2 = \operatorname{coker}(E(M) \to E(Q_1))$.

Remark 3.79. If $(I_j : j \in J)$ are injective modules then

$$\prod_{i \in J} I_j$$

is injective by using the limit property on the diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d] & N \arrow[dl] \arrow[ddl] \\ & \prod_{j \in J} I_j \arrow[d] & \\ & I_j & \end{tikzcd}$$

Remark 3.80. A direct sum of injectives need not be injective.

Theorem 3.81 (Bass). *Let R be a commutative ring. Then R is Noetherian if and only if every direct sum of injectives is again injective.*

Sketch of proof.

($\Longleftarrow$) Suppose R is not Noetherian; suppose we have a chain of ideals $I_1 \subsetneqq I_2 \subsetneqq \dots$. Let $E_n = E(R/I_n)$. Then

$$E = \bigoplus_{n=1}^{\infty} E_n$$

is not injective. Indeed, let

$$I = \bigcup_{n=1}^{\infty} I_n \subseteq R$$

and consider $f_n \colon I \to E(R/I_n)$ given by the composition $I \hookrightarrow R \to R/I_n \hookrightarrow E(R/I_n)$. These f_n yield a map

$$\begin{aligned} f \colon I &\to \prod_{n=1}^{\infty} E(R/I_n) \\ x &\mapsto (f_1(x), f_2(x), \dots) \end{aligned}$$

Note, however that f actually maps into

$$E = \bigoplus_{n=1}^{\infty} E(R/I_n) \subseteq \prod_{n=1}^{\infty} E(R/I_n)$$

since $x \in I$ implies $x \in I_n$ for all sufficiently large n, and thus that $f_n(x) = 0$ for all sufficiently large n. Now, if E is injective, then there is $\beta \colon R \to E$ such that the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d, "f"] & R \arrow[dl, dashed, "\beta"] \\ & E & \end{tikzcd}$$

Consider then $\beta(1)$; by definition of E there is $m \in \mathbb{N}$ such that

$$\beta(1) \in E_1 \oplus E_2 \oplus \ldots \oplus E_m \oplus (0) \oplus (0) \oplus \ldots$$

So

$$\beta(r) = r\beta(1) \subseteq E_1 \oplus E_2 \oplus \ldots \oplus E_m \oplus (0) \oplus (0) \oplus \ldots$$

for all $r \in R$. But then for $x \in I_{m+1} \setminus I_m$, we have $f_{m+1}(x) \in E_{m+1} \neq (0)$; so

$$\beta(x) = f_{m+1}(x) \notin E_1 \oplus E_2 \oplus \ldots \oplus E_m \oplus (0) \oplus (0) \oplus \ldots$$

a contradiction. So E is not injective.

($\Longrightarrow$) One checks the following:

Exercise 3.82. If M is finitely generated then

$$\hom_R\left(M, \bigoplus_{i \in I} N_i\right) \cong \bigoplus_{i \in I} \hom_R(M, N_i)$$

The idea is then that if R is Noetherian and $J \subseteq R$ is an ideal then J is finitely generated. If the N_i are injective, then $\hom(J, N_i) \to \hom(R, N_i)$ is surjective for all i; so

$$\begin{tikzcd}
\hom\big(J_1, \bigoplus_{i\in I} N_i\big) \arrow[r, "\cong"] \arrow[d, two heads] & \bigoplus_{i\in I} \hom(J, N_i) \arrow[d, two heads] \\
\hom\big(R, \bigoplus_{i\in I} N_i\big) \arrow[r, "\cong"] & \bigoplus_{i\in I} \hom(R, N_i)
\end{tikzcd}$$

TODO 4. *What does this mean?*

Then Baer's criterion gives that

$$\bigoplus_{i\in I} N_i$$

is injective. □ Theorem 3.81

Bass' theorem is very useful when studying injectives over a Noetherian ring.

Definition 3.83. An injective module E is *decomposable* if $E = E' \oplus E''$ where E' and E'' are non-zero; else it is *indecomposable*.

For a commutative Noetherian ring R we have that every injective R-module E is of the form

$$E \cong \bigoplus_{j\in J} E_j$$

where E_j is injective and indecomposable. Moreover, there is a bijection from $\operatorname{Spec}(R)$ to the isomorphism classes of indecomposable injectives given by $\mathfrak{p} \mapsto E(R/\mathfrak{p})$. Why? Well, if E is indecomposable and injective, we may pick $x \in E$ with maximal annihilator. (Recall $\operatorname{Ann}(x) = \{\, r \in R : rx = 0 \,\}$.) The usual trick for ideals in a Noetherian ring maximal with respect to some property yields that $\operatorname{Ann}(x) = \mathfrak{p}$ is prime. So

$$\begin{tikzcd}
R/\mathfrak{p} \arrow[r, "\cong"] & Rx \arrow[r, hook] \arrow[d, hook] & E(R/\mathfrak{p}) \arrow[dl, "\cong"] \\
& E &
\end{tikzcd}$$

TODO 5. *What does this mean?*

4 Complexes

We work in $\mathcal{A}$ an abelian category; we can always assume that this is R-**Mod** by Mitchell's embedding theorem.

Definition 4.1. A *chain complex* $C_\bullet$ is a family $(C_n : n \in \mathbb{Z})$ with $C_n \in \operatorname{Ob}(\mathcal{A})$ and morphisms $d_n \colon C_n \to C_{n-1}$ such that $d_{n-1} \circ d_n \colon C_n \to C_{n-2} = 0$. We call the d_n the *differentials* of $C_\bullet$. We then define $Z_n(C_\bullet) = \ker(d_n) \subseteq C_n$ to be the *n-cycles* of $C_\bullet$; we define $B_n(C_\bullet) = \operatorname{im}(d_{n+1}) \subseteq C_n$ to be the *n-boundaries* of $C_\bullet$. So $(0) \subseteq B_n(C_\bullet) \subseteq Z_n(C_\bullet) \subseteq C_n$. We define $H_n(C_\bullet) = Z_n(C_\bullet)/B_n(C_\bullet)$ to be the n^{th} *homology grape of $C_\bullet$*.

Dually, we define a *cochain complex* $C^\bullet$ is a family of $(C^n : n \in \mathbb{Z})$ and morphisms $d^n \colon C^n \to C^{n+1}$ such that $d^{n+1} \circ d^n = 0$ for all $n \in \mathbb{Z}$. We define $Z^n(C^\bullet) = \ker(d^n) \subseteq C^n$ to be the *n-cocycles*; we define $B^n(C^\bullet) = \operatorname{im}(d^{n-1}) \subseteq C^n$ to be the *n-coboundaries*. We define $H^n(C^\bullet) = Z^n(C)/B^n(C)$ to be the n^{th} *cohomology grape of $C^\bullet$*.

Remark 4.2. $H_n(C) = (0)$ if and only if $C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1}$ is exact at C_n.

Remark 4.3. $(C_n : n \in \mathbb{Z})$ is a chain complex if and only if $B^n = C_{-n}$ with $d^n = d_{-n} \colon C_{-n} \to C_{-n-1}$.

Example 4.4 (de Rham complex). Suppose $\varphi\colon R \to A$ is an R-algebra. Recall the Kähler differentials were $\Omega_{A/R}$ the free A-module generated by symbols da for $a \in A$ modulo the relations

- $d(a + rb) = da + rdb$ for all $r \in R$ and $a, b \in A$.
- $d(ab) = adb + bda$ for all $a, b \in A$.
- $dr = 0$ for all $r \in R$.

Now define

$$\Omega^i_{A/R} = \Lambda^i \Omega_{A/R} = \bigotimes_{j=1}^{i} \Omega_{A/R} / \left\langle a_1 \otimes \ldots \otimes a_i = \operatorname{sgn}(\sigma) a_{\sigma(1)} \otimes \ldots \otimes a_{\sigma(i)} \right\rangle$$

(We also take $\Omega^i_{A/R} = 0$ for $i < 0$.) Given $m_1, \ldots, m_i \in \Omega_{A/R}$ we let $m_1 \wedge \ldots \wedge m_i$ denote the image of $m_1 \otimes \ldots \otimes m_i$ in $\Lambda^i \Omega_{A/R} = \Omega^i_{A/R}$. Note that

- $\Omega^0_{A/R} = A$.
- $\Omega^1_{A/R} = \Omega_{A/R}$.
- We have a map $d\colon A \to \Omega_{A/R}$ given by $a \mapsto da$; we call this $d^0\colon \Omega^0_{A/R} \to \Omega^1_{A/R}$.
- We have another map $d^1\colon \Omega^1_{A/K} \to \Omega^2_{A/R}$ given by $d^1(adb) = da \wedge db$; in particular, we get $d^1 \circ d^0 = 0$.
- In general, these yield a map $d^n\colon \Omega^n_{A/R} \to \Omega^{n+1}_{A/R}$ satisfying

  $$d^n(\omega \wedge \eta) = d^i\omega \wedge \eta + (-1)^i \omega \wedge d^{n-i}\eta$$

  for all $\omega \in \Omega^i_{A/R}$ and all $\eta \in \Omega^{n-i}_{A/R}$. In particular, we take

  $$d^n(\omega_1 \wedge \ldots \wedge \omega_n) = (d^1\omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_n) - (\omega_1 \wedge d^1\omega_2 \wedge \omega_3 \wedge \ldots \wedge \omega_n) + (\omega_1 \wedge \omega_2 \wedge d^1\omega_3 \wedge \ldots \wedge \omega_n) - \ldots$$

- In particular, for $\omega \in \Omega^{n-1}_{A/R}$ and $\eta \in \Omega^1_{A/R}$, we have

  $$\begin{aligned}(d^{n+1} \circ d^n)(\omega \wedge \eta) &= d^{n+1}(d^{n-1}\omega \wedge \eta + (-1)^{n-1}\omega \wedge d\eta) \\ &= d^{n+1}(d^{n-1}\omega \wedge \eta) + (-1)^{n-1} d^{n+1}(\omega \wedge d\eta) \\ &= d^n(d^{n-1}(\omega)) \wedge \eta + (-1)^n d^{n-1}\omega \wedge d^1\eta + (-1)^{n-1} d^{n-1}\omega \wedge d^1\eta + (-1)^{n-1}(-1)^{n-1}\omega \wedge d^2(d^1\eta) \\ &= 0\end{aligned}$$

  by an inductive argument.

  TODO 6. *Really?*

Exercise 4.5. Suppose k is a field of characteristic 0; let $A = k[x_1, \ldots, x_n]$. Then

$$0 \to k \to \Omega^0_{A/k} \to \Omega^2_{A/k} \to \cdots \to \Omega^n_{A/k} \to 0$$

is exact.

Definition 4.6. Let $C_\bullet$ and $C'_\bullet$ be two chain complexes; say $C_\bullet = (C_n, d_n)$ and $C'_\bullet = (C'_n, d'_n)$. A *morphism of chain complexes* is a collection of maps $f_n\colon C_n \to C'_n$ such that the following diagram commutes:

$$\begin{tikzcd} C_n \arrow[r, "d_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[d, "f_{n-1}"] \\ C'_n \arrow[r, "d'_n"] & C'_{n-1} \end{tikzcd}$$

Thus if $\mathcal{C}$ is an abelian category then we can set $\operatorname{Ch}(\mathcal{C})$ to be the *category of chain complexes in* $\mathcal{C}$. Similarly, we define Co-Ch$(\mathcal{C})$ the *category of cochain complexes in* $\mathcal{C}$.

In fact $\mathrm{Ch}(\mathcal{C})$ and $\mathrm{Co\text{-}Ch}(\mathcal{C})$ are abelian categories. The only non-trivial part is checking then $\ker(f)$ and $\mathrm{coker}(f)$ are objects in $\mathrm{Ch}(\mathcal{C})$ for $f\colon C_\bullet \to C'_\bullet$. One can assume that $\mathcal{C} = R$-**Mod**, by Mitchell's embedding theorem. Note then that the following diagram commutes:

$$\begin{tikzcd}
\ker(f_n) \arrow[r, "d_n\restriction\ker(f_n)"] \arrow[d] & \ker(f_{n-1}) \arrow[d] \\
C_n \arrow[r, "d_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[d, "f_{n-1}"] \\
C'_n \arrow[r, "d'_n"] \arrow[d] & C'_{n-1} \arrow[d] \\
\mathrm{coker}(f_n) \arrow[r, "\overline{d'_n}"] & \mathrm{coker}(f_{n-1})
\end{tikzcd}$$

since if $x' = x'' + f_n(u)$ in C'_n then

$$d'_n(x') = d'_n(x'') + (d'_n \circ f_n)(u) = d'_n(x'') + (f_{n-1} \circ d_n)(u)$$

and $d'_n(x') = d'_n(x'')$ in $\mathrm{coker}(f_{n-1})$. One also checks that monomorphisms and epimorphisms are normal; hence $\mathrm{Ch}(\mathcal{C})$ is an abelian category.

Remark 4.7. One can show that a morphism $C_\bullet \to C'_\bullet$ takes $Z_n(C_\bullet)$ to $Z_n(C'_\bullet)$ and $B_n(C_\bullet)$ to $B_n(C'_\bullet)$; in particular, we get a map $H_n(C_\bullet)$ to $H_n(C'_\bullet)$.

Definition 4.8. A morphism $u\colon C_\bullet \to D_\bullet$ is called a *quasi-isomorphism* if for every $n \in \mathbb{Z}$ we have that the induced map $H_n(C_\bullet) \to H_n(D_\bullet)$ is an isomorphism.

Proposition 4.9. *the following are equivalent:*

1. *The chain complex $C_\bullet$ is exact at each C_n.*
2. *$H_n(C_\bullet) = 0$ for all $n \in \mathbb{Z}$.*
3. *$C_\bullet$ is quasi-isomorphic to $\cdots \to 0 \to 0 \to 0 \to \cdots$, the zero chain complex.*

Definition 4.10. A chain complex $C_\bullet$ is *bounded* if $C_n = 0$ for all but finitely many n. We say $C_\bullet$ is *bounded above* if $C_n = 0$ for all sufficiently large n; likewise with *bounded below.* We use $\mathrm{Ch}_b(\mathcal{C})$, $\mathrm{Ch}_-(\mathcal{C})$, and $\mathrm{Ch}_+(\mathcal{C})$ to denote the full subcategories of $\mathrm{Ch}(\mathcal{C})$ consisting of chains that are bounded, bounded below, and bounded above, respectively; Similarly, we get $\mathrm{Co\text{-}Ch}^b$, $\mathrm{Co\text{-}Ch}^-$, and $\mathrm{Co\text{-}Ch}^+$.

Remark 4.11. Since $\mathrm{Ch}(\mathcal{C})$ (respectively $\mathrm{Co\text{-}Ch}(\mathcal{C})$) is an abelian category, it makes sense to talk about short exact sequences of chain complexes

$$0 \to A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet \to 0$$

(where "0" denotes the zero chain complex $\cdots \to 0 \to 0 \to 0 \to \cdots$). Examining the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \ker(f_n) \arrow[r, "d_n\restriction\ker(f_n)"] \arrow[d, "i_n"] & \ker(f_{n-1}) \arrow[r] \arrow[d, "i_{n-1}"] & \cdots \\
\cdots \arrow[r] & A_n \arrow[r, "d_n"] \arrow[d, "f_n"] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & B_n \arrow[r, "d'_n"] & B_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

we see that $f\colon A \to B$ is a monomorphism if and only if $\cdots \to \ker(f_n) \to \ker(f_{n-1}) \to \cdots$ is the zero complex. Examining the diagram

$$
\begin{tikzcd}
\cdots \arrow[r] & A_n \arrow[r, "a_n"] \arrow[d, "f_n"] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & B_n \arrow[r, "b_n"] \arrow[d, "g_n"] & B_{n-1} \arrow[r] \arrow[d, "g_{n-1}"] & \cdots \\
\cdots \arrow[r] & C_n \arrow[r, "c_{n-1}"] & C_{n-1} \arrow[r] & \cdots
\end{tikzcd}
$$

we see that $A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet$ is exact at $B_\bullet$ if and only if $g_n \circ f_n = 0$ for all $n \in \mathbb{Z}$ and $\ker(g_n)/\operatorname{im}(f_n) = 0$ for all $n \in \mathbb{Z}$.

4.1 Long exact sequence

If $0 \to A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet \to 0$ is a short exact sequence in $\mathrm{Ch}(\mathcal{C})$ then there are *connecting morphisms* $\delta_n\colon H_n(C_\bullet) \to H_{n-1}(A_\bullet)$ such that

$$
\begin{tikzcd}
 & \cdots & H_{n+1}(C_\bullet) \arrow[dll, "\delta_{n+1}"'] \\
H_n(A_\bullet) \arrow[r, "f"] & H_n(B_\bullet) \arrow[r, "g"] & H_n(C_\bullet) \arrow[dll, "\delta_n"'] \\
H_{n-1}(A_\bullet) \arrow[r, "f"] & H_{n-1}(B_\bullet) \arrow[r, "g"] & H_{n-1}(C_\bullet) \arrow[dll, "\delta_{n-1}"'] \\
H_{n-2}(A_\bullet) & \cdots &
\end{tikzcd}
$$

is exact. Dually, if $0 \to A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \to 0$ is a short exact sequence in $\text{Co-Ch}(\mathcal{C})$ then there are $\delta^n\colon H^n(C) \to H^{n+1}(A)$ such that

$$
\begin{tikzcd}
 & \cdots & H^{n-1}(C^\bullet) \arrow[dll, "\delta^{n-1}"'] \\
H^n(A^\bullet) \arrow[r, "f"] & H^n(B^\bullet) \arrow[r, "g"] & H^n(C^\bullet) \arrow[dll, "\delta^n"'] \\
H^{n+1}(A^\bullet) \arrow[r, "f"] & H^{n+1}(B^\bullet) \arrow[r, "g"] & H^{n+1}(C^\bullet) \arrow[dll, "\delta^{n+1}"'] \\
H^{n+2}(A^\bullet) & \cdots &
\end{tikzcd}
$$

is exact. The key ingredient in the proof is the *snake lemma*.

Lemma 4.12 (Snake lemma)**.** *Suppose that $\mathcal{C}$ is an abelian category and suppose we have a commuting diagram with exact rows*

$$
\begin{tikzcd}
 & A' \arrow[r, "i'"] \arrow[d, "f"] & B' \arrow[r, "p'"] \arrow[d, "g"] & C' \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A \arrow[r, "i"] & B \arrow[r, "p"] & C &
\end{tikzcd}
$$

For clarity, we expand the diagram to get a commuting diagram containing the various kernels and cokernels:

$$\begin{tikzcd}
 & \ker(f) \arrow[r] \arrow[d] & \ker(g) \arrow[r] \arrow[d] & \ker(h) \arrow[d] & \\
 & A' \arrow[r, "i'"] \arrow[d, "f"] & B' \arrow[r, "p'"] \arrow[d, "g"] & C' \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A \arrow[r, "i"] \arrow[d] & B \arrow[r, "p"] \arrow[d] & C \arrow[d] & \\
 & \operatorname{coker}(f) \arrow[r] & \operatorname{coker}(g) \arrow[r] & \operatorname{coker}(h) &
\end{tikzcd}$$

Then there is $\delta\colon \ker(h) \to \operatorname{coker}(f)$ *as in the following (not necessarily commuting) diagram*

$$\begin{tikzcd}
 & \ker(f) \arrow[r] \arrow[d] & \ker(g) \arrow[r] \arrow[d] & \ker(h) \arrow[d] \arrow[dddll, dashed, rounded corners, "\delta"'] & \\
 & A' \arrow[r, "i'"] \arrow[d, "f"] & B' \arrow[r, "p'"] \arrow[d, "g"] & C' \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A \arrow[r, "i"] \arrow[d] & B \arrow[r, "p"] \arrow[d] & C \arrow[d] & \\
 & \operatorname{coker}(f) \arrow[r] & \operatorname{coker}(g) \arrow[r] & \operatorname{coker}(h) &
\end{tikzcd}$$

such that the sequence

$$\ker(f) \to \ker(g) \to \ker(h) \xrightarrow{\delta} \operatorname{coker}(f) \to \operatorname{coker}(g) \to \operatorname{coker}(h)$$

is exact. Moreover, if i' *is a monomorphism then* $0 \to \ker(f) \to \ker(g)$ *is exact; if* p *is an epimorphism then* $\operatorname{coker}(g) \to \operatorname{coker}(h) \to 0$ *is exact.*

Proof. Without loss of generality we assume $\mathcal{C} = R$-**Mod** for some R by Mitchell's embedding theorem. The only hard part then is finding δ and showing that

$$\ker(g) \xrightarrow{p'\restriction\ker(g)} \ker(h) \xrightarrow{\delta} \operatorname{coker}(f) \xrightarrow{\bar{i}} \operatorname{coker}(g)$$

is exact at $\ker(h)$ and at $\operatorname{coker}(f)$.

What is δ? Well, suppose $x \in \ker(h) \subseteq C'$. Take y such that $p'(y) = x$; then $g(y) \in B$. We claim that there is $a \in A$ such that $i(a) = g(y)$; we then define $\delta(x) = a + \operatorname{im}(f) \in \operatorname{coker}(f)$. Symbolically: $\delta = \overline{i^{-1} \circ g \circ (p')^{-1}}$.

Why is this defined and well-defined? Suppose we have $y_1, y_2 \in B'$ such that $p'(y_1) = p'(y_2) = x \in \ker(h)$; then $h(p'(y_1)) = h(p'(y_2)) = 0$. So, examining our diagram, we find that $p(g(y_1)) = p(g(y_2)) = 0$, and $g(y_1), g(y_2) \in \ker(p) = \operatorname{im}(i)$. So, since i is a monomorphism, there are unique $a_1, a_2 \in A$ such that $i(a_1) = g(y_1)$ and $i(a_2) = g(y_2)$.

Claim 4.13. $i(a_1) + \operatorname{im}(f) = i(a_2) + \operatorname{im}(f)$*; i.e.* $i(a_1 - a_2) \in \operatorname{im}(f)$*.*

Proof. Well, $y_1 - y_2 \in \ker(p') = \operatorname{im}(i')$; so there is $b \in A'$ such that $i'(b) = y_1 - y_2$. But then $i(f(b)) = g(i'(b)) = g(y_1 - y_2) = i(a_1 - a_2)$; so, by injectivity of i, we have $f(b) = a_1 - a_2$. $\square$ Claim 4.13

So δ is well-defined; it remains to check exactness of

$$\ker(g) \xrightarrow{p'\restriction \ker(g)} \ker(h) \xrightarrow{\delta} \mathrm{coker}(f) \xrightarrow{\bar{i}} \mathrm{coker}(g)$$

For exactness at $\ker(h)$, note that for $x \in \ker(g)$, we have

$$\delta(p'(x)) = \overline{i^{-1} \circ g \circ (p')^{-1}(p'(x))} = \overline{i^{-1}(g(x))} = \overline{i^{-1}(0)} = 0$$

So $\mathrm{im}(p' \restriction \ker(g)) \subseteq \ker(\delta)$. It remains to check that $\ker(\delta) \subseteq \mathrm{im}(p' \restriction \ker(g))$. Suppose $x \in \ker(\delta)$; we must find $y \in \ker(g)$ such that $x = p'(y)$. Well, since $x \in \ker(\delta)$, we have that $\overline{(i^{-1} \circ g \circ (p')^{-1})(x)} = 0$; i.e. if we fix a preimage z of x under p' (i.e. with $p'(z) = x$), then $i^{-1}(g(z)) \in \mathrm{im}(f)$. So there is $a \in A$ such that $i^{-1}(g(z)) = f(a)$; so $g(z) = i(f(a)) = g(i'(a))$. So $z - i'(a) \in \ker(g)$. But $p'(z - i'(a)) = p'(z) - p'(i'(a)) = x$; so $x \in \mathrm{im}(p' \restriction \ker(g))$. So $\mathrm{im}(p' \restriction \ker(g)) = \ker(\delta)$, and we have exactness at $\ker(h)$.

We now check exactness at $\mathrm{coker}(f)$. As usual, to show that $\mathrm{im}(\delta) \subseteq \ker(\bar{i})$, we note that

$$\begin{aligned}
\bar{i}(f(x)) &= \bar{i}(\overline{i^{-1}(g((p')^{-1}(x)))}) \\
&= \bar{i}(i^{-1}(g((p')^{-1}(x))) + \mathrm{im}(f)) \\
&= i(i^{-1}(g((p')^{-1}(x))) + \mathrm{im}(f)) \\
&= g((p')^{-1}(x)) + \mathrm{im}(i \circ f) + \mathrm{im}(g) \\
&= g((p')^{-1}(x)) + \mathrm{im}(g \circ i') + \mathrm{im}(g) \\
&= 0 + \mathrm{im}(g) \\
&= \bar{0}
\end{aligned}$$

It remains to check the reverse inclusion. Suppose $x \in \ker(\bar{i})$. Then $x \in \mathrm{coker}(f)$, so we may write $x = x_0 + \mathrm{im}(f)$ for some $x_0 \in A$; then since $\bar{i}(x) = 0$, we have that $i(x_0) + \mathrm{im}(g) = 0 + \mathrm{im}(g)$, and $i(x_0) = g(u)$ for some $u \in B'$. Hence if we knew that $t = p'(u) \in \ker(h)$, then we would get

$$\delta(t) = \overline{i^{-1}(g((p')^{-1}(t)))} = i^{-1}(g(u)) = \overline{x_0} = x$$

and we'd be done. It then suffices to show that $p'(u) \in \ker(h)$; i.e. that $h(p'(u)) = 0$. But $h(p'(u)) = p(g(u)) = p(i(x_0)) = 0$ by exactness of $A \xrightarrow{i} B \xrightarrow{p} C$; so we indeed get that $p'(u) \in \ker(h)$. □ Lemma 4.12

We now return to our goal of producing a long exact sequence of homology from a short exact sequence of chain complexes.

Proposition 4.14. *Suppose we have a short exact sequence $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ where $A_\bullet = (A_n, a_n)$, $B_\bullet = (B_n, b_n)$, and $C_\bullet = (C_n, c_n)$ are chain complexes. Then we get a long exact sequence of homology*

$$\begin{tikzcd}
 & \cdots & H_{n+1}(C_\bullet) \arrow[dll, rounded corners, "\delta_{n+1}"'] \\
H_n(A_\bullet) \arrow[r] & H_n(B_\bullet) \arrow[r] & H_n(C_\bullet) \arrow[dll, rounded corners, "\delta_n"'] \\
H_{n-1}(A_\bullet) \arrow[r] & H_{n-1}(B_\bullet) \arrow[r] & H_{n-1}(C_\bullet) \arrow[dll, rounded corners, "\delta_{n-1}"'] \\
H_{n-2}(A_\bullet) & \cdots &
\end{tikzcd}$$

We are now in a position to do so.

Proof. We get a commuting diagram with exact rows

$$\begin{tikzcd}
 & A_n \arrow[r] \arrow[d, "a_n"] & B_n \arrow[r] \arrow[d, "b_n"] & C_n \arrow[r] \arrow[d, "c_n"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r] & B_{n-1} \arrow[r] & C_{n-1} &
\end{tikzcd}$$

By a weakening of the snake lemma, we get that

$$Z_n(A_\bullet) \to Z_n(B_\bullet) \to Z_n(C_\bullet)$$

and

$$A_{n-1}/\operatorname{im}(a_n) \to B_{n-1}/\operatorname{im}(b_n) \to C_{n-1}/\operatorname{im}(c_n)$$

are exact for all $n \in \mathbb{Z}$. One checks that since $0 \to A_n \to B_n$ and $B_{n-1} \to C_{n-1} \to 0$ are exact, then so are

$$0 \to Z_n(A_\bullet) \to Z_n(B_\bullet) \to Z_n(C_\bullet)$$

and

$$A_{n-1}/\operatorname{im}(a_n) \to B_{n-1}/\operatorname{im}(b_n) \to C_{n-1}/\operatorname{im}(c_n) \to 0$$

for all $n \in \mathbb{Z}$.

Claim 4.15. *We get an induced map $a_n \colon A_n/\operatorname{im}(a_{n+1}) \to Z_{n-1}(A_\bullet) \subseteq A_{n-1}$.*

Proof. Since $a_n \circ a_{n+1} = 0$, we get that $\operatorname{im}(a_{n+1}) \subseteq \ker(a_n)$; hence we get an induced $a_n \colon A_n/\operatorname{im}(a_{n+1}) \to A_{n-1}$. But we likewise get $\operatorname{im}(a_n) \subseteq \ker(a_{n-1}) = Z_{n-1}(A_\bullet)$; so we indeed get an induced map $a_n \colon A_n/\operatorname{im}(a_{n+1}) \to Z_{n-1}(A_\bullet)$. ☐ Claim 4.15

Likewise we get $b_n \colon B_n/\operatorname{im}(b_{n+1}) \to Z_{n-1}(B_\bullet)$ and $c_n \colon C_n/\operatorname{im}(c_{n+1}) \to Z_{n-1}(C_\bullet)$; one checks that the following diagram commutes:

$$\begin{tikzcd}
 & A_n/\operatorname{im}(a_{n+1}) \arrow[r, "\overline{f_n}"] \arrow[d, "a_n"] & B_n/\operatorname{im}(b_{n+1}) \arrow[r, "\overline{g_n}"] \arrow[d, "b_n"] & C_n/\operatorname{im}(c_{n+1}) \arrow[r] \arrow[d, "c_n"] & 0 \\
0 \arrow[r] & Z_{n-1}(A_\bullet) \arrow[r, "f_{n-1}\restriction Z_{n-1}(A_\bullet)"] & Z_{n-1}(B_\bullet) \arrow[r, "g_{n-1}\restriction Z_{n-1}(B_\bullet)"] & Z_{n-1}(C_\bullet) &
\end{tikzcd}$$

So we have a commuting diagram with exact rows; so the snake lemma yields $\delta_n \colon \ker(c_n) \to \operatorname{coker}(a_n)$ such that

$$\ker(a_n) \to \ker(b_n) \to \ker(c_n) \xrightarrow{\delta_n} \operatorname{coker}(a_n) \to \operatorname{coker}(b_n) \to \operatorname{coker}(c_n)$$

But $H_n(A_\bullet) = \ker(a_n)/\operatorname{im}(a_{n+1})$ is just the kernel of our induced $a_n \colon A_n/\operatorname{im}(a_{n+1}) \to Z_{n-1}(A_\bullet)$; likewise we have $H_{n-1}(A_\bullet) = Z_{n-1}(A_\bullet)/B_{n-1}(A_\bullet) = Z_{n-1}(A_\bullet)/\operatorname{im}(a_n)$ is just the cokernel of our induced a_n. So we indeed get that the sequence

$$\begin{tikzcd}
H_n(A_\bullet) \arrow[r] & H_n(B_\bullet) \arrow[r] & H_n(C_\bullet) \arrow[dll, rounded corners, "\delta_n"'] \\
H_{n-1}(A_\bullet) \arrow[r] & H_{n-1}(B_\bullet) \arrow[r] & H_{n-1}(C_\bullet)
\end{tikzcd}$$

is exact for all $n \in \mathbb{Z}$. ☐ Proposition 4.14

4.2 Homotopies of complexes

Definition 4.16. Suppose $\alpha, \beta \colon A_\bullet \to B_\bullet$ are two morphisms between the chain complexes $A_\bullet = (A_n, a_n)$ and $B_\bullet = (B_n, b_n)$. We say α is *homotopic* to β (or α is *homotopy equivalent* to β, written $\alpha \sim \beta$) if for all $n \in \mathbb{Z}$ there is $h_{n-1} \colon A_{n-1} \to B_n$ (i.e. $h_{n-1} \in \hom_{\mathcal{C}}(A_{n-1}, B_n)$ with no (immediate) additional assumptions on h_{n-1}) such that for all $n \in \mathbb{Z}$ we have

$$\alpha_n - \beta_n = h_{n-1} \circ a_n + b_{n+1} \circ h_n$$

For illustrative purposes, a diagram with all the maps:

$$\begin{tikzcd}
A_{n+1} \arrow[r, "a_{n+1}"] & A_n \arrow[r, "a_n"] \arrow[dl, "h_n"'] \arrow[d, shift right, "\alpha_n"'] \arrow[d, shift left, "\beta_n"] & A_{n-1} \arrow[dl, "h_{n-1}"] \\
B_{n+1} \arrow[r, "b_{n+1}"'] & B_n \arrow[r, "b_n"'] & B_{n-1}
\end{tikzcd}$$

Remark 4.17. $\sim$ is indeed an equivalence relation.

Proof. For reflexivity, take $h_n = 0_{A_n, B_{n-1}}$ for all $n \in \mathbb{Z}$. For symmetry, given $(h_n : n \in \mathbb{Z})$ showing that $\alpha \sim \beta$, note that $(-h_n : n \in \mathbb{Z})$ shows that $\beta \sim \alpha$. For transitivity, given $(h_n : n \in \mathbb{Z})$ and $(\widetilde{h_n} : n \in \mathbb{Z})$ such that

$$\begin{aligned}\alpha_n - \beta_n &= h_{n-1} \circ a_n + b_{n+1} \circ h_n \\ \beta_n - \gamma_n &= \widetilde{h}_{n-1} \circ a_n + b_{n+1} \circ \widetilde{h}_n\end{aligned}$$

note that

$$\alpha_n - \gamma_n = (h_{n-1} + \widetilde{h}_{n-1}) \circ a_n + b_{n+1} \circ (h_n + \widetilde{h}_n)$$

□ Remark 4.17

Proposition 4.18. *If $\alpha, \beta \colon (A_n, a_n) \to (B_n, b_n)$ are homotopy equivalent then α and β induce the same maps $H_n(A_\bullet) \to H_n(B_\bullet)$.*

Proof. It suffices to show that if $\gamma \colon (A_n, a_n) \to (B_n, b_n)$ has $\gamma \sim 0$, then γ induces the 0 map $H_n(A_\bullet) \to H_n(B_\bullet)$. Suppose $\gamma_n = h_{n-1} \circ a_n + b_{n+1} \circ h_n$ for some $h_n \colon A_n \to B_{n+1}$. In diagram:

$$\begin{tikzcd}
A_{n+1} \arrow[r, "a_{n+1}"] \arrow[d, "\gamma_{n+1}"'] & A_n \arrow[r, "a_n"] \arrow[d, "\gamma_n"] \arrow[dl, "h_n"] & A_{n-1} \arrow[d, "\gamma_{n-1}"] \arrow[dl, "h_{n-1}"] \\
B_{n+1} \arrow[r, "b_{n+1}"] & B_n \arrow[r, "b_n"] & B_{n-1}
\end{tikzcd}$$

Well, $H_n(A_\bullet) = Z_n(A_\bullet)/B_n(A_\bullet) = \ker(a_n)/\operatorname{im}(a_{n+1})$, and likewise we have $H_n(B_\bullet) = \ker(b_n)/\operatorname{im}(b_{n+1})$; the induced map $\gamma \colon \ker(a_n)/\operatorname{im}(a_{n+1}) \to \ker(b_n)/\operatorname{im}(b_{n+1})$ is then given by $x + \operatorname{im}(a_{n+1}) \mapsto \gamma_n(x) + \operatorname{im}(b_{n+1})$. To show that γ induces the 0 map, we must show that $\gamma_n(\ker(a_n)) \subseteq \operatorname{im}(b_{n+1})$. Take $x \in A_n$ such that $a_n(x) = 0$. Then

$$\gamma_n(x) = h_{n-1}(a_n(x)) + b_{n+1}(h_n(x)) + \operatorname{im}(b_{n+1}) = h_{n-1}(0) + \operatorname{im}(b_{n+1}) = \operatorname{im}(b_{n+1})$$

as desired. □ Proposition 4.18

A key proposition:

Proposition 4.19. *Suppose $F_\bullet$ is*

$$\cdots \to F_i \xrightarrow{\varphi_i} F_{i-1} \xrightarrow{\varphi_{i-1}} \cdots \xrightarrow{\varphi_1} F_0 \to 0 \to 0 \to \cdots$$

and $G_\bullet$ is

$$\cdots \to G_i \xrightarrow{\psi_i} G_{i-1} \xrightarrow{\psi_{i-1}} \cdots \xrightarrow{\psi_1} G_0 \to 0 \to 0 \to \cdots$$

*i.e. two chain complexes in an abelian category $\mathcal{C}$. (We will work in R-**Mod**.) Suppose for all i we have F_i and G_i are projective objects. In addition, let*

$$\begin{aligned}M &= \operatorname{coker}(\varphi_1) = H_0(F_\bullet) \\ N &= \operatorname{coker}(\psi_1) = H_0(G_\bullet)\end{aligned}$$

and suppose that $H_i(G_\bullet) = 0$ for all $i > 0$. Then any $\beta \colon M \to N$ is induced by a chain map $\alpha \colon F_\bullet \to G_\bullet$. Moreover, α is uniquely determined by β up to homotopy equivalence.

Proof. We proceed by induction.

(Existence) We have two exact sequences

$$F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\pi_F} M \to 0$$

and

$$G_1 \xrightarrow{\psi_1} G_0 \xrightarrow{\pi_G} N \to 0$$

So, since F_0 is projective, there is some $\alpha_0 \colon F_0 \to G_0$ such that $\pi_G \circ \alpha_0 = \beta \circ \pi_F$; i.e. such that the following diagram commutes:

$$\begin{tikzcd} & F_0 \arrow[dl, dashed, "\alpha_0"'] \arrow[d, "\beta\circ\pi_F"] & \\ G_0 \arrow[r, two heads, "\pi_G"] & N \arrow[r] & 0 \end{tikzcd}$$

Now, $\alpha_0 \circ \varphi_1 \colon F_1 \to G_0$. Also

$$\pi_G \circ \alpha_0 \circ \varphi_1 = \beta \circ \pi_F \circ \varphi_1 = \beta \circ 0 = 0$$

by exactness; so $\operatorname{im}(\alpha_0 \circ \varphi_1) \subseteq \ker(\pi_G) = \operatorname{im}(\psi_1)$. So, since F_1 is projective, there is some $\alpha_1 \colon F_1 \to G_1$ such that $\psi_1 \circ \alpha_1 = \alpha_0 \circ \varphi_1$; i.e. such that the following diagram commutes:

$$\begin{tikzcd} & F_1 \arrow[dl, dashed, "\alpha_1"'] \arrow[d, "\alpha_0\circ\varphi_1"] & \\ G_1 \arrow[r, "\psi_1"] & \operatorname{im}(\psi_1) \arrow[r] & 0 \end{tikzcd}$$

Continuing in this manner, and using the fact that $H_i(G_\bullet) = 0$ for all $i > 0$, we get a chain map $\alpha \colon F_\bullet \to G_\bullet$. Moreover, $\alpha_0 \colon F_0/\operatorname{im}(\varphi_1) \to G_0/\operatorname{im}(\psi_1)$ has

$$\alpha_0(x + \operatorname{im}(\varphi_1)) = \alpha_0(x) + \operatorname{im}(\alpha_0 \circ \varphi_1) = \alpha_0(x) + \operatorname{im}(\psi_1 \circ \alpha_1)$$

for $x \in F_0$. But $\pi_G \circ \alpha_0 = \beta \circ \pi_F$; so $\pi_G(\alpha_0(x)) = \beta(x + \operatorname{im}(\varphi_1))$, and

$$\alpha_0(x + \operatorname{im}(\varphi_1)) = \alpha_0(x) + \operatorname{im}(\psi_1) = \beta(x) + \operatorname{im}(\psi_1)$$

So β is induced by the chain map $\alpha \colon F_\bullet \to G_\bullet$.

(Uniqueness) Suppose $\alpha, \alpha' \colon F_\bullet \to G_\bullet$ both induce β; we must show that $\alpha \sim \alpha'$. This reduces to showing that if $\gamma \colon F_\bullet \to G_\bullet$ induces $0_{M,N} \colon M \to N$, then $\gamma \sim 0$; we may thus assume that $\beta \colon M \to N$ is the 0 map. Our picture is

$$\begin{tikzcd} F_1 \arrow[r, "\varphi_1"] \arrow[d, "\gamma_1"] & F_0 \arrow[r, "\pi_F"] \arrow[d, "\gamma_0"] \arrow[dl, dashed, "h_0"] & M \arrow[r] \arrow[d, "0"] & 0 \\ G_1 \arrow[r, "\psi_1"] & G_0 \arrow[r, "\pi_G"] & N \arrow[r] & 0 \end{tikzcd}$$

where $h_0 \colon F_0 \to G_1$ is the map we wish to find.

Claim 4.20. $\operatorname{im}(\gamma_0) \subseteq \operatorname{im}(\psi_1) = \ker(\pi_G)$.

Proof. Well, $\pi_G \circ \gamma_0 = 0 \circ \pi_F = 0$; so $\operatorname{im}(\gamma_0) \subseteq \ker(\pi_G) = \operatorname{im}(\psi_1)$. □ Claim 4.20

So, since F_0 is projective, there is $h_0 \colon F_0 \to G_1$ such that $\gamma_0 = \psi_1 \circ h_0$; in (commuting) diagram:

$$\begin{tikzcd} & F_0 \arrow[dl, dashed, "h_0"'] \arrow[d, "\gamma_0"] & \\ G_1 \arrow[r, "\psi_1"] & \operatorname{im}(\psi_1) \arrow[r] & 0 \end{tikzcd}$$

We must now produce $h_1 \colon F_1 \to G_2$ such that $\psi_2 \circ h_1 + h_0 \circ \varphi_1 = \gamma_1$. But $\gamma_0 = \psi_1 \circ h_0$; so

$$\psi_1 \circ (h_0 \circ \varphi_1 - \gamma_1) = \psi_1 \circ h_0 \circ \varphi_1 - \psi_1 \circ \gamma_1 = \gamma_0 \circ \varphi_1 - \psi_1 \circ \gamma_1 = 0$$

since γ is a morphism of chain complexes. So $\operatorname{im}(h_0 \circ \varphi_1 - \gamma_1) \subseteq \ker(\psi_1) = \operatorname{im}(\psi_2)$. So, since F_1 is projective, we get $h_1 \colon F_1 \to G_2$ such that $-h_0 \circ \varphi_1 + \gamma_1 = \psi_2 \circ h_1$, as in the following commuting diagram:

$$\begin{tikzcd} & F_1 \arrow[dl, dashed, "h_1"'] \arrow[d, "-h_0\circ\varphi_1+\gamma_1"] & \\ G_2 \arrow[r, "\psi_2"] & \operatorname{im}(\psi_2) \arrow[r] & 0 \end{tikzcd}$$

Then $\gamma_1 = \psi_2 \circ h_1 + h_0 \circ \varphi_1$. Continuing in this manner, we get a homotopy $\gamma \sim 0$. $\square$ Proposition 4.19

4.3 Projective resolution

Suppose $\mathcal{C}$ is an abelian category with enough projectives (respectively, enough injectives); i.e. for all $C \in \operatorname{Ob}(\mathcal{C})$ there is a projective $P \in \operatorname{Ob}(\mathcal{C})$ and an epi $P \twoheadrightarrow C$ (respectively, an injective I and a mono $C \hookrightarrow I$). Then we can make a *projective resolution* of $C \in \operatorname{Ob}(\mathcal{C})$: an exact sequence

$$\cdots \to P_2 \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0 \to C \to 0$$

with each P_i projective.

Why must this exist? We work in R-**Mod**. Then there is a projective P_0 with an epi $\varphi_0 \colon P_0 \twoheadrightarrow C$; we get a short exact sequence $0 \to K_0 \to P_0 \to C \to 0$. Let $K_0 = \ker(\varphi_0)$. Then there is a projective P_1 and an epi $\varphi \colon P_1 \twoheadrightarrow K_0$; then

$$P_1 \xrightarrow{\varphi_1} P_0 \xrightarrow{\varphi_0} C \to 0$$

is exact since $\operatorname{im}(\varphi_1) = K_0 = \ker(\varphi_0)$. Let $K_1 = \ker(\varphi_1)$; then

$$0 \to K_1 \to P_1 \to P_0 \to C \to 0$$

is exact. We can find a projective P_2 and an epi $\varphi_2 \colon P_2 \twoheadrightarrow K_1$. Then

$$P_2 \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0 \to C \to 0$$

is exact. And so on.

Similarly, if we have enough injectives, we get an injective resolution of C: an exact sequence

$$0 \to C \to I_0 \to I_1 \to I_2 \to \cdots$$

with each I_i injective.

Theorem 4.21. *Let $C \in \operatorname{Ob}(\mathcal{C})$. If*

$$\cdots \to P_2 \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0 \xrightarrow{\varphi_0} C \to 0$$

and

$$\cdots \to Q_2 \xrightarrow{\psi_2} Q_1 \xrightarrow{\psi_1} Q_0 \xrightarrow{\psi_0} C \to 0$$

are two projective resolutions of C. Then

1. *The chain complexes $P_\bullet$ and $Q_\bullet$ given by*

$$\cdots \to P_2 \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0 \to 0 \to \cdots$$

and

$$\cdots \to Q_2 \xrightarrow{\psi_2} Q_1 \xrightarrow{\psi_1} Q_0 \to 0 \to \cdots$$

respectively are homotopy equivalent.

2. *If $\mathcal{D}$ is an abelian category and $F \colon \mathcal{C} \to \mathcal{D}$ is an additive functor, then for all i we have $H_i(FP_\bullet) \cong H_i(FQ_\bullet)$.*

Remark 4.22. $FP_\bullet$ and $FQ_\bullet$ given by

$$\cdots \to FP_2 \xrightarrow{F\varphi_2} FP_1 \xrightarrow{F\varphi_1} FP_0 \to 0 \to \cdots$$

and

$$\cdots \to FQ_2 \xrightarrow{F\psi_2} FQ_1 \xrightarrow{F\psi_1} FQ_0 \to 0 \to \cdots$$

are indeed chain complexes, since

$$(F\varphi_i) \circ (F\varphi_{i+1}) = F(\varphi_i \circ \varphi_{i+1}) = F(0) = 0$$

since F is additive.

Proof of Theorem 4.21. By our last result, there are $\alpha\colon P_\bullet \to Q_\bullet$ and $\beta\colon Q_\bullet \to P_\bullet$ such that α, β induce $\mathrm{id}_C\colon C \to C$; we get the following commuting diagram:

$$\begin{tikzcd}
{} \arrow[r] & P_2 \arrow[r] \arrow[d, "\alpha_2"] & P_1 \arrow[r] \arrow[d, "\alpha_1"] & P_0 \arrow[r] \arrow[d, "\alpha_0"] & C \arrow[r] \arrow[d, "\mathrm{id}_C"] & 0 \\
{} \arrow[r] & Q_2 \arrow[r] \arrow[d, "\beta_2"] & Q_1 \arrow[r] \arrow[d, "\beta_1"] & Q_0 \arrow[r] \arrow[d, "\beta_0"] & C \arrow[r] \arrow[d, "\mathrm{id}_C"] & 0 \\
{} \arrow[r] & P_2 \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

So $\beta \circ \alpha\colon P_\bullet \to P_\bullet$ induces $\mathrm{id}_C\colon C \to C$. But $\mathrm{id}_{P_\bullet}\colon P_\bullet \to P_\bullet$ also induces $\mathrm{id}_C\colon C \to C$. So $\beta \circ \alpha \sim \mathrm{id}_{P_\bullet}$. Similarly, we get that $\alpha \circ \beta \sim \mathrm{id}_{Q_\bullet}$. We get the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "\varphi_2"] \arrow[d, "\beta_2\circ\alpha_2"] & P_1 \arrow[r, "\varphi_1"] \arrow[d, "\beta_1\circ\alpha_1"] \arrow[dl, "h_1"] & P_0 \arrow[r] \arrow[d, "\beta_0\circ\alpha_0"] \arrow[dl, "h_0"] & 0 \arrow[r] & \cdots \\
\cdots \arrow[r] & P_2 \arrow[r, "\varphi_2"] & P_1 \arrow[r, "\varphi_1"] & P_0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

So there are $h_i\colon P_i \to P_{i+1}$ such that $\beta_i \circ \alpha_i - \mathrm{id}_{P_i} = \varphi_{i+1} \circ h_i + h_{i-1} \circ \varphi_i$. Applying F everywhere, we find that

$$F(\beta_i) \circ F(\alpha_i) - \mathrm{id}_{F(P_i)} = F(\varphi_{i+1}) \circ F(h_i) + F(h_{i-1}) \circ F(\varphi_i)$$

So $F(h_i)\colon FP_i \to FP_{i+1}$ show that

$$F(\alpha)\colon FP_\bullet \to FQ_\bullet$$
$$F(\beta)\colon FQ_\bullet \to FP_\bullet$$

satisfy $F(\beta) \circ F(\alpha) \sim \mathrm{id}_{F(P_\bullet)}$. Similarly, we get $F(\alpha) \circ F(\beta) \sim \mathrm{id}_{F(Q_\bullet)}$. So $F(\beta) \circ F(\alpha)$ and $\mathrm{id}_{F(P_\bullet)}$ induce the same map (i.e. the identity map) from $H_i(FP_\bullet) \to H_i(FP_\bullet)$. Similarly, $F(\alpha) \circ F(\beta)$ induces the identity on $H_i(FQ_\bullet)$ for all i. So $\beta \circ \alpha\colon P_\bullet \to P_\bullet$ induces $\mathrm{id}_C\colon C \to C$. But $\mathrm{id}_{P_\bullet}\colon P_\bullet \to P_\bullet$ also induces $\mathrm{id}_C\colon C \to C$. So $\beta \circ \alpha \sim \mathrm{id}_{P_\bullet}$. □ Theorem 4.21

We then say that the map $P_\bullet \to Q_\bullet$ is a *quasi-isomorphism*; i.e. the induced maps $H_i(P_\bullet) \to H_i(Q_\bullet)$ are isomorphisms.

5 Derived functors

Suppose we have $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ exact, and suppose that F is a right-exact additive functor. (e.g. in R-**Mod**, if M is a right R-module, we could take $F = M \otimes_R -\colon R\text{-}\mathbf{Mod} \to \mathbf{Ab}$.) We know

$$0 \to K \to FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC \to 0$$

is exact for some K; we'd like to understand K. e.g. if $N_1 \xhookrightarrow{i} N_2$ in R-**Mod**, what is

$$\ker(M \otimes_R N_1 \xrightarrow{\mathrm{id} \otimes_R i} M \otimes_R N_2)?$$

As we'll see, there is a first left-derived functor $L_1 F$ satisfying

$$L_1 FC \xrightarrow{\delta} FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC \to 0$$

In fact the object $L_1 FC$ is independent of f and g; it merely requires that $0 \to A \to B \to C \to 0$ be exact.

Definition 5.1. Suppose $\mathcal{C}$ and $\mathcal{D}$ are abelian categories; suppose $\mathcal{C}$ has enough projectives. Suppose $F \colon \mathcal{C} \to \mathcal{D}$ is right-exact and additive. Suppose $A \in \mathrm{Ob}(\mathcal{C})$; let

$$\cdots P_2 \to P_1 \to P_0 \to A \to 0$$

be a projective resolution. From this we obtain a chain complex $P_\bullet$ consisting of

$$\cdots P_2 \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0 \xrightarrow{\varphi_0} 0 \to 0 \to \cdots$$

to which we can apply F to get another chain complex $FP_\bullet$ consisting of

$$\cdots FP_2 \xrightarrow{F\varphi_2} FP_1 \xrightarrow{F\varphi_1} FP_0 \xrightarrow{F\varphi_0} 0 \to \cdots$$

We then define $L_i F(A) = H_i(FP_\bullet)$; $L_i F$ is called the i^{th} *left-derived functor of* F.

Why is this well-defined? Well, if $P_\bullet$ and $P'_\bullet$ are two chain complexes arising from projective resolutions of A, then there are $u \colon P_\bullet \to P'_\bullet$ and $v \colon P'_\bullet \to P_\bullet$ with $v \circ u \sim \mathrm{id}_{P_\bullet}$ and $u \circ v \sim \mathrm{id}_{P'_\bullet}$. Then $F(u) \colon FP_\bullet \to FP'_\bullet$ and $F(v) \colon FP'_\bullet \to FP_\bullet$ have $F(u) \circ F(v) = F(u \circ v) \sim F(\mathrm{id}_{P'_\bullet}) = \mathrm{id}_{FP'_\bullet}$. Similarly, we have $F(v) \circ F(u) \sim \mathrm{id}_{FP_\bullet}$. So $F(u)$ yields a quasi-isomorphism; in particular, we have $H_i(FP_\bullet) \cong H_i(FP'_\bullet)$, and $L_i F(A)$ is well-defined.

If $f \colon A \to B$, what is $L_1 F(f)$? Well, if

$$\cdots P_2 \to P_1 \to P_0 \to A \to 0$$

and

$$\cdots Q_2 \to Q_1 \to Q_0 \to B \to 0$$

are projective resolutions of A and B respectively, then there is $\theta \colon P_\bullet \to Q_\bullet$ such that θ induces f in $H_0(P_\bullet) \to H_0(Q_\bullet)$. We then set $L_i F(f)$ to be the map $H_i(FP_\bullet) \to H_i(FQ_\bullet)$ induced by $F(\theta) \colon FP_\bullet \to FQ_\bullet$. One checks that this is well-defined; one uses the fact that given two chain complexes $P_\bullet$ and $P'_\bullet$ arising from projective resolutions of A, we have that θ gives a canonical isomorphism $H_i(P_\bullet) \to H_i(P'_\bullet)$.

We saw that $L_i FA$ is independent of choice of projective resolution; we also have

Theorem 5.2. *$L_0 F = F$.*

Proof. There is $\varphi \colon P_0 \twoheadrightarrow A$ such that

$$P_2 \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0 \xrightarrow{\varphi} A \to 0$$

is exact. But F is right-exact; so if $K = \ker(\varphi)$, then since $0 \to K \to P_0 \to A \to 0$ is exact, we get that

$$FK \to FP_0 \to FA \to 0$$

is exact. We also have that $P_1 \to P_0 \to A \to 0$ is exact; so

$$FP_1 \to FP_0 \to FA \to 0$$

is exact. What is $L_0 F$? The 0^{th} homology of

$$\cdots FP_1 \xrightarrow{F\varphi_1} FP_0 \to 0 \to \cdots$$

i.e. $L_0 FA = FP_0 / \operatorname{im}(FP_1)$. But $\operatorname{im}(FP_1) = \ker(F\varphi)$; so $L_0 FA \cong FP_0 / \ker(F\varphi) \cong A$. ☐ Theorem 5.2

Theorem 5.3. *If A is projective then $L_iFA = 0$ for all $i \geq 1$.*

Proof. Notice

$$\cdots \to 0 \to 0 \to A \xrightarrow{\mathrm{id}_A} A \to 0$$

is a projective resolution of A; we get the chain complex $P_\bullet$ consisting of

$$\cdots \to 0 \to 0 \to A \to 0 \to \cdots$$

Applying F, we get the chain complex

$$\cdots \to 0 \to 0 \to FA \to 0 \to \cdots$$

So $H_i(FP_\bullet) = 0$ for all $i \geq 1$; so $L_iFA = 0$ for all $i \geq 1$. $\square$ Theorem 5.3

Theorem 5.4. *Suppose F is right-exact and additive; suppose $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a short exact sequence. Then there is a long exact sequence*

$$\begin{tikzcd}
 & & \cdots \arrow[dll, "\delta_3"'] & \\
L_2FA \arrow[r, "L_2Ff"] & L_2FB \arrow[r, "L_2Fg"] & L_2FC \arrow[dll, "\delta_2"'] & \\
L_1FA \arrow[r, "L_1Ff"] & L_1FB \arrow[r, "L_1Fg"] & L_1FC \arrow[dll, "\delta_1"'] & \\
FA \arrow[r, "Ff"] & FB \arrow[r, "Fg"] & FC \arrow[r] & 0
\end{tikzcd}$$

where $\delta \colon L_iFC \to L_{i-1}FA$.

Proof. Fix chain complexes $P_\bullet$ and $Q_\bullet$ arising from projective resolutions of A and C, respectively; we'd like to find a projective resolution $\cdots U_2 \to U_1 \to U_0 \to B \to 0$ of B such that the following diagram has exact columns:

$$\begin{tikzcd}
\cdots & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d, "\theta_2"] & P_1 \arrow[r] \arrow[d, "\theta_1"] & P_0 \arrow[r] \arrow[d, "\theta_0"] & A \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r] & U_2 \arrow[r] \arrow[d, "\tau_2"] & U_1 \arrow[r] \arrow[d, "\tau_1"] & U_0 \arrow[r] \arrow[d, "\tau_0"] & B \arrow[r] \arrow[d, "g"] & 0 \\
\cdots \arrow[r] & Q_2 \arrow[r] \arrow[d] & Q_1 \arrow[r] \arrow[d] & Q_0 \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
\cdots & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

i.e. such that $0 \to P_\bullet \xrightarrow{\theta} U_\bullet \xrightarrow{\tau} Q_\bullet \to 0$ is exact and θ induces $f \colon A \to B$ on $H_0(P_\bullet) \to U_\bullet)$ and τ induces $g \colon B \to C$ on $H_0(U_\bullet) \to H_0(Q_\bullet)$.

Claim 5.5. *We can find such $U_\bullet$, θ, and τ.*

Proof. At the first stage, we need to find U_0 and maps θ_0, τ_0 and χ_0 such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & \\
P_0 \arrow[r, "\varphi_0"] \arrow[d, "\theta_0"] & A \arrow[r] \arrow[d, "f"] & 0 \\
U_0 \arrow[r, "\chi_0"] \arrow[d, "\tau_0"] & B \arrow[r] \arrow[d, "g"] & 0 \\
Q_0 \arrow[r, "\psi_0"] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 & 0 &
\end{tikzcd}$$

How do we find such U_0, θ_0, and τ_0? Well, Q_0 is projective; so we have $h_0\colon Q_0 \to B$ such that $g \circ h_0 = \psi_0$; i.e. such that the following diagram commutes:

$$\begin{tikzcd}
& Q_0 \arrow[dl, dashed, "h_0"'] \arrow[d, "\psi_0"] & \\
B \arrow[r, two heads, "g"'] & C \arrow[r] & 0
\end{tikzcd}$$

Let $k_0 = f \circ \varphi_0\colon P_0 \to B$. Let $U_0 = P_0 \oplus Q_0$; let

$$\begin{aligned}
\chi_0 = k_0 + h_0 &\colon P_0 \oplus Q_0 \to B \\
\theta_0 = i &\colon P_0 \to P_0 \oplus Q_0 \\
\tau_0 = \pi &\colon P_0 \oplus Q_0 \to Q_0
\end{aligned}$$

Working in R-**Mod**, we note that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & \\
P_0 \arrow[r, "\varphi_0"] \arrow[d, "\theta_0"] & A \arrow[r] \arrow[d, "f"] & 0 \\
U_0 \arrow[r, "\chi_0"] \arrow[d, "\tau_0"] & B \arrow[r] \arrow[d, "g"] & 0 \\
Q_0 \arrow[r, "\psi_0"] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 & 0 &
\end{tikzcd}$$

Indeed, for the top square, if $p \in P_0$ then going one way we get

$$p \mapsto \varphi_0 \mapsto f(\varphi_0(p)) = k_0(p)$$

and going the other way we get

$$p \mapsto (p, 0) \mapsto k_0(p)$$

For the bottom square, if $(p, q) \in P_0 \oplus Q_0$, then going one way we get

$$(p, q) \mapsto k_0(p) + h_0(q) \mapsto g(k_0(p)) + g(h_0(q)) = \psi_0(q) + g(f(\varphi_0(p))) = \psi_0(q)$$

and going the other way we get

$$(p, q) \mapsto q \mapsto \psi_0(q)$$

We do one more iteration. We now wish to find U_1 and maps θ_1, τ_1, and χ_1 such that the following diagram commutes:

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
& P_1 \arrow[r, "\varphi_1"] \arrow[d, "\theta_1"] & P_0 \arrow[r, "\varphi_0"] \arrow[d, "\theta_0"] & A \arrow[r] \arrow[d, "f"] & 0 \\
& U_1 \arrow[r, "\chi_1"] \arrow[d, "\tau_1"] & U_0 \arrow[r, "\chi_0"] \arrow[d, "\tau_0"] & B \arrow[r] \arrow[d, "g"] & 0 \\
& Q_1 \arrow[r, "\psi_1"] \arrow[d] & Q_0 \arrow[r, "\psi_0"] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

We let $U_1 = P_1 \oplus Q_1$, and define the maps by

$$\chi_1 = k_1 + h_1 \colon P_1 \oplus Q_1 \to \ker(\chi_0)$$
$$\theta_1 = i \colon P_1 \to P_1 \oplus Q_1$$
$$\tau_1 = \pi \colon P_1 \oplus Q_1 \to Q_1$$

One checks that the diagram does indeed commute. Continuing in this way, we get the desired result. □ Claim 5.5

Now, apply F to $0 \to P_\bullet \xrightarrow{\theta} U_\bullet \xrightarrow{\tau} Q_\bullet \to 0$.

Claim 5.6. *$0 \to FP_i \to FU_i \to FQ_i \to 0$ is exact for all i.*

Proof. It suffices to show that if

$$0 \to P \to U \to Q \to 0$$

is a short exact sequence of projective objects, then

$$0 \to FP \to FU \to FQ \to 0$$

is exact. Why? Well, since Q is projective, we have a section $s \colon Q \to U$ of τ:

$$\begin{tikzcd}
& & & Q \arrow[dl, dashed, "s"'] \arrow[d, "\mathrm{id}_Q"] & \\
0 \arrow[r] & P \arrow[r, "\theta"] & U \arrow[r, "\tau"] & Q \arrow[r] & 0
\end{tikzcd}$$

Then $\mathrm{id}_U - s \circ \tau \colon U \to U$ satisfies

$$\tau \circ (\mathrm{id}_U - s \circ \tau) = \tau - \tau \circ s \circ \tau = \tau - \mathrm{id}_Q \circ \tau = 0$$

So $\mathrm{id}_U - s \circ \tau$ maps to $\ker(\tau) = \mathrm{im}(\theta)$, and there is $t \colon U \to P$ such that $\theta \circ t = \mathrm{id}_U - s \circ \tau$:

$$\begin{tikzcd}
& & U \arrow[dl, dashed, "t"'] \arrow[d, "\mathrm{id}_U - s\circ\tau"] & & \\
0 \arrow[r] & P \arrow[r, "\theta"] & U \arrow[r, "\tau"] & Q \arrow[r] & 0
\end{tikzcd}$$

We now apply F. Since F is right exact, we get

$$FP \xrightarrow{F\theta} FU \xrightarrow{F\tau} FQ \to 0$$

is exact; we also have $Ft \colon FU \to FP$ and $Fs \colon FQ \to FU$. I think at this point we just use the fact that since Q is projective, we have a retraction of θ, which then lifts to a retraction of $F\theta$.

TODO 7. *Do we still need all the work with s and t?*

□ Claim 5.6

So $0 \to FP_\bullet \to FU_\bullet \to FQ_\bullet \to 0$ is exact; so we get a long exact sequence of homology

$$\begin{tikzcd}
 & & \cdots \arrow[dll, rounded corners, "\delta_3"'] \\
H_2(FP_\bullet) \arrow[r] & H_2(FU_\bullet) \arrow[r] & H_2(FQ_\bullet) \arrow[dll, rounded corners, "\delta_2"'] \\
H_1(FP_\bullet) \arrow[r] & H_1(FU_\bullet) \arrow[r] & H_1(FQ_\bullet) \arrow[dll, rounded corners, "\delta_1"'] \\
H_0(FP_\bullet) \arrow[r] & H_0(FU_\bullet) \arrow[r] & H_0(FQ_\bullet) \arrow[r] & 0
\end{tikzcd}$$

i.e. we have an exact sequence

$$\begin{tikzcd}
 & & \cdots \arrow[dll, rounded corners, "\delta_3"'] \\
L_2FA \arrow[r, "L_2Ff"] & L_2FB \arrow[r, "L_2Fg"] & L_2FC \arrow[dll, rounded corners, "\delta_2"'] \\
L_1FA \arrow[r, "L_1Ff"] & L_1FB \arrow[r, "L_1Fg"] & L_1FC \arrow[dll, rounded corners, "\delta_1"'] \\
FA \arrow[r, "Ff"] & FB \arrow[r, "Fg"] & FC \arrow[r] & 0
\end{tikzcd}$$

as desired. □ Theorem 5.4

Remark 5.7. We have been using the fact that if $\mathcal{C}, \mathcal{D}$ are abelian categories and $F \colon \mathcal{C} \to \mathcal{D}$ is additive, then $F(0_{\mathcal{C}}) = 0_{\mathcal{D}}$. The reason for this is that if $A = F(0_{\mathcal{C}})$, then $\mathrm{id}_A = F(\mathrm{id}_{0_{\mathcal{C}}}) = F(0) = 0$. So $F(A)$ is an initial object, since any $f \in \hom_{\mathcal{D}}(A, B)$ satisfies $f = f \circ \mathrm{id}_A = f \circ 0 = 0$; likewise we get that A is a terminal object, and hence that $F(A) = 0_{\mathcal{D}}$.

Remark 5.8. Suppose $\mathcal{C}$ has enough injectives. Suppose $G \colon \mathcal{C} \to \mathcal{D}$ is additive and left-exact. If $C \in \mathrm{Ob}(\mathcal{C})$, we get an injective resolution

$$0 \to C \to I^0 \to I^1 \to \cdots$$

Applying G, we get

$$0 \to GC \to GI^0 \to GI^1 \to \cdots$$

and hence we get a chain complex $I^\bullet$ given by

$$0 \to GI^0 \to GI^1 \to \cdots$$

We then define $R^iG(C) = H^i(GI^\bullet)$. We get

1. $R^0G = G$.

2. C injective implies $R^iGC = 0$ for $i > 0$.

3. If $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact, then we get a long exact sequence of homology

$$\begin{tikzcd}
0 \arrow[r] & GA \arrow[r] & GB \arrow[r] & GC \arrow[dll, rounded corners, "\delta^0"'] \\
 & R^1GR \arrow[r] & \cdots
\end{tikzcd}$$

4. Given

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] & B \arrow[r, "g'"] & C' &
\end{tikzcd}$$

   We get that the following diagram commutes:

$$\begin{tikzcd}
R^iGC \arrow[r, "\delta^i"] \arrow[d, "R^iG\gamma"] & R^{i+1}GA \arrow[d, "R^{i+1}G\alpha"] \\
R^iGC' \arrow[r, "(\delta')^i"] & R^{i+1}GA'
\end{tikzcd}$$

Remark 5.9. The above results allow us to recover L_iFA for all $i \geq 0$ and for all $A \in \mathrm{Ob}(\mathcal{C})$. Indeed, we have $L_0FA = FA$; suppose now that we know L_iFA for all $i < n$. Then we can put A in a short exact sequence $0 \to K \to P \to A \to 0$ where P is projective; so we get a long exact sequence of homology

$$\begin{tikzcd}
 & & \cdots \arrow[dll, rounded corners] & \\
L_2FK \arrow[r] & 0 \arrow[r] & L_2FA \arrow[dll, rounded corners] & \\
L_1FK \arrow[r] & 0 \arrow[r] & L_1FA \arrow[dll, rounded corners] & \\
FK \arrow[r] & FP \arrow[r] & FA \arrow[r] & 0
\end{tikzcd}$$

So $L_2FA \cong L_1FK$, and $L_3FA \cong L_2FK$, and so forth. So knowing L_iFK gives us $L_{i+1}FA$ for all $i \geq 1$; we can obtain L_1FA from the exact sequence

$$0 \to L_1FA \to FK \to FP \to FA \to 0$$

6 Tor

Suppose R is a ring; consider R-**Mod**, the category of left R-modules. Suppose M is a right R-module and a left S-module (typically $S = \mathbb{Z}$). Then we get a functor $F\colon R\text{-}\mathbf{Mod} \to S\text{-}\mathbf{Mod}$ given by $N \mapsto M \otimes_R N$. Then F is right-exact and additive.

Definition 6.1. We define $\mathrm{Tor}_i^R(M, N) = L_iFN$. (i.e. $\mathrm{Tor}_i^R(M, -) = L_iF$.)

Remark 6.2. Tor measures how close M is to being flat.

Theorem 6.3. *the following are equivalent:*

1. *M is flat.*
2. $\mathrm{Tor}_i^R(M, N) = 0$ *for all* $i \geq 1$ *and all* $N \in \mathrm{Ob}(R\text{-}\mathbf{Mod})$.
3. $\mathrm{Tor}_1^R(M, N) = 0$ *for all* $N \in \mathrm{Ob}(R\text{-}\mathbf{Mod})$.

Proof.

(1) $\Longrightarrow$ (2) Take a left R-module N and a projective resolution

$$\cdots \to P_1 \to P_0 \to N \to 0$$

   Then since M is flat and the resolution is exact, we get that

$$\cdots \to M \otimes_R P_1 \to M \otimes_R P_0 \to M \otimes_R N \to 0$$

   is still exact. So $H_i(F(P_\bullet)) = 0$ for all $i \geq 1$; so $\mathrm{Tor}_i^R(M, N) = 0$ for all $i \geq 1$.

(2) $\implies$ (3) Immediate.

(3) $\implies$ (1) Suppose that $\operatorname{Tor}_1^R(M,N)=0$ for all $N\in\operatorname{Ob}(R\text{-}\mathbf{Mod})$; suppose $0\to A\to B$ is exact. Let C be the cokernel of $A\to B$; so $0\to A\to B\to C\to 0$ is exact. Applying $M\otimes_R -$ and taking the long exact sequence of homology, we find that

$$\begin{tikzcd} & M\otimes_R A \arrow[r] & M\otimes_R B \arrow[r] & M\otimes_R C \arrow[r] & 0 \\ & & & \operatorname{Tor}_1^R(M,C) \arrow[ull, rounded corners] & \end{tikzcd}$$

But $\operatorname{Tor}_1^R(M,C)=0$. So $0\to M\otimes_R A\to M\otimes_R B\to M\otimes_R C\to 0$ is exact; so M is flat. $\square$ Theorem 6.3

In algebraic geometry, Tor is used to give a measure of "intersection"; see Serre's formula.

Example 6.4. Consider $R=\mathbb{C}[x]$; consider $M=\mathbb{C}[x]/(f(x))$ and $N=\mathbb{C}[x]/(g(x))$. Then N fits into a short exact sequence

$$0\to(g(x))\to\mathbb{C}[x]\to\mathbb{C}[x]/(g(x))\to 0$$

Since $(g(x))$ is principal, we get that it is isomorphic as an $\mathbb{C}[x]$-module to $\mathbb{C}[x]$. So we get a free resolution of N

$$0\to\mathbb{C}[x]\xrightarrow{m}\mathbb{C}[x]\xrightarrow{\pi}N\to 0$$

(where $m(p)=pg$). Tensoring with M, we get a chain complex $C_\bullet$ given by

$$\cdots\to 0\to M\otimes_{\mathbb{C}[x]}\mathbb{C}[x]\to M\otimes_{\mathbb{C}[x]}\mathbb{C}[x]\to 0$$

So $\operatorname{Tor}_i^R(M,N)=H_i(C_\bullet)$. In particular, we have $H_i(C_\bullet)=0$ for all $i\geq 2$; so $\operatorname{Tor}_i^R(M,N)=0$ for all $i\geq 2$.

For $\operatorname{Tor}_0^R(M,N)$, note that the map $M\otimes_{C[x]}\mathbb{C}[x]\to M\otimes_{\mathbb{C}[x]}\mathbb{C}[x]$ can be expressed as a map $M\to M$ given by $a\mapsto g(x)a$. Then $\operatorname{Tor}_0^R(M,N)$ is the kernel of the zero map modulo the image of this map; i.e. $M/g(x)M$. Since $M=\mathbb{C}[x]/(f(x))$, we note that $M/g(x)M=\mathbb{C}[x]/(f(x),g(x))$. In particular, if $h=\gcd(f,g)$, then $M\otimes_R N=\operatorname{Tor}_0(M,N)=\mathbb{C}[x]/(h(x))$.

For $\operatorname{Tor}_1^R(M,N)$, we are interested in the homology at the left $M\otimes_{\mathbb{C}[x]}\mathbb{C}[x]$. But the incoming map is the 0 map; so $\operatorname{Tor}_i^R(M,N)=\ker(m)$. Writing $M=\mathbb{C}[x]/(f(x))$, we see that

$$\operatorname{Tor}_1^R(M,N)=\{\,a(x)+(f(x)):f(x)\mid a(x)g(x)\,\}=\{\,a(x)+(f(x)):f(x)\mid a(x)h(x)\,\}$$

Writing $f(x)=s(x)h(x)$, we see that $\operatorname{Tor}_1^R(M,N)=(s(x))/(f(x))$. Indeed, as we will see on assignment 4, we in general have that $\operatorname{Tor}_1^R(R/I,R/J)\cong I\cap J/IJ$.

Theorem 6.5 (Flatness criteria)**.** *Suppose R is a ring; suppose M is a right R-module. Then the following are equivalent:*

1. *M is flat.*
2. *$M\otimes_R I\to M=M\otimes_R R$ is injective for all left ideals $I\subsetneq R$.*
3. *$\operatorname{Tor}_1^R(M,R/I)=0$ for all left ideals $I\subsetneq R$.*

Proof.

(1) $\implies$ (2) Immediate.

(2) $\implies$ (3) Suppose we have a left ideal $I\subsetneq R$. Then the exact sequence $0\to I\to R\to R/I\to 0$ yields an exact sequence

$$\begin{tikzcd} & M\otimes_R I \arrow[r] & M\otimes_R R \arrow[r] & M\otimes_R R/I \arrow[r] & 0 \\ & & 0 \arrow[r] & \operatorname{Tor}_1^R(M,R/I) \arrow[ull, rounded corners] & \end{tikzcd}$$

But $M\otimes_R I\to M\otimes_R R$ is injective; so $\operatorname{Tor}_1^R(M,R/I)=0$.

(3) $\implies$ (1) Suppose (3) holds but M is not flat. Then there are left R-modules $N' \subseteq N$ such that $M \otimes_R N' \to M \otimes_R N$ is not injective. We make the following reductions:

Claim 6.6. *Without loss of generality, we may assume N' is finitely generated.*

Proof. Well, there is a non-zero $x \in M \otimes_R N'$ such that $\varphi(x) = 0$ in $M \otimes_R N$. Write

$$x = m_1 \otimes_R n_1 + \cdots + m_k \otimes_R n_k$$

where $n_1, \dots, n_k \in N'$ and $m_1, \dots, m_k \in M$. Let $N_0 \subseteq N'$ be $Rn_1 + \cdots + Rn_k$. Then x has some preimage $x_0 \in M \otimes_R N_0$ (under $M \otimes_R N_0 \to M \otimes_R N'$); then we have $N_0 \subseteq N' \subseteq N$ and the map $M \otimes_R N_0 \to M \otimes_R N$ factors through $M \otimes_R N'$, and in particular has $x_0 \neq 0$ in the kernel. So we can instead consider $N_0 \subseteq N$ and $x_0 \in \ker(M \otimes_R N_0 \to M \otimes_R N)$. □ Claim 6.6

Claim 6.7. *We may assume N is finitely generated.*

Proof. Consider $\varphi\colon M \otimes_R N_0 \to M \otimes_R N$; then $0 \neq x = m_1 \otimes_R n_1 + \cdots + m_k \otimes_R n_k \in \ker(\varphi)$. Notice that $M \otimes_R N$ is a free $\mathbb{Z}$-module on symbols (m, n) modulo relations of the form

$$\begin{aligned}(mr, n) - (m, rn) &= 0\\ (m_1 + m_2, n) - (m_1, n) - (m_2, n) &= 0\\ (m, n_1 + n_2) - (m, n_1) - (m, n_2) &= 0\end{aligned}$$

So if $x = 0$ in $M \otimes_R N$, then we can capture that fact using only finitely many relations from the above; say using (not in order)

$$\begin{gathered}(\widetilde{m_1} r_1, \widetilde{n_1}) - (\widetilde{m_1}, r_1 \widetilde{n_1})\\ \vdots\\ (\widetilde{m_s} r_s, \widetilde{n_s}) - (\widetilde{m_s}, r_s \widetilde{n_s})\\ (m_{11} + m_{21}, n_1') - (m_{11}, n_1') - (m_{21}, n_1')\\ \vdots\\ (m_{1j} + m_{2j}, n_j') - (m_{1j}, n_j') - (m_{2j}, n_j')\\ (m_1', n_{11} + n_{21}) - (m, n_{11}) - (m, n_{21})\\ \vdots\\ (m_t', n_{1t} + n_{2t}) - (m, n_{1t}) - (m, n_{2t})\end{gathered}$$

So we only need to take

$$\widehat{N} = R\widetilde{n_1} + \cdots + R\widetilde{n_s} + Rn_1' + \cdots + Rn_j' + Rn_{11} + Rn_{21} + \cdots + Rn_{1t} + Rn_{2t} + \underbrace{Rn_1 + \cdots + Rn_k}_{N_0}$$

Then $x_0 \in \ker(M \otimes_R N_0 \to M \otimes_R \widehat{N})$. □ Claim 6.7

We now have $N_0 \subseteq \widehat{N}$ both finitely generated with $M \otimes_R N_0 \to M \otimes_R \widehat{N}$ not injective. Write $N_0 = \langle n_1, \dots, n_k \rangle$; write $\widehat{N} = \langle n_1, \dots, n_k, u_1, \dots, u_m \rangle$. For $i \in \{1, \dots, m\}$, let $N_i = \langle n_1, \dots, n_k, u_1, \dots, u_i \rangle$; then

$$N_0 \subseteq N_1 \subseteq \cdots \subseteq N_m = \widehat{N}$$

Claim 6.8. *We may instead consider N_i and N_{i+1} for some $i \in \{1, \dots, m\}$.*

Proof. Since the composition

$$M \otimes_R N_0 \to \cdots \to M \otimes_R N_m = M \otimes_R \widehat{N}$$

is not injective, there is some $i \in \{1, \dots, m\}$ such that $M \otimes_R N_i \to M \otimes_R N_{i+1}$ is not injective. □ Claim 6.8

Note now that

$$N_{i+1}/N_i = \langle n_1, \dots, n_k, u_1, \dots, u_{i+1} \rangle / \langle n_1, \dots, n_k, u_1, \dots, u_i \rangle$$

is cyclic; so there is $\psi\colon R \twoheadrightarrow N_{i+1}/N_i$ given by $r \mapsto ru_{i+1} + N_i$. Let $I = \ker(\psi$. Then $N_{i+1}/N_i \cong R/I$; i.e. $0 \to N_i \to N_{i+1} \to R/I \to 0$ is exact. So we get a long exact sequence of homology

$$\begin{tikzcd}
M \otimes_R N_i \arrow[r] & M \otimes_R N_{i+1} \arrow[r] & M \otimes_R R/I \arrow[r] & 0 \\
& \cdots \arrow[r] & \mathrm{Tor}_1^R(M, R/I) \arrow[ull]
\end{tikzcd}$$

But $\mathrm{Tor}_1^R(M, R/I) = 0$ by hypothesis; so $0 \to M \otimes_R N_i \to M \otimes_R N_{i+1}$ is exact, a contradiction. $\square$ Theorem 6.5

Corollary 6.9. *Suppose k is a field; let $R = k[t]/(t^2)$, and suppose M is an R-module. Then M is flat if and only if $M/\bar{t}M \cong \bar{t}M$ (where $\bar{t} = t + (t^2)$).*

Proof. As previously shown, we get that M is flat if and only if $\mathrm{Tor}_1^R(M, R/I) = 0$ for all proper ideals I of R. Notice that $I = (0)$ or $I = (\bar{t})$, by the correspondence theorem. In the case $I = (0)$, we have $R/I = R$ is projective, and hence that $\mathrm{Tor}_1^R(M, R) = 0$.

So M is flat if and only if $\mathrm{Tor}_1^R(M, R/(\bar{t})) = 0$. Notice, however, that $R/(\bar{t}) \cong (k[t]/(t^2))/(t + (t^2)) \cong k[t]/(t) \cong k$. So M is flat if and only if $\mathrm{Tor}_1^R(M, k) = 0$. One checks that

$$\cdots \to R \to R \to R \xrightarrow{\pi} k \to 0$$

is a projective resolution of k (where $R \to R$ is given by $r \mapsto r\bar{t}$); hence the chain complex from which we derive $\mathrm{Tor}_i^R(M, k)$ is

$$\cdots \to M \otimes_R R \to M \otimes_R R \to M \otimes_R R \to 0$$

where the maps $M \otimes_R R \to M \otimes_R R$ can be expressed as the maps $M \to M$ given by $m \mapsto \bar{t}m$. So, unpacking our earlier statement that M is flat if and only if $\mathrm{Tor}_1^R(M, k) = 0$, we find that M is flat if and only if $\{\, m \in M : \bar{t}m = 0 \,\} = \{\, \bar{t}m : m \in M \,\} = \bar{t}M$; i.e. if and only if $\ker M \to \bar{t}M = \bar{t}M$, which by first isomorphism theorem is equivalent to $M/\bar{t}M \cong \bar{t}M$. $\square$ Corollary 6.9

Theorem 6.10. *Suppose R is a commutative ring; suppose $a \in R$ is not a zero divisor. Suppose M is flat and we have $m \in M$ such that $am = 0$. Then $m = 0$.*

Proof. Consider the short exact sequence $0 \to R \to R \to R/aR \to 0$ (with the map $R \to R$ given by $x \mapsto xa$). Tensoring with M, we find that

$$0 \to M \otimes_R R \to M \otimes_R M \to M \otimes_R R/aR \to 0$$

is exact; we can express this as a short exact sequence

$$0 \to M \to M \to M \otimes_R /aR \to 0$$

where the map $M \to M$ is $m \mapsto ma$. So the map $M \to M$ given by $m \mapsto am$ is injective; so if $am = 0$, then $m = 0$. $\square$ Theorem 6.10

The converse holds if R is a PID.

Theorem 6.11. *If R is a PID, then M is flat if and only if M is torsion-free; i.e. whenever $a \in R \setminus \{\, 0 \,\}$ and $am = 0$, we have $m = 0$.*

Proof.

$(\implies)$ Generally true.

($\Longleftarrow$) Suppose M is torsion-free; let $a \in R \setminus \{ 0 \}$. Then

$$0 \to M \to M \to M/aM \to 0$$

is exact (where the map $M \to M$ is $m \mapsto am$). Consider also the short exact sequence $0 \to R \to R \to R/aR \to 0$ (where the map $R \to R$ is $x \mapsto ax$); tensoring with M, we obtain a long exact sequence

$$\begin{tikzcd}
M \otimes_R R \arrow[r] & M \otimes_R R \arrow[r] & M \otimes_R R/aR \arrow[r] & 0 \\
 & 0 = \mathrm{Tor}_1^R(M, R) \arrow[r] & \mathrm{Tor}_1^R(M, R/aR) \arrow[ull] &
\end{tikzcd}$$

But M is torsion-free; so the map $M \otimes_R R \to M \otimes_R$ can be expressed as the map $M \to M$ given by $m \mapsto am$. So $\mathrm{Tor}_1^R(M, R/aR) = 0$ for all $a \neq 0$. So, since R is a PID, we have $\mathrm{Tor}_1^R(M, R/I) = 0$ for all ideals I of R. So M is flat. $\square$ Theorem 6.11

So for example in $\mathbb{Z}$, we have

- The injectives are the divisible $\mathbb{Z}$-modules (namely direct sums of $\mathbb{Q}$ and $C_p = \{ \exp(2\pi i j/p^k) : k \geq 0, j \geq 0 \}$).
- The projectives are the free $\mathbb{Z}$-modules.
- The flat $\mathbb{Z}$-modules are the torsion-free $\mathbb{Z}$-modules.

Some general facts:

Suppose R is commutative; suppose M and N are R-modules. Then

$$\mathrm{Tor}_0^R(M, N) = M \otimes_R N \cong N \otimes_R M \, \mathrm{Tor}_0^R(N, M)$$

Fact 6.12. *In general, we have* $\mathrm{Tor}_i^R(M, N) \cong \mathrm{Tor}_i^R(N, M)$.

Fact 6.13. *Suppose R and S are commutative; suppose A is an R-module, C is an S-module, and B is both an R-module and an S-module. If B is flat as an R-module and as an S-module, then* $\mathrm{Tor}_n^S(A \otimes_R B, C) \cong \mathrm{Tor}_n^R(A, B \otimes_S C)$.

In particular, for $n = 0$ we get $(A \otimes_R B) \otimes_S C \cong A \otimes_R (B \otimes_S C)$. Another special case is when S is a flat R-algebra, and we let $B = S$; we then get $\mathrm{Tor}_n^S(A \otimes_R S, C) \cong \mathrm{Tor}_n^R(A, C)$.

6.1 Ext

Suppose R is a ring; suppose M and N are left R-modules. We create $\mathrm{Ext}_R^i(M, N)$ as follows:

Define $G = \hom(M, -) \colon R\text{-}\mathbf{Mod} \to \mathbf{Ab}$; then G is additive and left-exact. We then set $\mathrm{Ext}_R^i(M, N) = R^i G(N)$. To compute $\mathrm{Ext}_R^i(M, N)$, we take an injective resolution

$$0 \to N \to I^0 \to I^1 \to \cdots$$

and obtain a cochain complex

$$0 \to \hom(M, I^0) \to \hom(M, I^1) \to \cdots$$

We then have $\mathrm{Ext}_R^i(M, N) = H^i(\hom(M, I^\bullet))$.

Example 6.14. Let $M = N = \mathbb{Z}/3\mathbb{Z}$; we compute $\mathrm{Ext}_{\mathbb{Z}}^i(M, N)$. We get an injective resolution

$$0 \to \mathbb{Z}/3\mathbb{Z} \to C_3 \to C_3 \to 0$$

where the map $C_3 \to C_3$ is $x \mapsto x^3$. Our cochain complex is then

$$0 \to \hom(\mathbb{Z}/3\mathbb{Z}, C_3) \xrightarrow{a} \hom(\mathbb{Z}/3\mathbb{Z}, C_3) \xrightarrow{b} 0 \to \cdots$$

Suppose $\psi \colon \mathbb{Z}/3\mathbb{Z} \to C_3$. Then $\psi \in \ker(a)$ if and only if $\psi(1)^3 = 1$ in C_3; i.e. if and only if $\psi(1) \in \{ 1, \exp(2\pi i/3), \exp(4\pi i/3) \}$. So $\ker(a) \cong \mathbb{Z}/3\mathbb{Z}$. So $\mathrm{Ext}_{\mathbb{Z}}^0(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z}$.

We also get that $\mathrm{Ext}_{\mathbb{Z}}^i(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) = 0$ for $i \geq 2$; it remains to find $\mathrm{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/3\mathbb{Z})$. But this is just

$$\ker(b)/\operatorname{im}(a) = \hom(\mathbb{Z}/3\mathbb{Z}, C_3)/\operatorname{im}(a) = \hom(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/3\mathbb{Z})/\operatorname{im}(a) \cong \mathbb{Z}/3\mathbb{Z}$$

An alternative description of Ext: consider $\widetilde{G} = \hom(-, N) \colon (R\text{-}\mathbf{Mod})^{\mathrm{op}} \to \mathbf{Ab}$. Then $\widetilde{G}$ is left-exact and additive. We can compute $R^i\widetilde{G}$ by taking an injective resolution of M in $(R\text{-}\mathbf{Mod})^{\mathrm{op}}$

$$0 \to M \to I^0 \to I^1 \to \cdots$$

i.e. an exact sequence

$$\cdots \to I^1 \to I^0 \to M \to 0$$

where the I^i are projective. So if we take a projective resoltuion

$$\cdots \to P^0 \to M \to 0$$

in R-**Mod** and apply $\hom(-, N)$, we get a cochain complex

$$0 \to \hom(P^0, N) \to \hom(P^1, N) \to \cdots$$

with $R^i\widetilde{G}(M) = H^i(\hom(P^\bullet, N))$.

Fact 6.15. $R^i\widetilde{G}(M) \cong \mathrm{Ext}^i_R(M, N)$.

6.1.1 Ext via Yoneda equivalence

If X and X' are two R-modules and we have two short exact sequences

$$\alpha \colon 0 \to A \to X \to B \to 0$$

and

$$\alpha' \colon 0 \to A \to X' \to B \to 0$$

then we write $\alpha \sim_Y \alpha'$ if there is $f \colon X \to X'$ such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\mathrm{id}"] & X \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & A \arrow[r] & X' \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

We then define $E^1(A, B)$ to be the set of equivalence classes of $\sim_Y$.

Fact 6.16. $E^1(A, B) \cong \mathrm{Ext}^1_R(A, B)$.

More generally, we can define an analogous equivalence relation on exact sequences

$$\alpha \colon 0 \to A \to X_1 \cdots \to X_n \to B \to 0$$

We let $E^n(A, B)$ be the collection of equivalence classes of exact sequences under the analogous equivalence relation.

Remark 6.17. We get a map $E^n(A, B) \times E^m(B, C) \to E^{n+m}(A, C)$ given by appending the sequences. Taking $A = B = C$, we find that

$$\bigoplus E^n(A, A)$$

is a graded ring.

TODO 8. *Missing stuff.*

Theorem 6.18 (Eilenberg-Watts)**.** *Suppose $F, G, H \colon R$-**Mod** $\to S$-**Mod** are additive. Suppose*

- *F is right-exact and commutes with direct sums.*
- *G is contravariant, left-exact, and converts direct sums into direct products.*
- *H has $S = \mathbb{Z}$, is left-exact, and commutes with projective limits.*

Then

- $F \cong M \otimes_R -$ *for some* (R, S)*-bimodule* M*.*
- $G \cong \hom(-, N)$ *for some* (R, S)*-bimodule* N*.*
- $H \cong \hom(M, -)$ *where* M *is an* R*-module.*

Example 6.19 (Grape cohomology). Fix a grape G and consider G-**Mod**, the category fo abelian grapes $(A, +)$ endowed with a G-action $G \times A \to A$. Consider $H\colon G\text{-}\mathbf{Mod} \to \mathbf{Ab}$ given by $A \mapsto \{\, a \in A : ga = a \text{ for all } g \in G \,\}$. For example, if $G = S_2$ and $A = \mathbb{Z} \oplus \mathbb{Z}$, we can set $(1, 2)(a, b) = (b, a)$, and thus get $A \in G$-**Mod**. In this case we have

$$HA = \{\, (a, b) : (1, 2)(a, b) = (a, b) \,\} = \mathbb{Z}(1, 1) \cong \mathbb{Z}$$

One can easily verify that H is left-exact. However, it is not right-exact: for example, if

- $G = \mathbb{Z}/2\mathbb{Z}$
- $B = \mathbb{Z}/4\mathbb{Z}$
- $C = \mathbb{Z}/2\mathbb{Z}$

then we can consider the quotient map $\varphi\colon B \to C$; then $H\varphi = 0$.

One can also easily verify that H commutes with projective limits. One also notes that G-**Mod** $\cong$ $\mathbb{Z}[G]$-**Mod** (where

$$\mathbb{Z}[G] = \left\{ \sum_{g \in G} n_g g : n_g \in \mathbb{Z} n_g = 0 \text{ for all but finitely many} g \right\}$$

is the grape algebra). So we can view H as a functor $\mathbb{Z}[G]$-**Mod** $\to$ **Ab**; by Eilenberg-Watts, we then get $H \cong \hom_{\mathbb{Z}[G]}(M, -)$ for some $\mathbb{Z}[G]$-module M. In fact we may take $M = \mathbb{Z}$ with the trivial G-action, which yields the $\mathbb{Z}[G]$-module structure

$$\left(\sum_g n_g g \right) m = \sum_g n_g g m = \left(\sum_g n_g \right) m$$

Indeed, given $\theta \in \hom_{\mathbb{Z}[G]}(\mathbb{Z}, A)$, we may let $a = \theta(1)$; then $g \cdot a = \theta(g \cdot 1) = \theta(1) = a$, and $a \in HA$. Conversely, if $a \in HA$, then $\theta\colon \mathbb{Z} \to A$ given by $\theta(n) = na$ has $\theta \in \hom_{\mathbb{Z}[G]}(\mathbb{Z}, A)$. So $\hom_{\mathbb{Z}[G]}(\mathbb{Z}, A) \cong HA$.

We may thus conclude that

$$H^i(G, A) := R^i H(A) \cong R^i \hom_{\mathbb{Z}[G]}(\mathbb{Z}, -)(A) = \operatorname{Ext}^i_{\mathbb{Z}[G]}(\mathbb{Z}, A)$$

Example 6.20. Let $G = \langle x : x^2 = 1 \rangle \cong \mathbb{Z}/2\mathbb{Z}$; let $A = \mathbb{Z} \oplus \mathbb{Z}$ with $x(a, b) = (b, a)$. Then $R = \mathbb{Z}[G] \cong \mathbb{Z}[x]/(x^2 - 1) = \mathbb{Z}[x]/(x + 1)(x - 1)$. So $H^i(G, A) = \operatorname{Ext}^i_R(\mathbb{Z}, A)$. Note that we get an exact sequence

$$\cdots \xrightarrow{\varphi_2} R \xrightarrow{\varphi_1} R \xrightarrow{\varphi_2} R \xrightarrow{\varphi_1} R \xrightarrow{\theta} \mathbb{Z} \to 0$$

where $\varphi_1(a) = a(x - 1)$ and $\varphi_2(a) = a(x + 1)$. We truncate and apply $\hom(-, A)$ to get a cochain complex

$$0 \to \hom_R(R, A) \to \hom_R(R, A) \to \cdots$$

i.e.

$$0 \to \mathbb{Z}^2 \to \mathbb{Z}^2 \to \mathbb{Z}^2 \to \cdots$$

Then

$$
\begin{aligned}
H^0 &= \{\, (a,a) : a \in \mathbb{Z} \,\} = \mathbb{Z}(1,1) = HA \\
H^1 &= \ker / \operatorname{im} \\
&= \{\, (a,b) : a = -b \,\} / \{\, (b-a, a-b) : a, b \in \mathbb{Z} \,\} \\
&= (0) \\
H^2 &= (0) \text{ (similarly)} \\
H^3 &= (0) \\
&\ \ \vdots
\end{aligned}
$$

So

$$
H^i(G,A) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ 0 & \text{else} \end{cases}
$$

What is the significance of this? In the assignment we are asked to show that

$$
H^1(G,A) \cong \{\, \text{crossed homomorphisms} \,\} / \{\, \text{principal crossed homomorphisms} \,\}
$$

Hence in this case we get that all crossed homomorphisms are principal; i.e. given $f\colon G \to A$ with $f(gh) = f(g) + gf(h)$, we have that f takes the form $f(g) = ga - a$ for some $a \in A$.

We now showcase another use of the above. Suppose now that $A \in \mathrm{Ob}(G\text{-}\mathbf{Mod})$; consider all grapes H such that we have

$$
1 \to A \xrightarrow{i} H \xrightarrow{\pi} G \to 1
$$

i.e. $A \trianglelefteq H$ and $H/A \cong G$. Then G acts on any such A by declaring $ga = hah^{-1} \in A$ where we pick $h \in H$ satisfying $\pi(h) = g$. We consider the case where this coincides with our original G-action. Then $H^2(G,A)$ is isomorphic to all such extensions $1 \to A \to H \to G \to 1$ modulo Yoneda equivalence. Note that we always have at least one such extension; namely $A \rtimes G$. So in our example, since $H^2(G,A) = 0$, we get

$$
1 \to \mathbb{Z}^2 \to H \to \mathbb{Z}/2\mathbb{Z} \to 1
$$

where $1 \neq x \in \mathbb{Z}/2\mathbb{Z}$ acts via permuting coordinates.

Example 6.21. Suppose k is a field of characteristic 0; let $\overline{k}$ be the algebraic closure. Let $G = \mathrm{Gal}(\overline{k}/k)$; then G acts on $(\overline{k})^*$ via $\sigma\lambda = \sigma(\lambda)$. Then $H^2(\mathrm{Gal}(\overline{k},k),(\overline{k})^*)$ is the *Brauer grape* of k, denoted $\mathrm{Br}(k)$; this gives the structure of all finite-dimensional division rings D over k with $Z(D) = k$. For example, it holds that

$$
\mathrm{Br}(\mathbb{R}) = \mathbb{Z}/2\mathbb{Z} \cong H^2(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathbb{C}^*) = H^2(\mathbb{Z}/2\mathbb{Z}, \mathbb{C}^*)
$$

Example 6.22 (Hochschild homology/cohomology). Suppose A is a ring; suppose M is an (A,A)-bimodule. We set

$$
\begin{aligned}
\mathrm{HH}_i(M) &= \mathrm{Tor}_i^{A \otimes A^{\mathrm{op}}}(A, M) \\
\mathrm{HH}^i(M) &= \mathrm{Ext}_i^{A \otimes A^{\mathrm{op}}}(A, M)
\end{aligned}
$$

There is also local cohomology and sheaf cohomology.