Assignment 1—PMATH 930

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What's purple and commutes?
An abelian grape.

1. I claim that the differential ideals of $(\mathbb{C}[t], \frac{\mathrm{d}}{\mathrm{d}t})$ are $\{0\}$ and $\mathbb{C}[t]$.

   ($\implies$) Suppose we are given a differential ideal I of $(\mathbb{C}[t], \frac{\mathrm{d}}{\mathrm{d}t})$; suppose $0 \neq f \in I$. If $\deg(f) > 0$ then by hypothesis $0 \neq \frac{\mathrm{d}f}{\mathrm{d}t} \in I$; so there is a non-zero element of I of degree $\deg(f) - 1$. Continuing inductively we may assume $\deg(f) = 0$ (and still $f \neq 0$); so $I \supseteq (f) = \mathbb{C}[t]$, and $I = \mathbb{C}[t]$.

   ($\impliedby$) It is clear that $\{0\}$ and $\mathbb{C}[t]$ are differential ideals. $\square$

2. **(Extension of rings)** I first verify the implicit claim that R/I embeds in S/J (since derivations for us require that the codomain contain the domain). Define $\Phi\colon R \to S/J$ by $r \mapsto r + J$; so Φ is the composition of the quotient map $S \to S/J$ and the inclusion $R \to S$, and is thus a ring homomorphism. But $\ker(\Phi) = J \cap R = I$; so by the first isomorphism theorem we get that $\operatorname{Ran}(\Phi) \cong R/I$, and R/I embeds in S/J.

   (Existence) Define a map $\delta^*\colon R \to S/J$ by $\delta^*(r) = \delta(r) + J$; so $\delta^* = \pi \circ \delta$ is a homomorphism of additive grapes (where $\pi\colon S \to S/J$ is the quotient map). Also by hypothesis we have $\delta^*(I) = \pi(\delta(I)) \subseteq \pi(J) = \{0 + J\}$; so by the universal property of quotients we get a homomorphism of additive grapes $\delta\colon R/I \to S/J$ such that $\delta(r + I) = \delta(r) + J$ for all $r \in R$. Also if $r_1 + I, r_2 + I \in R$ then

   $$\delta((r_1+I)(r_2+I)) = \delta(r_1r_2+I) = \delta(r_1r_2)+J = r_1\delta(r_2)+\delta(r_1)r_2+J = (r_1+I)\delta(r_2+I)+\delta(r_1+I)(r_2+I)$$

   (where in the last expression $r_i + I$ is viewed as an element of S/J via the Φ defined above). So δ satisfies the Leibniz rule, and δ is a derivation.

   (Uniqueness) Given two such derivations δ_1, δ_2 by the defining condition they're equal on all of R/I. $\square$

3. Suppose $V = \operatorname{Spec}(k[X]/I)$ where $I = (P_1, \ldots, P_\ell)$ for some $P_1, \ldots, P_\ell \in k[X]$ and $X = (X_1, \ldots, X_n)$. Let $J \subseteq K[X, Y]$ (where $Y = (Y_1, \ldots, Y_n)$) be the ideal generated by $P_j(X)$ and

   $$P_j^\delta(X) + \sum_{i=1}^{n} \frac{\partial P_j}{\partial X_i}(X) Y_i$$

   as j ranges over $\{1, \ldots, \ell\}$. For each $P \in I$ we can write $P = Q_1P_1 + \cdots + Q_\ell P_\ell$ for some $Q_1, \ldots, Q_n \in k[X_1, \ldots, X_n]$; then

   $$P(X) = \sum_{j=1}^{\ell} Q_i(X)P_i(X) \in J$$

and

$$
\begin{aligned}
&P^\delta(X) + \sum_{i=1}^n \frac{\partial P}{\partial X_i}(X) Y_i \\
&= \left(\sum_{j=1}^\ell Q_j P_j\right)^\delta (X) + \sum_{i=1}^n \frac{\partial \sum_{j=1}^\ell Q_i P_i}{\partial X_j}(X) Y_i \\
&= \sum_{j=1}^\ell (Q_j^\delta(X) P_j(X) + Q_j(X) P_j^\delta(X)) + \sum_{i=1}^n \sum_{j=1}^\ell \left(\frac{\partial Q_i}{\partial X_j}(X) P_i(X) + Q_i(X) \frac{\partial P_i}{\partial X_j}(X)\right) Y_i \\
&= \sum_{j=1}^\ell Q_j^\delta(X) \underbrace{P_j(X)}_{\in J} + \sum_{i=1}^n \sum_{j=1}^\ell \frac{\partial Q_i}{\partial X_j}(X) \underbrace{P_i(X)}_{\in J} + \sum_{j=1} Q_j(X) \underbrace{\left(P_j^\delta(X) + \sum_{i=1}^n \frac{\partial P_i}{\partial X_j}(X) Y_i\right)}_{\in J} \\
&\in J
\end{aligned}
$$

(Here we are using the fact that $(\cdot)^\delta)$ distributes over sums and $(Q_jP_j)^\delta = Q_j^\delta P_j + Q_j P_j^\delta$; the latter can be easily verified by distributing the polynomial multiplication and verifying the result for single terms.) So J contains, and is thus equal to, the ideal of $k[X,Y]$ generated by $P(X)$ and

$$\sum_{i=1}^n \frac{\partial P}{\partial X_i}(X) Y_i$$

as P ranges over I. But this is the defining ideal of τV; so $\tau V = \mathrm{Spec}(k[X,Y]/\sqrt{J})$, as desired. □

4. Suppose K is an existentially closed differential integral domain. Suppose $f, g \in K\{X\}$ with $\mathrm{ord}(f) > \mathrm{ord}(g)$ and $g \neq 0$. Let $n = \mathrm{ord}(f)$; write $f(X) = f^*(X, \delta X, \ldots, \delta^n X)$ for some $f^* \in K[X_0, \ldots, X_n]$. Consider $L = K(t_0, \ldots, t_{n-1})^{\mathrm{alg}}$ where the t_i are indeterminates; then $f^*(t_0, t_1, \ldots, t_{n-1}, y) \in K(t_0, \ldots, t_{n-1})[y]$ is non-constant since $\mathrm{ord}(f) = n$, and thus has a root in $s \in L$. We extend δ to L using Corollary (4) iteratively, and declaring

$$\delta(t_i) = \begin{cases} s & \text{if } i = n-1 \\ t_{i+1} & \text{else} \end{cases}$$

But then

$$f(t_0) = f^*(t_0, \delta t_0, \ldots, \delta^{n-1} t_0, \delta^n t_0) = f^*(t_0, t_1, \ldots, t_{n-1}, s) = 0$$

Write $g(X) = g^*(X, \delta X, \ldots, \delta^m X)$ for $g^* \in K[X_0, \ldots, X_m]$ and $m < n$. Then

$$g(t_0) = g^*(t_0, \delta t_0, \ldots, \delta^m t_0) = g^*(t_0, t_1, \ldots, t_m) \neq 0$$

since the t_i are algebraically independent over K.

So the formula $f(x) = 0 \wedge g(x) \neq 0$ has a realization in $(L, \delta) \models T$ (where T is the theory of differential integral domains). So by existential closure it has a realization in (K, δ). □

5. I discussed this question with Wilson Poulter and Sam Kim.

Suppose $f\colon K^n \to K$ is a 0-definable function; let $X = (X_1, \ldots, X_n)$. Consider the set of L_δ-formulas

$$\Sigma(x) = \{\, f(x) \neq F(x) : F \in \mathbb{Q}\langle X \rangle \,\}$$

I claim that Σ is not a type. Indeed, for any $L \succeq K$ and $a \in L^n$ we have that $y = f(a)$ defines $\{\, f(a) \,\}$ over $\{\, a \,\}$; so $f(a) \in \mathrm{dcl}(a) = \mathbb{Q}\langle a \rangle$, and there is $F(X) \in \mathbb{Q}\langle X \rangle$ such that $f(a) = F(a)$. So Σ isn't realized in any elementary extension, and is thus not a type. So by compactness there are $F_0, \ldots, F_{\ell-1} \in \mathbb{Q}\langle X \rangle$ such that $f(x) \neq F_0(x) \wedge \cdots \wedge f(x) \neq F_{\ell-1}(x)$ isn't realized in K^n; i.e. for all $a \in K^n$ we have $f(x) \in \{\, F_i(a) : i < \ell \,\}$. We then let $D_i \subseteq K^n$ be defined by

$$f(x) = F_i(x) \wedge \bigwedge_{j<i} f(x) \neq F_j(x)$$

Then the D_i partition K^n and $f \restriction D_i = F_i \restriction D_i$. □

6. I discussed this question with Wilson Poulter and Sam Kim.

I claim that $K \models \mathrm{DCF}_0$ is κ-saturated if and only if the following holds:

(*) Suppose $(F,\delta) \subseteq (K,\delta)$ is a differential subfield with $F = \mathbb{Q}\langle A\rangle$ for some $|A| < \kappa$; suppose (L,δ^*) is a differential field extension of (F,δ) such that either

(a) L is a finitely generated field extension of F (i.e. it takes the form $F(a_1,\ldots,a_n)$ for some $a_1,\ldots,a_n \in L$), or

(b) $L \cong F\langle X\rangle$ with X a single indeterminate.

Then the inclusion $F \hookrightarrow K$ extends to a δ-embedding $L \hookrightarrow K$. In diagram:

$$\begin{tikzcd} (L,\delta^*) \arrow[r, hook, dashed] & (K,\delta) \\ (F,\delta) \arrow[u, hook, "\subseteq"] \arrow[ur, hook, "\subseteq"'] & \end{tikzcd}$$

Remark 1 (Comments on the above condition).

(a) If $\kappa > \aleph_0$ we can simplify the hypothesis on F to be simply that $|F| < \kappa$.

(b) A consequence of our condition is the existence of a "differential-algebraically independent" set of size κ, for a suitable definition of differential-algebraic independence; I don't believe the converse holds.

(c) One might hope that we could get away with only requiring the condition hold for $L = F\langle X\rangle$; i.e. that given any small differential subfield there is $a \in K$ "differentially transcendental" over F. I don't think this is sufficient: I could reduce it to showing that given any prime ideal I of $F[X_0,\ldots,X_n]$ there is $a \in K$ such that $I = \{\, f \in F[X_0,\ldots,X_n] : f(a,\delta a,\ldots,\delta^n a) = 0 \,\}$, but I couldn't figure out how to do that. Of course one can find $a_0,\ldots,a_n$ such that $I = \{\, f \in F[X_0,\ldots,X_n] : f(a_0,\ldots,a_n) = 0 \,\}$ by κ-saturation of K as a model of ACF, but I couldn't see how to get $\delta a_i = a_{i+1}$. Alternatively one can pick some realization of your type and use existential closure of K and Noetherianity of $F[x_0,\ldots,x_n]$ to get $a \in K$ such that $I \subseteq \{\, f \in F[x_0,\ldots,x_n] : f(a,\delta a,\ldots,\delta^n a) = 0 \,\}$, but I'm not seeing how the other containment should follow.

That said I don't have a counterexample for any of this; I'm not sure how one would go about exhibiting a "large" differentially closed field omitting some type. My only thought is the omitting types theorem, but that doesn't produce large models.

Proof.

($\Longrightarrow$) Suppose K is κ-saturated; suppose we have F, K, L as above.

Case 1. Suppose $L = F(a_1,\ldots,a_n)$ for $a_1,\ldots,a_n \in L$. Then $\mathrm{tp}(a_1\cdots a_n/A)$ is a type in DCF_0 over A since it's realized in any differentially closed extension of L. Then by κ-saturation there are $b_1,\ldots,b_n \in K$ such that $\mathrm{tp}(b_1\cdots b_n/A) = \mathrm{tp}(a_1\cdots a_n/A)$; then since $F = \mathbb{Q}\langle A\rangle = \mathrm{dcl}(A)$ we get that $\mathrm{tp}(b_1\cdots b_n/F) = \mathrm{tp}(a_1\cdots a_n/F)$. So the map $L = F(a_1,\ldots,a_n) \to K$ that fixes F and maps a_i to b_i is a well-defined δ-embedding over F, as desired.

Case 2. Suppose $L \cong F\langle X\rangle$ for X a single indeterminate. Consider $\{\, f(x) \neq 0 : 0 \neq f \in F\{X\} \,\}$. This can be phrased as a set of formulas over A, since $F = \mathrm{dcl}(A)$, and it is in fact a type: it is realized for instance in any differentially closed extension of $F\{X\}$. So by κ-saturation it is realized in K, say by a. Then a satisfies no differential polynomials over F; so $F\langle a\rangle \cong F\langle X\rangle$, and the map $F\langle X\rangle \to K$ given by $X \mapsto a$ is a δ-embedding over F, as desired.

($\Longleftarrow$) Suppose $A \subseteq K$ has $|A| < \kappa$; let $F = \mathbb{Q}\langle A\rangle$. We will show that every $p \in S_1(F)$ is realized in K. By quantifier elimination it suffices to consider partial types consisting only of literals over F; i.e. formulas of the form $f(x) = 0$ and their negations, for $f \in F\{X\}$.

Suppose $p(x) \vdash f(x) \neq 0$ for all non-zero $f \in F\{X\}$. By hypothesis we get a δ-embedding $F\{X\} \hookrightarrow K$ over F; then the image of X is a realization of p by quantifier elimination. Suppose then that $p(x) \vdash f(x) = 0$ for some non-zero $f \in F\{X\}$. Take some realization a of p in some elementary extension of K.

Claim 2. *$F\langle a\rangle$ is a finitely generated field extension of F.*

Proof. Pick non-zero $f \in F\{X\}$ such that $f(a) = 0$ and $(\mathrm{ord}(f), \deg(f))$ is minimal in the lexicographic order among such; note that such non-zero f exist by assumption on $p = \mathrm{tp}(a/F)$. Write $f(X) = f^*(X, \delta X, \ldots, \delta^n X)$ for $f^* \in F[X_0, \ldots, X_n]$. By minimality of f and since $\frac{\partial f^*}{\partial X_n}$ is lesser we get that $\frac{\partial f^*}{\partial X_n}(a, \delta a, \ldots, \delta^n a) \neq 0$. I claim that $F\langle a\rangle = F(a, \delta a, \ldots, \delta^n a)$. To see this, we note that

$$0 = \delta(f^*(a, \delta a, \ldots, \delta^n a)) = \sum_{i=0}^{n} \frac{\partial f^*}{\partial X_i}(a, \delta a, \ldots, \delta^n a)\delta^{i+1} a + (f^*)^{\delta}(a, \delta a, \ldots, \delta^n a)$$

so since $\frac{\partial f^*}{\partial X_n}(a, \delta a, \ldots, \delta^n a) \neq 0$ we get

$$\delta^{n+1} a = \frac{-(f^*)^{\delta}(a, \delta a, \ldots, \delta^n a) - \sum_{i=0}^{n-1} \frac{\partial f^*}{\partial X_i}(a, \delta a, \ldots, \delta^n a)}{\frac{\partial f^*}{\partial X_n}(a, \delta a, \ldots, \delta^n a)}$$

We can then use the quotient rule to write $\delta^{n+k} a$ as a rational function of $a, \delta a, \ldots, \delta^n a$ for $k \geq 1$, since the denominator will always be a power of $\frac{\partial f^*}{\partial X_n}(a, \delta a, \ldots, \delta^n a) \neq 0$. So $\delta^{n+k} a \in F(a, \delta a, \ldots, \delta^n a)$ for all $k \geq 1$; so $F\{a\} = F(a, \delta a, \ldots, \delta^n a)$. □ Claim 2

Then by the hypothesis we get a δ-embedding $F\langle a\rangle \hookrightarrow K$ over F. Then if b is the image of a under this embedding, we get by quantifier elimination that $p = \mathrm{tp}(a/F) = \mathrm{tp}(b/F)$. So p is realized in K. □