Course notes for PMATH 753

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Lectures by Kenneth R. Davidson, Fall 2015

Contents

1 Preliminaries

I acknowledge the contributions of Ilia Chtcherbakov, Eric Yau, and Mitchell Haslehurst.

Course outline on learn.
Ken Davidson, MC 5324
TA: Boyu Li, MC 5418
Recommended: A course in functional analysis, by Conway
Supplementary:

- Lax, interesting application, but weird order
- Pederson
- Marcoux's course notes, also weird order

Assignments probably about every two weeks.

2 Point-set topology

If (X, d) is a metric space, recall we define

$$B_r(x) = \{\, y \in X : d(x, y) < r \,\}$$

to be the *open balls*. We say $U \subseteq X$ is *open* if and only if for all $x \in U$ there is $r > 0$ such that $B_r(x) \subseteq U$.

Definition 1. A *topological space* is a set X together with $\tau \subseteq \mathcal{P}(X)$ (whose elements are called *open sets*) satisfying

1. $\emptyset, X \in \tau$.
2. If $\mathcal{U} \subseteq \tau$, then
$$\bigcup \mathcal{U} \in \tau$$
3. If $U, V \in \tau$, then $U \cap V \in \tau$.

Example 2.

1. The *discrete topology* is $(X, \mathcal{P}(X))$. This is, in fact, a metric topology.
2. The *trivial topology* is $(X, \{\, \emptyset, X \,\})$.
3. Suppose $(X, <)$ is a total order. We define the *order topology* to be generated by
$$\begin{aligned} &X \\ L_x &= \{\, y \in X : y < x \,\} \\ G_x &= \{\, y \in X : y > x \,\} \end{aligned}$$
i.e. the open sets are
$$\bigcup_\alpha (L_{x_\alpha} \cap G_{y_\alpha}) \cup \bigcup_\beta L_{b_\beta} \cup \bigcup_\gamma G_{c_\gamma}$$
4. Let $X = C[0, 1]$. Let $x \in [0, 1]$, $a \in \mathbb{C}$, $r > 0$. Let
$$\begin{aligned} U_{x,a,r} &= \{\, f \in C[0, 1] : |f(x) - a| < r \,\} \\ U_{\{\, (x_i, a_i, r_i) : i < n \,\}} &= \bigcap \{\, U_{x_i, a_i, r_i} : i < n \,\} \end{aligned}$$
We declare unions of the latter to be open. This is the *topology for pointwise convergence*.

Definition 3. A set C in (X, τ) is *closed* if and only if $X \setminus C$ is open.

Definition 4. For a topological space (X, τ), a subset $A \subseteq X$, we define

- the *interior* of A is the largest open $U \subseteq A$:
$$A^\circ = \bigcup (\mathcal{P}(A) \cap \tau)$$
- the *closure* of A is the smallest closed $C \supseteq A$:
$$\overline{A} = \bigcap \{\, K \subseteq X : K \supseteq A, K^c \in \tau \,\}$$

Proposition 5.

1. *If $\mathcal{F}$ is a collection of closed sets, then*
$$\bigcap \mathcal{F}$$
is closed.

2. *If F, G are closed, then so is $F \cup G$.*

3. *For $A \subseteq X$, we have that $x \in \overline{A}$ if and only if for all open $U \ni x$, we have $U \cap A \neq \emptyset$.*

4. *$\overline{A} = ((A^c)^\circ)^c$.*

Proof.

1. For each $F \in \mathcal{F}$, we have F^c is open. So
$$\left(\bigcap \mathcal{F}\right)^c = \bigcup \{ F^c : F \in \mathcal{F} \}$$
is open, and
$$\bigcap \mathcal{F}$$
is closed.

2. $(F \cup G)^c = F^c \cap G^c$ is open, so $F \cup G$ is closed.

3. Suppose $x \in X$ and there is open $U \ni x$ such that $U \cap A = \emptyset$. Then U^c is closed and $A \subseteq U^c$. So $\overline{A} \subseteq U^c$, and $x \notin \overline{A}$.
Conversely, if $x \notin \overline{A}$, then $x \in (\overline{A})^c$. Setting $U = (\overline{A})^c$, we have $x \in U$ and $U \cap A \subseteq U \cap A^c = \emptyset$.

4.
$$\begin{aligned}(A^c)^\circ &= \bigcup \{ U \in \tau : U \cap A = \emptyset \} \\ &= (\overline{A})^c\end{aligned}$$
by previous item. Thus
$$((A^c)^\circ)^c = ((\overline{A})^c)^c = \overline{A}$$

☐ Proposition 5

Proposition 6. *If $\mathcal{S} \subseteq \mathcal{P}(X)$, then there is a smallest topology τ containing $\mathcal{S}$ given by $\emptyset, X$ and arbitrary unions of finite intersections of elements of $\mathcal{S}$.*

Proof. We check the properties.

1. By construction.

2. A union of unions is itself a union.

3. Well
$$\bigcup_\alpha (S_{\alpha,1} \cap \cdots \cap S_{\alpha,n_\alpha}) \cap \bigcup_\beta (T_{\beta,1} \cap \cdots \cap T_{\beta,m_\beta}) = \bigcup_\alpha \bigcup_\beta S_{\alpha,1} \cap \cdots \cap S_{\alpha,n_\alpha} \cap T_{\beta,1} \cap \cdots \cap T_{\beta,m_\beta}$$
(Check the set theory, if you don't believe it.)

☐ Proposition 6

Definition 7. If $\mathcal{S} \subseteq \mathcal{P}(X)$ generates τ as above, then $\mathcal{S}$ is a *subbase* of τ. If $\mathcal{S} \subseteq \mathcal{P}(X)$ and every $U \in \tau$ is the union of sets in $\mathcal{S}$, then $\mathcal{S}$ is a *base* for τ.

Example 8.

1. Suppose (X, d) is a metric space. Then $\{ B_r(X) : x \in X, r > 0 \}$ is a base for the metric topology.

2. In the special case of $(\mathbb{R}, d)$, we have that $\{ (r, s) : r, s \in \mathbb{Q} \}$ is a base.

Proposition 9. *Suppose $\{ \tau_\alpha \}$ is a collection of topologies on X. Then*

1.

$$\tau_{\min} = \bigcap_\alpha \tau_\alpha$$

is a topology on X.

2.

$$\tau_{\max} = \bigcup_\alpha \tau_\alpha$$

is a subbase for a topology on X.

Definition 10. If σ, τ are topologies on X, we say

- $\sigma < \tau$ if $\sigma \subseteq \tau$ (σ is *weaker than* τ).
- $\sigma > \tau$ if $\sigma \supseteq \tau$ (σ is *stronger than* τ).

Example 11. Let $X = C[0,1]$. Let τ be induced by the metric

$$d(f,g) = \|f-g\|_\infty = \sup_{x\in[0,1]} |f(x) - g(x)|$$

Consider the topology σ with base the sets

$$U = U_{\{\,(x_i,a_i,r_i):i<n\,\}} = \{\, f : |f(x_i) - a_i| < r_i, i < n \,\}$$

We claim that $\sigma \subseteq \tau$: Suppose $f \in U$ with $f(x_i) = b_i$, $|b_i - a_i| < r_i$. Then we can take

$$r = \min_{i<n} r_i - |a_i - b_i|$$

If $\|f-g\|_\infty < r$, then $|g(x_i) - b_i| < r$, and thus $|g(x_i) - a_i| < |g(x_i) - b_i| + |b_i - a_i| \leq r_i$.

Thus $\sigma \subseteq \tau$; they are not equal because $U \neq \emptyset$ is always unbounded. Indeed, for

$$U_{\{\,(x_i,a_i,r_i):i<n\,\}}$$

pick $y \notin \{\, x_i : i < n \,\}$. Then there is $g \in C[0,1]$ with $g(x_i) = a_i$ and $g(y)$ is arbitrarily large.

Definition 12.

- (X,τ) is *separable* if and only if there is a countable dense subset. i.e. a countable A such that $\overline{A} = X$.
- (X,τ) is *first-countable* if and only if for each $x \in X$, there is a collection $\mathcal{U}$ of open $U \ni x$ such that for all open $V \ni x$ there is $U \in \mathcal{U}$ such that $U \subseteq V$.
- (X,τ) is *second-countable* if and only if there is a countable base for the topology.

Example 13.

1. If (X,d) is a compact metric space, then X is separable.
2. If (X,d) is any metric space, then X is first-countable.
3. If (X,d) is a separable metric space, then X is second-countable.

   Proof. Suppose $\{\, x_i : i < \omega \,\} \subseteq X$ is a countable, dense set. Consider

   $$\left\{\, B_{\frac{1}{m+1}}(x_n) : m,n < \omega \,\right\}$$

   We claim that this is a base for the topology. Let U be open in (X,d). Let $x \in U$. Need to find m,n such that

   $$x \in B_{\frac{1}{m}}(x_n) \subseteq U$$

   Well, there is $r > 0$ such that $B_r(x) \subseteq U$. Pick m such that $\frac{1}{m} < \frac{r}{2}$. By density of $\{\, x_i : i < n \,\}$, we have some n such that $d(x_n, x) < \frac{1}{m}$. Then

   $$x \in B_{\frac{1}{m}}(x) \subseteq B_{\frac{2}{m}}(x) \subseteq B_r(x) \subseteq U$$

   as desired. ∎

Definition 14. Suppose (X,τ), (Y,σ) are topological spaces. We say $f\colon X\to Y$ is *continuous* if for all open $V\subseteq Y$, we have $f^{-1}(V)$ is open (in X). We say f is a *homeomorphism* if f is a bijection and f and f^{-1} are both continuous. We say f is *open* if for all open $U\subseteq X$ we have that $f(U)$ is open (in Y).

Example 15.

1. Suppose (X,τ) is a topological space. Consider the sequence of maps

$$(X,\text{discrete})\xrightarrow{f}(X,\tau)\xrightarrow{g}(X,\text{trivial})$$

   where $f=g=\mathrm{id}_X$. Then f and g are bijective and continuous but f^{-1}, g^{-1} are not continuous.

2. Any f from a discrete space into $\mathbb{R}$ is continuous. The only continuous functions from a trivial topology into $\mathbb{R}$ are constant.

3. The map

$$\begin{aligned} f\colon(-1,1)&\to\mathbb{R}\\ x&\mapsto\tan\Big(\frac{\pi}{2}x\Big)\end{aligned}$$

   is a homeomorphism.

Definition 16. Suppose (X,τ) is a topological space, $(x_n:n<\omega)$ is a sequence in X. We say $(x_n:n<\omega)$ *converges* if and only if for all

$$U\in\mathcal{O}(x)=\{\,U\in\tau:x\in U\,\}$$

there is an $N<\omega$ such that for all $N\leq N<\omega$ we have $x_n\in U$.

Example 17.

1. $X=\{\,a,b\,\}$, $\tau=\{\,\emptyset,\{\,a\,\},\{\,a,b\,\}\,\}$. Then $x_n\to a$ if and only if x_n is eventually a. On the other hand, every sequence converges to b. In particular, some sequences converge to a and b.

2. $X=[0,1)\cup\{\,a,b\,\}$ with $U\subseteq X$ open if all of the following hold:

   - $U\cap[0,1)$ is open in the metric topology.
   - If $a\in U$ or $b\in U$, then there is $\varepsilon>0$ such that $U\supseteq(1-\varepsilon,1)$.

   Then any sequence in $[0,1)$ that converges to 1 in the metric topology converges to both a and b in τ. As another example, the sequence

$$1-\frac{1}{2},a,1-\frac{1}{3},a,\ldots$$

   converges to a but not b.

Definition 18. (X,τ) is *Hausdorff* if for all $x\neq y$ in X there is open $U\ni x$, open $V\ni y$ such that $U\cap V=\emptyset$.

Example 19.

1. Metric spaces are Hausdorff.

2. The prior two examples are not Hausdorff.

Proposition 20. *If $C(X)$ (the set of continous maps $X\to\mathbb{C}$) separates points (i.e. for $x\neq y$ there is $f\in C_b(X)$ such that $f(x)\neq f(y)$), then X is Hausdorff.*

Proof. Say $x\neq y$. Then there is a continuous $f\colon X\to\mathbb{C}$ such that $f(x)\neq f(y)$, by hypothesis. $\mathbb{C}$ is Hausdorff, so we may find open $U\ni f(x)$, open $V\ni f(y)$ such that $U\cap V=\emptyset$. Then $f^{-1}(U)$ and $f^{-1}(V)$ are open sets containing x and y, respectively, and $f^{-1}(U)\cap f^{-1}(V)=\emptyset$. $\square$ Proposition 20

2.1 Nets

Main message: sequences are not enough.

Example 21. Let $X = \mathbb{N} \times \mathbb{N}$. Define τ by:

- For $m + n \geq 1$, the set $\{ (m, n) \}$ is open.
- An open $U \ni (0, 0)$ must have a finite $F \subseteq \mathbb{N}$ such that for all $n \in \mathbb{N} \setminus F$, we have

  $$\{ m < \mathbb{N} : (m, n) \in U \}$$

  is a cofinite subset of $\mathbb{N}$.

Check that this defines a topology.

1. (X, τ) is Hausdorff: to house off (m_1, n_1), (m_2, n_2), and $(0, 0)$, use

   $$\begin{aligned} U_0 &= X \setminus \{ (m_1, n_1), (m_2, n_2) \} \\ U_1 &= \{ (m_1, n_1) \} \\ U_2 &= \{ (m_2, n_2) \} \end{aligned}$$

2. $(0, 0) \in \overline{X \setminus \{ (0, 0) \}}$ since every non-empty open set has an element besides $(0, 0)$.
3. No sequence $((m_k, n_k) : k < \omega)$ in $X \setminus \{ (0, 0) \}$ converges to $(0, 0)$.

   Proof. Suppose $x_k = (m_k, n_k)$ is a sequence.

   Case 1. Suppose $(n_k : k < \omega)$ is bounded. Then there must be a constant subsequence $(n_{k_i} : i < \omega)$. Then

   $$U = X \setminus \{ (m, n) : n = n_0 \}$$

   is open and contains $(0, 0)$. But $x_{k_i} \notin U$ for all $i < \omega$. So $x_k \not\to (0, 0)$.

   Case 2. Suppose otherwise. Then there is a subsequence $(n_{k_i} : i < \omega)$ such that $(n_{k_i} : i < \omega) \to \infty$. Then

   $$U = X \setminus \{ x_{k_i} : i < \omega \}$$

   is open because only finitely many x_{k_i} have $n_{k_i} = n$. But $x_{k_i} \notin U$ for $i < \omega$. So $x_k \not\to (0, 0)$.

   □

Definition 22. A *directed set* is a set Λ with a binary relation $\leq$ such that

1. $\lambda \leq \lambda$ for all $\lambda \in \Lambda$.
2. If $\lambda_1 \leq \lambda_2$ and $\lambda_2 \leq \lambda_3$, then $\lambda_1 \leq \lambda_3$ for all $\lambda_i \in \Lambda$.
3. Directedness: if $\lambda_1, \lambda_2 \in \Lambda$ then there is $\lambda_3 \in \Lambda$ such that $\lambda_1 \leq \lambda_3$ and $\lambda_2 \leq \lambda_3$.

(We do not require antisymmetry; some authors do.)

Definition 23. A *net* is a function $x \colon \Lambda \to X$ (usually written $(x_\lambda : \lambda \in \Lambda)$). A net *converges* to $x \in X$ if for all $U \in \mathcal{O}(x)$ there is $\lambda_0 \in \Lambda$ such that $x_\lambda \in U$ for all $\lambda \geq \lambda_0$.

Definition 24. A *subnet* Γ of Λ is a function $\varphi \colon \Gamma \to \Lambda$ which is *cofinal*: for all $\lambda_0 \in \Lambda$ there is $\gamma_0 \in \Gamma$ such that $\varphi(\gamma) \geq \lambda_0$ for all $\gamma \geq \gamma_0$.

In practice, such φ will usually be monotonic.

Example 25. We now return to Example 21.

4. There is a net in $X \setminus \{0, 0\}$ converging to $(0,0)$. Let $\Lambda = \mathcal{O}((0,0))$ ordered by $U \leq V$ if $U \supseteq V$. This is clearly a directed set. We then define a net as follows: for $U \in \Lambda$, let x_U be the smallest element of $U \setminus \{0, 0\}$ in the usual well-ordering of $\mathbb{N}^2$.

   Claim 26. $(x_U : U \in \Lambda) \to (0,0)$.

   Proof. Suppose $V \in \mathcal{O}((0,0))$. If $U \geq V$, then $U \subseteq V$, and $x_U \in U \subseteq V$. □ Claim 26

5. Take the sequence
$$((0,1),(1,0),(0,2),(1,1),(2,0),\dots) = (x_k : k < \omega)$$
   This does not converge to $(0,0)$. Define $\varphi\colon \Lambda \to \omega$ by $\varphi(U) = i$ if $x_U = x_i$. This map is cofinal, since if $N < \omega$, we can $V \in \Lambda$ such that $U \geq V \implies \varphi(U) \geq N$ by taking $V = X \setminus \{(0,0), x_0, \dots, x_{N-1}\}$. So Λ is a subnet of the sequence $(x_n : n < \omega)$.

Proposition 27. *Suppose $A \subseteq X$. Then $x \in \overline{A}$ if and only if there is a net $(x_\lambda : \lambda \in \Lambda)$ in A converging to x.*

Proof.

($\implies$) Well, $x \in \overline{A}$ if and only if $U \cap A \neq \emptyset$ for all $U \in \mathcal{O}(x)$. We can make $\mathcal{O}(x)$ into a directed set by reverse containment, as before. Use the axiom of choice to pick $x_U \in U \cap A$ for each $U \in \mathcal{O}(x)$. Then $(x_U : U \in \mathcal{O}(x))$ converges to x, since for all $V \in \mathcal{O}(x)$, if $U \geq V$ then $x_U \in U \subseteq V$.

($\impliedby$) This implies that every $U \in \mathcal{O}(x)$ contains an element of A.

□ Proposition 27

Proposition 28. *$f\colon (X,\tau) \to (Y,\sigma)$ is continuous if and ony if for any net $(x_\lambda : \lambda \in \Lambda) \to x$ in X, we have that $(f(x_\lambda) : \lambda \in \Lambda) \to f(x)$ in Y.*

Proof.

($\implies$) Suppose $(x_\lambda : \lambda \in \Lambda) \to x$. Let $V \ni f(x)$ be open. Then $U = f^{-1}(V)$ is open, and $x \in U$. So there is $\lambda_0 \in \Lambda$ such that for all $\lambda \geq \lambda_0$, we have $x_\lambda \in U$, and thus $f(x_\lambda) \in f(U) \subseteq V$. Thus $(f(x_\lambda) : \lambda \in \Lambda) \to f(x)$.

($\impliedby$) Suppose f is not continuous. Then there is V open in Y such that $U = f^{-1}(V)$ is not open, and thus U^c is not closed. Thus there is $x \in U$ such that $x \in \overline{U^c}$. By the previous proposition, there is a net $(x_\lambda : \lambda \in \Lambda)$ in U^c converging to x. But $f(x_\lambda) \in V^c$, so, by previous proposition, we have $f(x_\lambda) \not\to f(x) \in V$.

□ Proposition 28

Example 29. Not that the sequential characterization of continuity does not apply in general. Consider the space from before $(\mathbb{N}^2, W)$. Consider $f\colon (\mathbb{N}^2, W) \to (\mathbb{N}^2, \text{discrete})$ given by $\mathrm{id}_{\mathbb{N}^2}$. Then $(x_i : i < \omega) \to x$ if and only if $x \neq (0,0)$ and $x_i = x$ eventually. Thus $(f(x_i) : i < \omega) \to f(x)$. But f is discontinuous, since $f^{-1}(\{0,0\}) = \{0,0\}$ is not open.

2.2 Axiom of choice

Definition 30. A set A is *well-ordered* if it has a total order $<$ and every non-empty subset has a least element. A *partial order* is reflexive, antisymmetric, and transitive. It is called *inductive* if every chain (totally ordered subset) has an upper bound.

Definition 31. The *axiom of choice* says that if X is a set, then there is a $c\colon 2^X \setminus \{\emptyset\} \to X$ such that $c(A) \in A$ for all $A \neq \emptyset$.

Definition 32. The *well-ordering principle* states that every set can be well-ordered.

Definition 33. *Zorn's lemma* states that if every chain in a partial order has an upper bound, then there is a maximal element. (Note: this doesn't mean the maximal element is comparable to everything; merely that no element is larger than it.)

Remark 34. If A is well-ordered, then there is a least element.

Definition 35. An *initial segment* is

$$I = I(b) = \{ a \in A : a < b \}$$

Theorem 36. *The following are equivalent:*

1. *The axiom of choice*
2. *The well-ordering principle*
3. *Zorn's lemma*

Proof.

(2) $\implies$ (1) Place a well-ordering on X; we can then define $c(A)$ to be the least element of A.

(3) $\implies$ (2) Let

$$\mathcal{W} = \{ (F, <_F) : F \subseteq X, <_F \text{ a well-ordering of } F \}$$

Say $(F, <_F) \leq (G, <_G)$ if $F \subseteq G$ and $<_F = <_G |_{F \times F}$, and F is an initial segment of G. Let

$$\mathcal{C} = \{ (F_\alpha, <_{F_\alpha}) : \alpha \in I \}$$

be a chain, with $(F_\alpha, <_{F_\alpha}) \leq (F_\beta, <_{F_\beta})$ for $\alpha < \beta$. Let

$$G = \bigcup_{\alpha \in I} F_\alpha$$
$$<_G = \bigcup_{\alpha \in I} <_{F_\alpha}$$

Now, if $\emptyset \neq A \subseteq G$, we have

$$A = \bigcup_{\alpha \in I} (A \cap F_\alpha) \neq 0$$

so there is $\alpha \in I$ such that $A \cap F_\alpha \neq \emptyset$; then we have a least $a \in A \cap F_\alpha$. Now, for any $b \in A$, we have that $b \in F_\alpha$, in which case $a \leq b$ by our choice of a; or that $b \notin F_\alpha$, in which case $b \in F_\beta$ for some $\beta > \alpha$, so F_α is an initial segment of F_β, and $b > a$. So in fact a is the least element of A, and $<_G$ is a well-ordering. Then $(G, <_G)$ is an upper bound of $\mathcal{C}$. So $\mathcal{W}$ is inductive, and thus has a maximal element $(F, <_F)$ by Zorn's lemma.

If $F \neq X$, we could pick $a \in X \setminus F$ and define a well-ordering of $F \cup \{ a \}$ by $b < a$ for all $b \in F$, contradicting our choice of $(F, <_F)$ as a maximal element of $\mathcal{W}$. So $X = F$, and we have a well-ordering of X.

(1) $\implies$ (3) Let $(P, \leq)$ be an inductive partial order. Suppose there is no maximal element. Then for all $x \in P$, we have that

$$U_x = \{ y \in P : x < y \} \neq \emptyset$$

Then there is $f \colon P \to P$ such that $f(x) \in U_x$ for all x. Since $(P, \leq)$ is inductive, we have that for each chain $\mathcal{C}$, that

$$U_{\mathcal{C}} = \{ x \in P : x \text{ is an upper bound for } \mathcal{C} \} \neq \emptyset$$

Then there is a map

$$g \colon \{ \mathcal{C} : \mathcal{C} \text{ is a chain of } P \} \to P$$

such that $g(\mathcal{C})$ is an upper bound of $\mathcal{C}$ for each chain $\mathcal{C}$. Define $h = f \circ g$; then $h(\mathcal{C})$ is strictly greater than every element of $\mathcal{C}$ for all chains $\mathcal{C}$.

Define a well-ordering on P by $a_1 = h(\emptyset)$, $a_2 = h(\{ a_1 \})$, $a_3 = h(\{ a_1, a_2 \})$, and so on. Consider subsets $A \subseteq P$ such that

1. $(A, \leq)$ is a well-ordering.
2. If $I \subsetneq A$ is an initial segment of A, then the least element of $A \setminus I$ is $h(I)$.

Call such A a *conforming set*.

Claim 37. *If A, B are two conforming sets, then either $A \subseteq B$ or $B \subseteq A$, and it is an initial segment.*

Proof. Let $\mathcal{H}$ be the set of initial segments common to A and B. Let

$$J = \bigcup_{I \in \mathcal{H}} I$$

be the largest initial segment common to A and B. Then, if both A and B were proper supersets of J, we would have $h(J) \in A \cap B$, and $J \cup \{ h(J) \}$ would be a strictly larger initial segment common to A and B, a contradiction. □ Claim 37

Now, let X be the union of all the conforming sets; then X is well-ordered by $\leq$, and each A is an initial segment. So $(X, \leq)$ is a maximal conforming set. But $X \cup \{ h(X) \}$ is a strictly larger conforming set, a contradiction.

□ Theorem 36

2.3 Compactness

Definition 38. Suppose (X, τ) is a topological space. We say $A \subseteq X$ is *compact* if every open cover of A has a finite subcover.

Theorem 39. *The following are equivalent:*

1. *X is compact.*
2. *Every collection of closed sets $\mathcal{C}$ with the finite intersection property (that every finite intersection is non-empty) satisfies*

$$\bigcap \mathcal{C} \neq \emptyset$$

3. *Every net in X has a convergent subnet.*

Proof.

(1) $\implies$ (2) Suppose $\{ C_\alpha : \alpha \in I \}$ has the finite intersection property but

$$\bigcap_{\alpha \in I} C_\alpha = \emptyset$$

Then $\{ C_\alpha^c : \alpha \in I \}$ is an open cover of X with no finite subcover, a contradiction.

(2) $\implies$ (3) Let $(x_\lambda : \lambda \in \Lambda)$ be a net. For $\gamma \in \Lambda$, let $C_\gamma = \overline{\{ x_\lambda : \lambda \geq \gamma \}}$. Then given any $\{ \gamma_1, \dots, \gamma_n \} \subseteq \Lambda$, we some γ such that $\gamma_i \leq \gamma$ for all i; then

$$x_\gamma \in \bigcap_{i=1}^{n} C_{\gamma_i}$$

So $\{ C_\gamma : \gamma \in \Lambda \}$ has the finite intersection property, and, by assumption, we have some

$$x \in \bigcap_{\gamma \in \Lambda} C_\gamma$$

Let

$$\Gamma = \{ (\lambda, U) : \lambda \in \Lambda, U \in \mathcal{O}(x) \}$$

Define an order on Γ by $(\lambda, U) \leq (\mu, V)$ if $\lambda \leq \mu$ and $U \supseteq V$. Want to define $\varphi \colon \Gamma \to \Lambda$ which is cofinal such that $x_{\varphi(\lambda,U)} \in C_\lambda \cap U$. Well, $C_\lambda = \overline{\{ x_\gamma : \gamma \geq \lambda \}}$ intersects U since $x \in U$. Thus $\{ x_\gamma : \gamma \geq \lambda \}$ also intersects U, since U is open, and there is $\gamma \geq \lambda$ such that $x_\gamma \in U$. Let

$$Y_{\lambda,U} = \{ x_\gamma : \gamma \geq \lambda \} \cap U \neq \emptyset$$

By axiom of choice, there is $\varphi \colon \Gamma \to \Lambda$ such that $\varphi(\lambda, U) \in Y_{\lambda,U}$.

Claim 40. $(x_{\varphi(\lambda,U)} : (\lambda, U) \in \Gamma) \to x$.

Proof. If $v \in \mathcal{O}(x)$, pick λ_0 arbitrary. Then if $(\lambda, U) \geq (\lambda_0, V)$, then $x_{\varphi(\lambda,U)} \in U \subseteq V$. So $(x_{\varphi(\lambda,U)} : (\lambda, U) \in \Gamma) \to x$, as desired. □ Claim 40

To check cofinality, suppose $\lambda_0 \in \Lambda$. Then $\varphi(\lambda, U) \geq \lambda \geq \lambda_0$ if $\lambda \geq \lambda_0$. Pick arbitrary $U_0 \in \mathcal{O}(x)$. Then if $(\lambda, U) \geq (\lambda_0, U_0)$, we have $\varphi(\lambda, U) \geq \lambda_0$.

(3) $\implies$ **(1)** Let $\{ U_\alpha : \alpha \in A \}$ be an open cover. Suppose there is no finite subcover. Then for each $F \subseteq_{\text{fin}} A$, say $F = \{ \alpha_1, \ldots, \alpha_n \}$, we have

$$U_{\alpha_1} \cup \cdots \cup U_{\alpha_n} \neq X$$

so

$$C_F = U_{\alpha_1}^c \cap U_{\alpha_2}^c \cap \cdots \cap U_{\alpha_n}^c \neq \emptyset$$

Let $\Lambda = \{ F \subseteq A : |F| < \aleph_0 \}$ be ordered by $F \leq G$ if $F \subseteq G$. By axiom of choice, pick $x_F \in C_F$ for each $F \in \Lambda$. Then $(x_F : F \in \Lambda$ is a net. By assumption, there is a subnet Γ with $\varphi \colon \Gamma \to \Lambda$ cofinal such that $(x_{\varphi(\gamma)} : \gamma \in \Gamma) \to x \in X$. Thus for all $\alpha \in A$ there is $\gamma_0 \in \Gamma$ such that $\varphi(\gamma) \geq \{ \alpha \}$ if $\gamma \geq \gamma_0$. i.e. $\varphi(\gamma) = F \ni \alpha$ if $\gamma \geq \gamma_0$. Thus

$$x_{\varphi(\gamma)} = x_F \in C_F \supseteq C_{\{ \alpha \}} = U_\alpha^c$$

But U_α^c is closed. So

$$x = \lim_{\gamma \in \Gamma} x_{\varphi(\gamma)} \in U_\alpha^c$$

and $x \notin U_\alpha$ for any α. But the U_α cover X, a contradiction.

□ Theorem 39

Proposition 41. *Suppose $f \colon (X, \tau) \to (Y, \sigma)$ is continuous. Suppose $C \subseteq X$ is compact. Then $f(C)$ is compact.*

Proof. Let $\{ V_\alpha : \alpha \in A \}$ be an open cover of $f(C)$. Set $U_\alpha = f^{-1}(V_\alpha)$; these are open in X by continuity, and

$$\bigcup_{\alpha \in A} U_\alpha \supseteq C$$

Then there is a subcover

$$C \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$$

and thus

$$f(C) \subseteq f(U_{\alpha_1}) \cup \cdots \cup f(U_{\alpha_n}) \supseteq V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}$$

□ Proposition 41

Definition 42 (Product topology)**.** Suppose $((X_\alpha, \tau_\alpha) : \alpha \in A)$ are topological spaces, put a topology on

$$X = \prod_{\alpha \in A} X_\alpha$$

(whose elements take the form $x = (x_\alpha : \alpha \in A)$ where $x_\alpha \in X_\alpha$) by using the weakest topology such that all

$$\begin{aligned} \pi_\alpha \colon X &\to X_\alpha \\ x &\mapsto x_\alpha \end{aligned}$$

are continuous. i.e. if $U \subseteq X_\alpha$ is open, then

$$\pi_\alpha^{-1}(U) = \{ x \in X : x_\alpha \in U \} = U \times \prod_{\beta \neq \alpha} X_\beta$$

is open. So if $\alpha_1, \ldots, \alpha_n \in A$ and each U_{α_i} open in X_{α_i}, then

$$U_{\alpha_1} \times U_{\alpha_2} \times \cdots \times U_{\alpha_n} \times \prod_{\beta \notin \{ \alpha_1, \ldots, \alpha_n \}} X_\beta$$

is open, and these sets form a basis for the topology.

Remark 43. For $U_\alpha \subseteq X_\alpha$, we have that

$$\prod_{\alpha \in A} U_\alpha$$

is open if $U_\alpha = X_\alpha$ except finitely often. The converse holds except if some $U_\alpha = \emptyset$.

Theorem 44 (Tychonoff)**.** *The product of compact spaces is compact.*

Proof. Let $((X_\alpha, \tau_\alpha) : \alpha \in A)$ be compact topological spaces. Let

$$X = \prod_{\alpha \in A} X_\alpha$$

Suppose X is not compact; suppose there is an open cover $\mathcal{U}$ with no finite subcover. We plan to use Zorn's lemma to find a maximal open cover with no finite subcover. The order we use is set inclusion: $\mathcal{U} \leq \mathcal{V}$ if $\mathcal{U} \subseteq \mathcal{V}$. We are given that Λ, the collection of open covers with no finite subcover, is non-empty. Now, suppose

$$\{ \mathcal{U}_\alpha : \alpha \in T \}$$

is a chain in Λ (with T a totally ordered set and $\alpha \leq \beta$ in T implies $\mathcal{U}_\alpha \leq \mathcal{U}_\beta$). Let

$$\mathcal{U} = \bigcup_{\alpha \in T} \mathcal{U}_\alpha$$

Then this, too, is an open cover. Furthermore, if $\mathcal{U}$ had a finite subcover $X \subseteq U_1 \cup \cdots \cup U_n$, then we could find α_i such that $U_i \in \mathcal{U}_{\alpha_i}$. Letting α be the maximum of the α_i, we have that the U_i are all in the $\mathcal{U}_\alpha$, and are a finite subcover of $\mathcal{U}_\alpha$, a contradiction. So $\mathcal{U}$ has no finite subcover, and $\mathcal{U} \in \Lambda$ is an upper bound for

$$\{ \mathcal{U}_\alpha : \alpha \in T \}$$

Thus by Zorn's lemma there is a maximal open cover $\mathcal{U}_0$ with no finite subcover.

Properties of $\mathcal{U}_0$:

1. If $U \in \mathcal{U}_0$ and $V \subseteq U$ is open, then $V \in \mathcal{U}_0$.
2. If $U_1, U_2 \in \mathcal{U}_0$, then $U_1 \cup U_2 \in \mathcal{U}_0$.
3. If V_1, V_2 are open with $V_1 \cap V_2 \in \mathcal{U}_0$, then one of V_1 and V_2 is in $\mathcal{U}_0$.

   Proof. If $V_1 \notin \mathcal{U}_0$ then $\mathcal{U}_0 \cup \{ V_1 \}$ has a finite subcover

   $$X \subseteq V_1 \cup U_1 \cup U_2 \cup \cdots \cup U_n = V_1 \cup W_1$$

   where $W_1 \in \mathcal{U}_0$. If $V_2 \notin \mathcal{U}_0$, then

   $$X \subseteq V_2 \cup W_2$$

   where $W_2 \in \mathcal{U}_0$. But then

   $$X \subseteq (V_1 \cap V_2) \cup W_1 \cup W_2$$

   A contradiction. So $V_1 \in \mathcal{U}_0$ or $V_2 \in \mathcal{U}_0$. □

For $\alpha \in A$, let

$$\mathcal{W}_\alpha = \left\{ U \subseteq X_\alpha : U \text{ open}, U \times \prod_{\beta \neq \alpha} X_\beta \in \mathcal{U}_0 \right\}$$

If we had

$$\bigcup W_\alpha = X_\alpha$$

then $\mathcal{W}_\alpha$ is an open cover. Then, since X_α is compact, we have a finite subcover

$$X_\alpha \subseteq U_1 \cup \cdots \cup U_n$$

So

$$X \subseteq \left(U_1 \times \prod_{\beta \neq \alpha} X_\beta \right) \cup \cdots \cup \left(U_n \times \prod_{\beta \neq \alpha} X_\beta \right)$$

a contradiction. So

$$C_\alpha = \left(\bigcup \mathcal{W}_\alpha \right)^c \neq \emptyset$$

By axiom of choice, there is $x = (x_\alpha : \alpha \in A) \in X$ such that $x_\alpha \in C_\alpha$ for all $\alpha \in A$. Now, $\mathcal{U}_0$ covers X, so there is $U \in \mathcal{U}$ such that $x \in U$. Thus there is a basic open set $V \subseteq U$ with $x \in V$. Then

$$\begin{aligned} \mathcal{U}_0 \ni V & \\ &= (V_{\alpha_1} \times V_{\alpha_2} \times \cdots \times V_{\alpha_n}) \times \prod_{\beta \notin \{ \alpha_1, \ldots, \alpha_n \}} X_\beta \\ &= \pi_{\alpha_1}^{-1}(V_{\alpha_1}) \cap \pi_{\alpha_2}^{-1}(V_{\alpha_2}) \cap \cdots \cap \pi_{\alpha_n}^{-1}(V_{\alpha_n}) \end{aligned}$$

By the third property, there is i_0 such that

$$\pi_{\alpha_{i_0}}(V_{\alpha_{i_0}}) = V_{\alpha_{i_0}} \times \prod_{\beta \neq \alpha_{i_0}} X_\beta \in \mathcal{U}_0$$

and thus

$$V_{\alpha_{i_0}} \in \mathcal{W}_{\alpha_{i_0}}$$

So

$$x_{\alpha_{i_0}} \in V_{\alpha_{i_0}} \subseteq \bigcup \mathcal{W}_{\alpha_{i_0}}$$

contradicting our choice of $x_{\alpha_{i_0}}$. □ Theorem 44

Remark 45. Tychonoff's theorem implies the axiom of choice.

Proof. Suppose X_α are non-empty sets. Define $Y_\alpha = X_\alpha \sqcup \{ p_\alpha \}$ and define τ_α on Y_α by

$$\tau_\alpha = \{ \emptyset, \{ p_\alpha \}, X_\alpha, Y_\alpha \}$$

These are compact because τ is finite. Thus

$$\prod Y_\alpha$$

is compact by Tychonoff's theorem. Let

$$C_\alpha = \pi_\alpha^{-1}(X_\alpha) = X_\alpha \times \prod_{\beta \neq \alpha} Y_\beta$$

Then these are closed. For $F = \{ \alpha_1, \ldots, \alpha_n \}$, set

$$C_F = C_{\alpha_1} \cap \cdots \cap C_{\alpha_n}$$

Pick $x_i \in X_{\alpha_i}$ because $X_{\alpha_i} \neq \emptyset$ for $1 \leq i \leq n$. Let

$$x = (x_1, \ldots, x_n, p_\beta : \beta \notin \{ \alpha_1, \ldots, \alpha_n \}) \in C_{\alpha_1} \cap \cdots \cap C_{\alpha_n}$$

So $\{ C_\alpha \}$ has the finite intersection property. So their intersection contains some x; then x satisfies $x_\alpha \in X_\alpha$ for all α, and we have a choice function. □ Remark 45

Definition 46. (X, τ) is *normal* if points are closed and whenever A, B are closed in X with $A \cap B = \emptyset$, then there is open $U \supseteq A$, $V \supseteq B$ such that $U \cap V = \emptyset$.

Example 47.

1. Metric spaces: can set

$$U = \{ x \in X : d(x, A) < d(x, B) \}$$
$$V = \{ x \in X : d(x, B) < d(x, A) \}$$

2. If X is compact and Hausdorff, then X is normal. Given A, B, fix $a \in A$. Suppose $b \in B$. Then, since X is Hausdorff, there are $U_b \ni a$ and $V_b \ni b$ open and disjoint. Then, by compactness

$$B \subseteq V_{b_1} \cup \cdots \cup V_{b_n} = V_a$$

and

$$a \in U_{b_1} \cap \cdots \cap U_{b_n} = U_a$$

Then

$$A \subseteq \bigcup_{a \in A} U_a$$

Again by compactness, we have

$$A \subseteq U_{a_1} \cup \cdots \cup U_{a_n}$$

and then

$$B \subseteq V_{a_1} \cap \cdots \cap V_{a_n}$$

Theorem 48 (Urysohn's lemma)**.** *Suppose (X, τ) is normal. Suppose A, B are disjoint closed sets. Then there is continuous $f \colon X \to [0, 1]$ such that $f \restriction A = 0$, $f \restriction B = 1$.*

Theorem 49 (Tietze's extension theorem)**.** *Suppose X is normal, $A \subseteq X$ is closed, and $f \colon A \to \mathbb{R}$ is continuous. Then there is $g \colon X \to \mathbb{R}$ continuous such that $g \restriction A = f$.*

Proof of Theorem 48. By normality, there is open $A_{\frac{1}{2}} \supseteq A$ such that $\overline{A_{\frac{1}{2}}} \cap B = \emptyset$. Also, we have open $A_{\frac{3}{4}} \supseteq \overline{A_{\frac{1}{2}}}$ such that $\overline{A_{\frac{3}{4}}} \cap B = \emptyset$, and we have open $A_{\frac{1}{4}} \supseteq A$ such that $A_{\frac{1}{4}} \cap A^c_{\frac{1}{2}} = \emptyset$. Continuing this way, we define A_y for all dyadic rationals $y \in (0, 1)$. We then take

$$f(x) = \begin{cases} \inf\{ y : x \in A_y \} & \text{such a } y \text{ exists} \\ 1 & \text{else} \end{cases}$$

Then this is the desired function. ☐ Theorem 48

3 Banach spaces

Definition 50. Let V be a vector space over $\mathbb{F} \in \{ \mathbb{R}, \mathbb{C} \}$. A *norm* on V is a map $\|\cdot\| \colon V \to [0, \infty)$ such that

1. for $v \in V$, we have $\|v\| = 0$ if and only if $v = 0$
2. $\|tv\| = |t|\|v\|$ for $t \in \mathbb{F}$, $v \in V$ (called "positive homogeneous")
3. $\|v + w\| \leq \|v\| + \|w\|$ for $v, w \in V$.

A normed vector space $(V, \|\cdot\|)$ is called a *Banach space* if it is *complete*; i.e. every Cauchy sequence $(v_n : n \in \mathbb{N})$ (i.e. for all $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n, m \geq N$, we have $\|v_m - v_n\| < \varepsilon$) converges (i.e. there is $v \in V$ such that $\|v_n - v\| \to 0$).

Remark 51. $(V, \|\cdot\|)$ is a metric space with $d(v, w) = \|v - w\|$.

Example 52.

1. Suppose X is compact, Hausdorff. Consider

$$C(X) = \{\, f\colon X \to \mathbb{C} \mid f \text{ continuous} \,\}$$
$$C_{\mathbb{R}}(X) = \{\, f\colon X \to \mathbb{R} \mid f \text{ continuous} \,\}$$

   with the norm

$$\|f\|_\infty = \sup_{x\in X} |f(x)| < \infty$$

   Then $(f_n : n \in \mathbb{N}) \to f$ if and only if $\|f - f_n\| \to 0$, which holds if and only if $(f_n : n \in \mathbb{N}) \to f$ uniformly. But recall that the uniform limit of continuous functions is continuous. So $C(X)$ is complete.

2. For $1 \le p < \infty$, consider

$$\ell_p = \left\{ (a_n : n \in \mathbb{N}) : \text{all} a_n \in \mathbb{C}, \|(a_n : n \in \mathbb{N})\|_p = \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{\frac{1}{p}} \right\}$$
$$\ell_\infty = \{\, (a_n : n \in \mathbb{N}) : \sup_{n\in\mathbb{N}} |a_n| = \|(a_n : n \in \mathbb{N})\|_\infty < \infty \,\}$$
$$L^p(0,1) = \{\, f \text{ Lebesgue measurable} : \left(\int |f|^p dm\right)^{\frac{1}{p}} < \infty \,\}$$
$$L^p(\mu) = \{\, f\ \mu\text{-measurable} : \left(\int |f|^p d\mu\right)^{\frac{1}{p}} < \infty \,\}$$

   Proposition 53 (Hölder's inequality). *For $p \ge 1$ and $q \ge 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ (we allow $p = 1$ and $q = \infty$), then if $f \in L^p(\mu)$, $g \in L^q(\mu)$, then*

$$\left| \int fg d\mu \right| \le \|f\|_p \|g\|_q$$

   Proposition 54 (Minkowski's inequality). *For $f, g \in L^p(\mu)$, we have $\|f + g\|_p \le \|f\|_p + \|g\|_p$.*

   Proposition 55. *$L^p(\mu)$ is complete.*

3. Consider $C^{(n)}[0,1]$ (the set of continuously n-differentiable functions) with the norm

$$\|f\|_{C^{(n)}} = \max_{0\le i\le n} \|f^{(i)}\|_\infty$$

   If

$$Df = \sum_{i=0}^{n} a_i(x) f^{(i)}(x)$$

   for $a_i \in C[0,1]$, then D is a linear map from $C^{(n)}[0,1]$ to $C[0,1]$.

4. Hilbert spaces: $\mathcal{H}$ together with an inner product

$$\langle \cdot, \cdot \rangle \colon \mathcal{H}^2 \to \mathbb{F}$$

   that is linear in x, conjugate-linear in y, and positive-definite. i.e.

$$\begin{aligned}
\langle ax_1 + x_2, y\rangle &= a\langle x_1, y\rangle + \langle x_2, y\rangle \\
\langle x, ay_1 + y_2\rangle &= a\langle x, y_1\rangle + \langle x, y_2\rangle \\
\langle x, x\rangle &\ge 0 \\
\langle x, x\rangle = 0 &\iff x = 0 \\
\langle y, x\rangle &= \overline{\langle x, y\rangle}
\end{aligned}$$

Note that for all $t \in \mathbb{R}$, we have

$$
\begin{aligned}
0 &\leq \langle x + ty, x + ty \rangle \\
&= \langle x, x \rangle + \bar{t}\langle x, y \rangle + t\langle y, x \rangle + |t|^2 \langle y, y \rangle
\end{aligned}
$$

I believe $t = 1$ shows that $\langle x, y \rangle = \overline{\langle y, x \rangle}$ given the other axioms. Taking $t = 1$ also shows the triangle inequality of $\|x\| = \langle x, x \rangle^{\frac{1}{2}}$.

Taking

$$
t = \frac{-\langle x, y \rangle}{\|y\|^2}
$$

shows the Cauchy-Schwarz inequality: that

$$
|\langle x, y \rangle| \leq \|x\| \|y\|
$$

A *Hilbert space* is a complete inner product space. Examples include ℓ_2, $L^2(\mu)$.

5. Another example of a Hilbert space. Suppose Ω is an open, connected, bounded subset of $\mathbb{C}$. Let

$$
L^2_a(\Omega) = \left\{ f \text{ analytic on } \Omega : \|f\|^2 = \int_\Omega |f(z)|^2 dm_2 < \infty \right\}
$$

If $z \in \Omega$, there is r such that $\overline{B_r(z)} \subseteq \Omega$. Then

$$
\begin{aligned}
\int_{\overline{B_r(z)}} f(w)dw &= \int_0^r \int_0^{2\pi} f(z + r\exp(i\theta))rd\theta dr \\
&= \int_0^r 2\pi r \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z + r\exp(i\theta))}{r\exp(i\theta)} ri\exp(i\theta)d\theta dr \\
&= \int_0^r 2\pi r \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z} dwdr \\
&= \int_0^r 2\pi r f(z)dr \\
&= (\pi r^2) f(z)
\end{aligned}
$$

by Cauchy's integral formula. Thus

$$
f(z) = \frac{1}{\pi r^2} \int_{\overline{B_r(z)}} f(w)dw
$$

So

$$
\begin{aligned}
|f(z)| &\leq \frac{1}{\pi r^2} \int_{\overline{B_r(z)}} |f| dm_2 \\
&= \frac{1}{\pi r^2} \int_\Omega |f| \chi_{\overline{B_r(z)}} dm_2 \\
&\leq \frac{1}{\pi r^2} \|f\| \left(\int_{\overline{B_r(z)}} dm_2 \right)^{\frac{1}{2}} \\
&= \frac{\|f\|}{\sqrt{\pi r^2}}
\end{aligned}
$$

If $w \in \overline{B_{\frac{r}{2}}(z)}$, then $\overline{B_{\frac{r}{2}}(w)} \subseteq \Omega$, and

$$
|f(w)| \leq \frac{\|f\|}{\sqrt{\frac{\pi r^2}{4}}}
$$

for all $w \in \overline{B_{\frac{r}{2}}(z)}$. Suppose $(f_n : n \in \mathbb{N})$ is a Cauchy sequence in $L^2_a(\Omega)$. For $w \in \overline{B_{\frac{r}{2}}(z)}$, we have

$$\begin{aligned} |f_n(w) - f_m(w)| &= |(f_n - f_m)(w)| \\ &\leq \frac{1}{\sqrt{\frac{\pi r^2}{4}}} \|f_n - f_m\| \end{aligned}$$

Thus $f_n \restriction \overline{B_{\frac{r}{2}}(z)}$ is uniformly Cauchy; thus $(f_n : n \in \mathbb{N}) \to f$ uniformly in $\overline{B_{\frac{r}{2}}(z)}$; thus the limit is analytic in $B_{\frac{r}{2}}(z)$.

3.1 General constructions in Banach spaces

Proposition 56. *Let X, Y be normed vector spaces over $\mathbb{F}$; let $T\colon X \to Y$ be a linear map. Then the following are equivalent:*

1. $$\|T\| = \sup_{\|x\| \leq 1} \|Tx\|_Y < \infty$$

   (T is bounded).

2. *T is uniformly continuous.*

3. *T is continuous.*

4. *T is continuous at* 0.

Proof.

(1) $\implies$ **(2)** If $\|T\| < \infty$, then

$$\begin{aligned} \|Tx - Ty\| &= \|T(x - y)\| \\ &\leq \|T\|\|x - y\| \end{aligned}$$

Then for any $\varepsilon > 0$ we may let $\delta = \frac{\varepsilon}{\|T\|}$; then $\|x - y\| < \delta \implies \|Tx - Ty\| < \varepsilon$.

(2) $\implies$ **(3)** $\implies$ **(4)** Trivial.

$\neg$**(1)** $\implies$ $\neg$**(4)** If

$$\sup_{\|x\| \leq 1} \|Tx\| = \infty$$

pick $x_n \in X$ such that $\|x_n\| \leq 1$ and $\|Tx_n\| > n^2$. Let $y_n = \frac{1}{n} x_n$. Then

$$\|y_n\| \leq \frac{1}{n} \to 0$$

and thus $y_n \to 0$. But $\|Ty_n\| > n$; so $Ty_n \not\to 0$, and T is not continuous.

□ Proposition 56

Write $\mathcal{B}(X, Y)$ for the set of all bounded linear maps from X to Y. Then $\|T\|$ is defined; it is, in fact, a norm:

$$
\begin{aligned}
\|T\| &\geq 0 \\
\|T\| = 0 &\implies Tx = 0 \text{ for all} x \\
\|tT\| &= \sup_{\|x\|\leq 1} \|tTx\| \\
&= |t| \sup_{\|x\|\leq 1} \|Tx\| \\
&= |t|\|T\| \\
\|S+T\| &= \sup_{\|x\|\leq 1} \|(S+T)x\| \\
&\leq \sup_{\|x\|\leq 1} (\|Sx\| + \|Tx\|) \\
&\leq \sup_{\|x\|\leq 1} \|Sx\| + \sup_{\|x\|\leq 1} \|Tx\| \\
&= \|S\| + \|T\|
\end{aligned}
$$

Proposition 57. *Suppose X, Y are normed vector spaces; suppose Y is complete. Then $\mathcal{B}(X, Y)$ is a Banach space.*

Proof. Suppose $(T_n : n \in \mathbb{N})$ is a Cauchy sequence. Then, for $x \in X$, if $\|x\| \leq 1$, then we have $\|T_m x - T_n x\| \leq \|T_m - T_n\| < \varepsilon$ for sufficiently large m, n. Thus $(T_n x : n \in \mathbb{N})$ is Cauchy, uniform on $B_1(x)$. Thus $T_n \to T$ uniformly on $B_1(x)$. Thus T is linear and uniformly continuous. So it is bounded, and $\mathcal{B}(X, Y)$ is complete. □ Proposition 57

Definition 58. We set $\mathcal{B}(X) = \mathcal{B}(X, X)$. We set $X^* = \mathcal{B}(X, \mathbb{F})$ to be the *dual space of* X.

Theorem 59. *If $1 \leq p < \infty$ with*

$$\frac{1}{p} + \frac{1}{q} = 1$$

then $\ell_p^* = \ell_q$.

Proof. Pairing: given $a = (a_n : n \in \mathbb{N}) \in \ell_p$ and $b = (b_n : n \in \mathbb{N}) \in \ell_q$, we set

$$\varphi_b(a) = b(a) = \langle a, b \rangle = \sum_{n=1}^{\infty} a_n b_n$$

(Note that this is the *definition* of $\langle \cdot, \cdot \rangle$, and that this is bilinear, rather than sesquilinear.)

Hölder's inequality then yields

$$|\varphi_b(a)| = \left| \sum_{n=1}^{\infty} a_n b_n \right| \leq \|a\|_p \|b\|_q$$

So $|\varphi_b(a)| \leq \|a\|_p \|b\|_q$, and we have $\|\varphi_b\| \leq \|b\|_q$.

Case 1. Suppose $p = 1$ and $q = \infty$. Then $\|b\|_\infty = \sup_{n \in \mathbb{N}} |b_n|$. Letting

$$e_n = (0, \dots, 0, 1, 0, \dots)$$

we then have $\|e_n\|_p = 1$. Then

$$\|\varphi_b\| \geq \sup_{n \in \mathbb{N}} |\varphi_b(e_n)| = \sup |b_n| = \|b\|_\infty \geq \|\varphi_b\|$$

Case 2. Take

$$a_n = \begin{cases} \frac{\overline{b_n}}{|b_n|} |b_n|^{q-1} & 1 \leq n \leq N \\ 0 & n > N \end{cases}$$

Let $a = (a_n : n \in \mathbb{N})$. Then

$$\begin{aligned}\|a\|_p^p &= \sum_{n=1}^{N} |a_n|^p \\ &= \sum_{n=1}^{N} |b_n|^{p(q-1)} \\ &= \sum_{n=1}^{N} |b_n|^q \\ &\le \|b\|_q^q\end{aligned}$$

Without loss of generality we may assume $\|b\|_q = 1$. Then $\|a\|_p \le 1$. Then

$$\varphi_b(a) = \sum_{n=1}^{N} |b_n|^{q-1} |b_n| = \sum_{n=1}^{N} |b_n|^q \to \|b\|_q^q = 1$$

Thus $\|\varphi_b\| \ge \|b\|_q$; thus $\|\varphi_b\| = \|b\|_q$.

Now let $\varphi \in \ell_p^*$. Let $b_n = \varphi(e_n)$. Let $b = (b_n : n \in \mathbb{N})$. We know that $|\varphi(a)| \le \|\varphi\|\|a\|$. Note that, as above, we have

$$\|a\|_p = \left(\sum_{n=1}^{N} |b_n|^q \right)^{\frac{1}{p}}$$

And

$$\begin{aligned}\|\varphi\|\|a\|_p &\ge |\varphi(a)| \\ &= \left| \sum_{n=1}^{N} a_n b_n \right| \\ &= \sum_{n=1}^{N} |b_n|^{q-1} |b_n| \\ &= \sum_{n=1}^{N} |b_n|^q\end{aligned}$$

Thus

$$\|\varphi\| \ge \left(\sum_{n=1}^{N} |b_n|^q \right)^{q-\frac{1}{p}}$$

Thus something which immediately implies we're done.

□ Theorem 59

Example 60. Let

$$c_0 = \{\, a = (a_1, a_2, \dots) : \lim_{n\to\infty} a_n = 0 \,\}$$

Set

$$\|a\| = \sup_{n\ge 1} |a_n|$$

Then

$$c_0^* = \{\, \varphi \colon c_0 \to \mathbb{C} : \|\varphi\| = \sup_{\|a\|\le 1} |\varphi(a)| < \infty \,\}$$

Set $e_n = (0, \dots, 0, 1, 0, 0, \dots)$. For $a \in c_0$, we then have

$$a = \lim_{N\to\infty} \sum_{n=1}^{N} a_n e_n$$

Let $\varphi(e_n) = x_n \in \mathbb{C}$; then $|x_n| \leq \|\varphi\|$. Let

$$\overrightarrow{a}_N = \frac{\overline{x_1}}{|x_1|}e_1 + \frac{\overline{x_2}}{|x_2|}e_2 + \cdots + \frac{\overline{x_N}}{|x_N|}e_N$$

Then

$$\|\overrightarrow{a}_N\| = \max\left|\frac{\overline{x_i}}{|x_i|}\right| \leq 1$$

(where

$$\frac{\overline{0}}{|0|}$$

is taken to be 0). Then

$$\begin{aligned}\varphi(\overrightarrow{a}_N) &= \frac{\overline{x_1}}{|x_1|}x_1 + \frac{\overline{x_2}}{|x_2|}x_2 + \cdots + \frac{\overline{x_N}}{|x_N|}x_N \\ &= \sum_{i=1}^{N}|x_i| \\ &\leq \|\varphi\|\|\overrightarrow{a}_N\| \\ &\leq \|\varphi\|\end{aligned}$$

Let $N \to \infty$. Then

$$\|(x_n : n \in \mathbb{N})\|_1 = \sum_{i=1}^{\infty}|x_i| \leq \|\varphi\|$$

So $(x_n : n \in \mathbb{N}) \in \ell_1$. Conversely, if $x = (x_n : n \in \mathbb{N}) \in \ell_1$, define

$$\varphi_x(a) = \sum a_n x_n$$

which then converges absolutely, as

$$|a_n x_n| \leq \|a\||x_n|$$

Then

$$|\varphi_x(a)| \leq \sum_{n=1}^{\infty}\|a\||x_n| = \|a\|\|x\|_1$$

So φ_x is continuous, and $\|\varphi_x\| \leq \|x\|_1 \leq \|\varphi_x\|$ (as shown above). So $\|\varphi_x\| = \|x\|_1$. So $c_0^* = \ell_1$.

We now look at $\mathcal{B}(c_0)$ and $\mathcal{B}(c_0, \ell_\infty)$. If $T \in \mathcal{B}(c_0, \ell_\infty)$, then

$$TE_n = t_n = \begin{pmatrix} t_{1n} \\ t_{2n} \\ \vdots \end{pmatrix}$$

and

$$\|t_n\|_\infty = \sup_{i \geq 1}|t_{in}| \leq \|T\|\|e_n\| = \|T\|$$

Then

$$T\sum_{n=1}^{N} a_n e_n = \sum_{n=1} a_n T e_n = \sum_{n=1}^{N} a_n \overrightarrow{t_n}$$

and

$$T\sum_{n=1}^{\infty} a_n e_n = \lim_{n\to\infty}\sum_{n=1}^{N} a_n T e_n = \lim_{n\to\infty}\sum_{n=1}^{N} a_n \overrightarrow{t}_n$$

where this latter limit exists if T is continuous. We can think of T as having an $\infty \times \infty$ matrix $(t_{ij} : i \geq 1, j \geq 1)$ with columns $t_n \in \ell_\infty$ and

$$\sup_{n \in \mathbb{N}}\|t_n\|_\infty = \|T\|$$

Observe that the n^{th} entry of $T(a_1, a_2, \dots)$ is

$$\sum_{j=1}^{\infty} t_{nj} a_j$$

So the linear map

$$\varphi_n(a) = \langle T\overrightarrow{a}, \delta_n \rangle = \sum_{j\in\mathbb{N}} t_{n_j} a_j$$

is continuous, where $\delta_n = (0, \dots, 0, 1, 0, \dots) \in \ell_1$. Then

$$\|\varphi_n\| \leq \|T\| \|\delta_i\|_1 = \|T\|$$

Let $\overrightarrow{r_n} = (t_{nj} : j \in \mathbb{N}) \in \ell_1$; then $\|r_n\|_1 \leq \|T\|$.

On the other hand, suppose $\{ r_n : n \in \mathbb{N} \} \subseteq \ell_1$ satisfies

$$\sup_{n\in\mathbb{N}} \|r_n\| = R < \infty$$

Then $T\colon c_0 \to \ell_\infty$ given by

$$(T\overrightarrow{a})_n = \langle \overrightarrow{a}, r_n \rangle = \sum_{j\in\mathbb{N}} r_{nj} a_j$$

Then

$$|(T\overrightarrow{a})_n| \leq \|a\| \|r_n\|_1 \leq R\|a\|$$

So $\operatorname{Ran}(T) \subseteq \ell_\infty$, and $\|T\| = R$. When do we have $T \in \mathcal{B}(c_0)$? Need $t_n = Te_n \in c_0$. If each $t_n \in c_0$, then

$$T \sum_{i=1}^{N} a_i e_i = \sum_{i=1}^{N} a_i \overrightarrow{t}_i \in c_0$$

If $\overrightarrow{a} \in c_0$, then

$$Ta = \lim_{N\to\infty} T \sum_{i=1}^{N} a_i e_i$$

which exists by continuity. But c_0 is closed inside ℓ_∞. So $T\overrightarrow{a} \in c_0$. Thus $T \in \mathcal{B}(c_0)$ if and only if the rows of T have bounded ℓ_1 norm and the columns of T are in c_0.

Proposition 61. *If X is a Banach space and $M \subseteq X$ is a closed subspace, then M is a Banach space.*

Proof. For $m \in M$, we have $\|m\|_M = \|m\|_X$, so M is a normed vector space. Then M is a closed subset of a complete metric space, and M is complete. ☐ Proposition 61

Example 62. Let $A(\mathbb{D})$ the *disc algebra* be the set of $f(z)$ that are continuous on $\overline{\mathbb{D}}$ and analytic on $\mathbb{D}$. Set

$$\|f\| = \|f\|_\infty = \sup_{z\in\overline{\mathbb{D}}} |f(z)|$$

Clearly $A(\mathbb{D}) \subseteq C(\overline{\mathbb{D}})$. For $f_n \in A(\mathbb{D})$ with $(f_n : n \in \mathbb{N}) \to f$ in $C(\overline{\mathbb{D}})$ (i.e. uniform convergence). Then f is analytic on $\mathbb{D}$ because uniform limits of analytic functions are analytic. Also observe that for $f \in A(\mathbb{D})$, the maximum modulus principle yields

$$\|f\|_\infty = \sup_{|z|=1} |f(z)|$$

We can consider $A(\mathbb{D}) \subseteq C(\mathbb{T})$, where

$$\mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \}$$

Consider $R\colon A(\mathbb{D}) \to C(\mathbb{T})$ given by $R(f) = f \restriction \mathbb{T}$. Then $\|Rf\| = \|f\|$; i.e. R is an *isometry.* So $R(A(\mathbb{D}))$ is a subspace of $C(\mathbb{T})$ with the same norm, same linear structure as $A(\mathbb{D})$. (They are *isometrically isomorphic.*) So we can consider $A(\mathbb{D})$ as a subspace of $C(\mathbb{T})$. It is, in fact, closed, as we know it is a complete subspace.

Interest: Fourier series. For $f \in C(\mathbb{T})$, we can set

$$\widehat{f}(n) = \frac{1}{2\pi} f(\exp(i\theta)) \exp(-in\theta) d\theta$$

for $n \in \mathbb{Z}$. For $f \in A(\mathbb{D})$, we have

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

Thus has radisu of convergence ≥ 1. Also f is continuous on $\overline{\mathbb{D}}$, so

$$f_r(z) = \sum_{n=0}^{\infty} a_n r^n z^n = f(rz)$$

satisfies $f_r \to f$ uniformly. Recall

Theorem 63 (Abel's theorem). *For $f \in C(\mathbb{T})$, recall we can write*

$$f \sim \sum_{n=-\infty}^{\infty} \widehat{f}(n) r^n \exp(2n\theta)$$

though this doesn't always converge. However, if $z = r\exp(i\theta)$ for $0 \leq r < 1$, then

$$f(z) = \sum_{n=-\infty}^{\infty} \widehat{f}(n) r^n \exp(in\theta)$$

is harmonic on $\mathbb{D}$

Then $f_r(\exp(i\theta)) \to f(\exp(i\theta))$ uniformly. If $\widehat{f}(n) = 0$ for $n < 0$, then

$$f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n$$

is analytic, so $f \in A(\mathbb{D})$. Thus

$$\begin{aligned} A(\mathbb{D}) &= \{ f \in C(\mathbb{T}) : \widehat{f}(n) = 0 \text{ for } n < 0 \} \\ &= \bigcap_{n=-1}^{-\infty} \ker \varphi_n \end{aligned}$$

where $\varphi_n(f) = \widehat{f}(n)$.

Definition 64 (Quotient spaces)**.** Suppose X is a Banach space, $M \subseteq X$ is a closed subspace. Set

$$X/M = \{ \dot{x} = x + M : x \in X \}$$

be the collection of cosets of X/M with the quotient vector space structure. Define the quotient norm to be

$$\|\dot{x}\| = \inf_{m \in M} \|x + m\|$$

Proposition 65. *X/M is a Banach space.*

Proof. We check that it is a norm:

1. Clearly $\|\dot{x}\| \geq 0$. If $\|\dot{x}\| = 0$, then there is $(m_n : n \in \mathbb{N})$ in M such that $\|x + m_n\| \to 0$. i.e. $x + m_n \to 0$. But $x + m_n \in x + M$ and $x + M$ is closed. So $0 \in M$ and $\dot{x} = \dot{0}$.

2. For $t \neq 0$, we have

$$\begin{aligned}
\|t\dot{x}\| &= \inf_{m \in M} \|tx + m\| \\
&= \inf_{m' \in M} \|t(x + m')\| \\
&= |t| \inf_{m' \in M} \|x + m'\| \\
&= |t| \|\dot{x}\|
\end{aligned}$$

3. Note that

$$\begin{aligned}
\|\dot{x} + \dot{y}\| &= \inf_{m \in M} \|x + y + m\| \\
&= \inf_{m,n \in M} \|x + m + y + n\| \\
&\leq \inf_{m \in M, n \in M} (\|x + m\| + \|y + n\|) \\
&= \|x\| + \|y\|
\end{aligned}$$

We now check completeness. Suppose $(\dot{x}_n : n \in \mathbb{N})$ is a Cauchy sequence in X/M. Drop to a subsequence such that $\|\dot{x}_{n-1} - \dot{x}_n\| < 2^{-n}$. Recursively choose $y_n \in X$ such that

1. $\dot{y}_n = \dot{x}_n$
2. $\|y_{n-1} - y_n\| < 2^{-n}$

We pick $y_1 = x_1$. Now,

$$\frac{1}{4} > \|\dot{x}_1 - \dot{x}_2\| = \inf_{m \in M} \|y_1 - x_2 - m\|$$

Pick m_2 such that

$$\|y_1 - (x_2 + m_2)\| < \frac{1}{4}$$

and set $y_2 = x_2 + m_2$. Given y_n, note that

$$\frac{1}{2^{n+1}} > \|\dot{x}_n - \dot{x}_{n+1}\| = \|\dot{y_n} - \dot{x}_{n+1}\| = \inf_{m \in M} \|y_n - (x_{n+1} + m)\|$$

and pick $m \in M$ such that

$$\frac{1}{2^{n+1}} > \|y_n - (x_{n+1} + m)\|$$

Then set $y_{n+1} = x_{n+1} + m$.

Then $(y_n : n \in \mathbb{N})$ is Cauchy in X. So

$$y = \lim_{n \to \infty} y_n$$

exists by completeness of X. Check then that $\dot{x}_n = \dot{y}_n \to \dot{y}$. So X/M is complete, and it is a Banach space.

□ Proposition 65

Proposition 66. *For a Banach space X, M a proper closed subspace, we have that the map $Q \colon X \to X/M$ by $Qx = \dot{x}$ has $\|Q\| = 1$ and $\ker(Q) = M$.*

Proof. Well,

$$\begin{aligned}
\|Qx\| &= \|\dot{x}\| \\
&= \inf_{m \in M} \|x + m\| \\
&\leq \|x + 0\| \\
&= \|x\|
\end{aligned}$$

Thus $\|Q\| \le 1$, and Q is continuous. But $M \neq X$; so there is $x \in X$ such that $\dot{x} \neq 0$. Then

$$\|\dot{x}\| = \inf_{m \in M} \|x + m\|$$

For $\varepsilon > 0$ there is $m \in M$ such that $\|x + m\| < \|\dot{x}\| + \varepsilon$. Then

$$\|Q\| \ge \frac{\|Q(x+m)\|}{\|x+m\|} > \frac{\|\dot{x}\|}{\|\dot{x}\| + \varepsilon}$$

which approaches 1 as $\varepsilon \to 0$. □ Proposition 66

Observe that $\ker Q = \{ x : \dot{x} = \dot{0} \} = M$.

Example 67. Consider $C(X)$ where X is a compact Hausdorff space. Suppose $E \subseteq X$ is closed. Set $I(E) = \{ f \in C(X) : f \restriction E = 0 \}$; this is a closed subspace. Consider $C(X)/I(E)$. For $g \in C(X)$, we have

$$\|\dot{g}\| = \inf_{f \restriction E = 0} \|g + f\|_\infty \ge \inf_{f \restriction E = 0} \sup_{x \in E} |g(x) + f(x)| = \sup_{x \in E} |g(x)| = \|g \restriction E\|$$

Suppose $\|g \restriction E\| = 1$. Let

$$h(z) = \begin{cases} z & |z| \le 1 \\ \frac{z}{|z|} & |z| > 1 \end{cases}$$

Then $\|h\|_\infty = 1$, so $\|h \circ g\|_\infty = 1$. Also $h \circ g \restriction E = g$. Set $f = g - h \circ g \in I(E)$. Then

$$\|g - f\| = \|h \circ g\| = \|g \restriction E\| = 1$$

Thus $\|g\| = \|g \restriction E\|$. We then have the following map $g \in C(X) \mapsto g \restriction E \in C(E)$ which factors as $g \mapsto \dot{g} \in C(X)/I(E)$ followed by an isometry. Tietze's extension theorem then says that R maps onto $C(E)$. Thus $C(X) \cong C(...)/I(E) = C(E)$ something

3.2 More on Hilbert spaces

Definition 68. Suppose $\mathcal{H}$ is a Hilbert space; suppose $x, y \in \mathcal{H}$. We write $x \perp y$ (x is *orthogonal* to y) if $\langle x, y \rangle = 0$.

Remark 69 (Pythagorean law). In this case we have

$$\begin{aligned} \|x + y\|^2 &= \langle x + y, x + y \rangle \\ &= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle \\ &= \|x\|^2 + \|y\|^2 \end{aligned}$$

Definition 70. We say $\{ e_\alpha : \alpha \in I \}$ is *orthonormal* if

$$\langle e_\alpha, e_\beta \rangle = \delta_{\alpha\beta} = \begin{cases} 1 & \alpha = \beta \\ 0 & \text{else} \end{cases}$$

Remark 71. If $\{ e_1, \ldots, e_n \}$ is orthonormal, then

$$\left\| \sum_{i=1}^n a_i e_i \right\| = \left(\sum_{i=1}^n |a_i|^2 \right)^{\frac{1}{2}}$$

As motivation for our definitions of convergence, note that in $\mathbb{R}$, we have that an infinite sum doesn't converge, converges conditionally (in which case rearrangements can converge to anything), or converges absolutely (in which case it is rearrangement-invariant). In $\mathbb{R}^n$ we have a similar situation except that in the case of conditional convergence, there is an affine subspace of vectors to which it can converge.

Definition 72. In a Banach space X, a sum

$$\sum_{n=1}^{\infty} x_n$$

converges absolutely if

$$\sum_{n=1}^{\infty} \|x_n\| < \infty$$

It *converges unconditionally* if all rearrangements converge to the same sum. It *converges conditionally* if

$$y_n = \sum_{i=1}^{n} x_k$$

converges but not unconditionally.

Remark 73. If

$$L = \sum_{n=1}^{\infty} \|x_n\| < \infty$$

then for any $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that

$$\sum_{n=1}^{\infty} \|x_n\| > L - \varepsilon$$

So

$$s_n = \sum_{i=1}^{n} x_i$$

are Cauchy, since for $n, m \geq N$ we have

$$\|s_n - s_m\| \leq \sum_{i=m+1}^{n} \|x_i\| < \varepsilon$$

If

$$\sum_{i=1}^{\infty} x_{\pi(i)}$$

is a rearrangement, then there is M such that $\{ 1, \dots, N \} \subseteq \{ \pi(1), \dots, \pi(M) \}$. To look at all rearrangements at once, let $\Lambda = \{ F \subseteq_{\text{fin}} \mathbb{N} \}$ where $F \leq G$ if $F \subseteq G$. Set

$$s_F = \sum_{n \in F} x_n$$

Then if $s_F \to x$, this means that every rearrangement converges to x. In our case, if $F, G \supseteq \{ 1, \dots, N \}$ then

$$\|s_F - s_G\| \leq \sum_{i \in F \triangle G} \|x_i\| < \varepsilon$$

so it is Cauchy, and thus converges. So absolutely convergent implies unconditionally convergent.

Theorem 74. *Suppose $\mathcal{H}$ is a Hilbert space with $\{ e_\alpha : \alpha \in I \}$ is an orthonormal set. Let*

$$\mathcal{M} = \overline{\operatorname{span}\{ e_\alpha : \alpha \in I \}}$$

Then

1. *$\{ e_\alpha : \alpha \in I \}$ is linearly independent. Moreover, we have*

$$\operatorname{dist}(e_\alpha, \operatorname{span}\{ e_\beta : \beta \neq \alpha \}) = 1$$

2. *Suppose $x \in \mathcal{H}$. Let $x_\alpha = \langle x, e_\alpha \rangle$. Then*

$$\sum_{\alpha \in I} |x_\alpha|^2 \leq \|x\|^2$$

(This is the Bessel inequality.)

3. *If*

$$\sum_{\alpha \in I} |x_\alpha|^2 < \infty$$

then

$$\sum_{\alpha \in I} x_\alpha e_\alpha$$

converges unconditionally.

4. $$Px = \sum_{\alpha \in I} x_\alpha e_\alpha$$

is a continuous linear map $\mathcal{H} \to \mathcal{M}$ with $P^2 = P$ and $\|P\| = 1$.

5. *$Px = 0$ if and only if $x \perp \mathcal{M}$.*

6. *If $x \in \mathcal{M}$, then*

$$\|x\| = \left(\sum_{\alpha \in I} |x_\alpha|^2 \right)^{\frac{1}{2}}$$

Proof.

1. $$\operatorname{dist}(e_\alpha, \operatorname{span}\{\, e_\beta : \beta \neq \alpha \,\}) = \inf \|e_\alpha - \sum_{\text{finite}} y_\beta e_\beta\| = \inf \Big(1 + \sum |y_\beta|^2 \Big)^{\frac{1}{2}} = 1$$

2. Let $F \subseteq_{\text{fin}} I$, with

$$s_F = \sum_{\alpha \in F} x_\alpha e_\alpha$$

Then

$$\begin{aligned} 0 &\leq \|x - s_F\|^2 \\ &= \langle x, x \rangle - 2 \operatorname{Re} \langle x, s_F \rangle + \langle s_F, s_F \rangle \\ &= \|x\|^2 - 2 \operatorname{Re} \left(\sum_{\alpha \in F} \langle x, x_\alpha e_\alpha \rangle \right) + \left\langle \sum_{\alpha \in F} x_\alpha e_\alpha, \sum_{\alpha \in F} x_\alpha e_\alpha \right\rangle \\ &= \|x\|^2 - 2 \operatorname{Re} \sum_{\alpha \in F} \overline{x_\alpha} \langle x, e_\alpha \rangle + \sum_{\alpha \in F} x_\alpha \overline{x_\alpha} \\ &= \|x\|^2 - \sum_{\alpha \in F} |x_\alpha|^2 \end{aligned}$$

So

$$\sum_{\alpha \in F} |x_\alpha|^2 \leq \|x\|^2$$

and

$$\sum_{\alpha \in I} |x_\alpha|^2 = \sup_{F \subseteq_{\text{fin}} I} \sum_{\alpha \in F} |x_\alpha|^2 \leq \|x\|^2$$

Note that

$$\{\, \alpha : x_\alpha \neq 0 \,\} = \bigcup_{n \geq 1} \left\{ \alpha : |x_\alpha| \geq \frac{1}{n} \right\}$$

is therefore countable, since each unionand has cardinality

$$\leq \frac{\|x\|^2}{\frac{1}{n^2}} = n^2 \|x\|^2 < \infty$$

3. Suppose

$$L = \sum |x_\alpha|^2 < \infty$$

For $F \subseteq_{\text{fin}} I$, let

$$s_F = \sum_{\alpha \in F} x_\alpha e_\alpha$$

Then

$$\sup_{F \subseteq_{\text{fin}} I} \|s_F\|^2 = \sup_{F \subseteq_{\text{fin}} I} \sum_{\alpha \in F} |x_\alpha|^2 = \sum |x_\alpha|^2 = L$$

Pick F_0 such that

$$\sum_{\alpha \in F_0} |x_\alpha|^2 > L - \varepsilon$$

If $F, G \supseteq F_0$ then

$$\begin{aligned}
\|s_F - s_G\|^2 &= \left\| \sum_{\alpha \in F \setminus G} x_\alpha e_\alpha - \sum_{\alpha \in G \setminus F} x_\alpha e_\alpha \right\|^2 \\
&= \sum_{\alpha \in F \triangle G} |x_\alpha|^2 \\
&\leq \sum_{\alpha \in F \cup G} |x_\alpha|^2 - \sum_{\alpha \in F_0} |x_\alpha|^2 \\
&< L - (L - \varepsilon) \\
&= \varepsilon
\end{aligned}$$

So $\{\, s_F \,\}$ is Cauchy, and thus converges unconditionally.

Example 75.

$$\sum_{n=1}^{\infty} \frac{1}{n} e_n$$

converges unconditionally since

$$\left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{\frac{1}{2}} < \infty$$

but not absolutely since

$$\sum_{n=1}^{\infty} \left\| \frac{1}{n} e_n \right\| = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

4. By (2) and (3), if $x \in \mathcal{H}$, then $(x_\alpha : \alpha \in I)$ is square-summable and

$$Px = \sum_{\alpha} x_\alpha e_\alpha$$

is well-defined, and further that

$$\|Px\|^2 = \sum |x_\alpha|^2 \leq \|x\|^2$$

So $\|P\| \leq 1$ and P is linear; so P is continuous. Let $y = Px \in \overline{\operatorname{span}\{ e_\alpha : \alpha \in I \}} = \mathcal{M}$. Then

$$\langle y, e_\alpha \rangle = \lim_F \langle s_F, e_\alpha \rangle = \lim_{F \supseteq \{ \alpha \}} \left\langle x_\alpha e_\alpha + \sum_{\beta \in F \setminus \{ a \}} x_\beta e_\beta, e_\alpha \right\rangle = x_\alpha$$

Thus

$$Py = \sum x_\alpha e_\alpha = y$$

i.e. $P(Px) = Px$ and $P^2 = P$. If $y \in \operatorname{span}\{ e_\alpha : \alpha \in I \}$, say

$$y = \sum_{\alpha \in F} y_\alpha e_\alpha$$

then $y_\beta = 0$ for $\beta \notin F$, and

$$s_G = \sum_{\alpha \in G} y_\alpha e_\alpha = \sum_{\alpha \in G \cap F} y_\alpha e_\alpha = y$$

if $G \supseteq F$.

5.

$$\begin{aligned} Px = 0 &\iff \langle x, e_\alpha \rangle = 0 \text{ for all } \alpha \\ &\iff x \perp \sum_{\alpha \in F} a_\alpha e_\alpha \text{ for all } F \subseteq_{\text{fin}} I, \text{ all } (a_\alpha : \alpha \in F) \\ &\iff x \perp \mathcal{M} \end{aligned}$$

by continuity. So $\ker(P) = \mathcal{M}^\perp$.
If $x \in \mathcal{H}$ with $y = Px \in \mathcal{M}$, then $x - y = (I - P)x$, and

$$\langle x - y, e_\alpha \rangle = \langle x, e_\alpha \rangle - \langle y, e_\alpha \rangle = 0$$

for all α. So $x - y \perp \mathcal{M}$. But $x = y + (x - y)$, and

$$\|x\|^2 \geq \|y\|^2 + \|x - y\|^2$$

$\mathcal{M}^\perp$ is the *orthogonal complement of* $\mathcal{M}$. We call P the *orthogonal projection* of $\mathcal{H}$ onto $\mathcal{M}$.

6. If $y \in \mathcal{M}$,

$$s_F = \sum_{\alpha \in F} y_\alpha e_\alpha$$

then

$$\|s_F\|^2 = \sum_{\alpha \in F} |y_\alpha|^2$$

Then $s_F \to y$, so

$$\|y\|^2 = \lim \sum_{\alpha \in F} |y_\alpha|^2 = \sum |y_\alpha|^2$$

□ Theorem 74

Definition 76. An *orthonormal basis* for a Hilbert space $\mathcal{H}$ is an orthonormal set $\{ e_\alpha : \alpha \in I \}$ such that $\mathcal{H} = \overline{\operatorname{span}\{ e_\alpha : \alpha \in I \}}$.

Theorem 77. *Every Hilbert space has an orthonormal basis.*

Proof. Order all orthonormal sets by inclusion. Suppose

$$\mathcal{C} = \{ \mathcal{E}_\beta \}$$

with $\beta_1 < \beta_2 \implies \mathcal{E}_{\beta_1} \subseteq \mathcal{E}_{\beta_2}$ is a chain. Then

$$\mathcal{E} = \bigcup_{\beta \in \mathcal{C}} \mathcal{E}_\beta$$

is a set of vectors. Suppose $e, f \in \mathcal{E}$, say $e \in \mathcal{E}_{\beta_1}$, $f \in \mathcal{E}_{\beta_2}$. Say $\beta_1 \leq \beta_2$; then $e, f \in \mathcal{E}_{\beta_2}$, and $\langle e, f \rangle = 0$. So $\mathcal{E}$ is an orthonormal basis, and is an upper bound of $\mathcal{C}$. By Zorn's lemma, we have that $\mathcal{H}$ has a maximal orthonormal set $\mathcal{E} = \{ e_\alpha : \alpha \in I \}$. Let $\mathcal{M} = \overline{\operatorname{span}\{ e_\alpha : \alpha \in I \}}$.

Claim 78. $\mathcal{M} = \mathcal{H}$.

Proof. Suppose otherwise; suppose we have $x \notin \mathcal{M}$. Let $y = Px \in \mathcal{M}$; let $z = (I - P)x \in \mathcal{M}^\perp$; then $z \neq 0$, and

$$x = y + z$$

Let

$$e = \frac{z}{\|z\|}$$

Then $\mathcal{E} \cup \{ e \}$ is orthonormal, contradicting maximality of $\mathcal{E}$. So $\mathcal{M} = \mathcal{H}$. □ Claim 78

□ Theorem 77

Corollary 79. *Every closed subspace $\mathcal{M}$ of a Hilbert space $\mathcal{H}$ is the range of an orthogonal projection.*

Proof. $\mathcal{M}$ is a Hilbert space so there is an orthonormal basis $\{ e_\alpha : \alpha \in I \}$ for $\mathcal{M}$. Define

$$Px = \sum \langle x, e_\alpha \rangle e_\alpha$$

as before. □ Corollary 79

Theorem 80. *If $\mathcal{H}$ is a Hilbert space and $\varphi \in \mathcal{H}^*$ (i.e. φ is a continuous linear functional), then there is a unique $y \in \mathcal{H}$ such that $\varphi(x) = \langle x, y \rangle$ and $\|y\| = \|\varphi\|$.*

Proof. Let $\{ e_\alpha : \alpha \in I \}$ be an orthonormal basis for $\mathcal{H}$. Define $a_\alpha = \varphi(e_\alpha)$. For $F \subseteq_{\text{fin}} I$, look at

$$\varphi\left(\sum_{\alpha \in F} \overline{a_\alpha} e_\alpha\right) = \sum_{\alpha \in F} \overline{a_\alpha} \varphi(e_\alpha) = \sum_{\alpha \in F} |a_\alpha|^2 \leq \|\varphi\| \left\| \sum_{\alpha \in F} \overline{a_\alpha} e_\alpha \right\| = \|\varphi\| \left(\sum_{\alpha \in F} |a_\alpha|^2 \right)^{\frac{1}{2}}$$

So

$$\left(\sum_{\alpha \in F} |a_\alpha|^2 \right)^{\frac{1}{2}} \leq \|\varphi\|$$

and

$$\sup_{F \subseteq_{\text{fin}} I} \left(\sum_{\alpha \in F} |a_\alpha|^2 \right)^{\frac{1}{2}} \leq \|\varphi\|$$

Define

$$y = \sum \overline{a_\alpha} e_\alpha$$

Then

$$\|y\| = \left(\sum |a_\alpha|^2 \right)^{\frac{1}{2}} \leq \|\varphi\|$$

For $x \in \mathcal{H}$, write

$$x = \sum x_\alpha e_\alpha$$

Then

$$\sum |x_\alpha|^2 = \|x\|^2 < \infty$$

Then

$$\begin{aligned}
\langle x, y\rangle &= \left\langle \sum x_\alpha e_\alpha, \sum y_\beta e_\beta \right\rangle \\
&= \lim_F \left\langle \sum_{\alpha\in F} x_\alpha e_\alpha, \sum y_\beta e_\beta \right\rangle \\
&= \lim_F \sum_{\alpha\in F} x_\alpha \overline{y_\alpha} \\
&= \sum x_\alpha \overline{y_\alpha} \text{ by Cauchy-Schwarz} \\
&= \sum a_\alpha x_\alpha
\end{aligned}$$

But

$$\varphi(x) = \varphi\left(\sum x_\alpha e_\alpha\right) = \sum x_\alpha \varphi(e_\alpha) = \sum a_\alpha x_\alpha$$

So $\langle x, y\rangle = \varphi(x)$. Also

$$\|\varphi\| = \sup_{\|x\|\le 1} |\varphi(x)| = \sup_{\|x\|\le 1} |\langle x, y\rangle| \le \sup_{\|x\|\le 1} \|x\|\|y\| = \|y\|$$

by Cauchy-Schwarz. So $\|y\| = \|\varphi\|$. $\square$ Theorem 80

Remark 81. The map $\varphi \mapsto y$ is conjugate-linear. So $\mathcal{H}^*$ is anti-isomorphic to $\mathcal{H}$.

Definition 82. The *dimension* of $\mathcal{H}$ ($\dim(\mathcal{H})$) is the cardinality of an orthonormal basis.

Proposition 83. $\dim(\mathcal{H})$ *is well-defined.*

Proof. If $\dim(\mathcal{H}) < \infty$, then the cardinality of a basis is well-defined. So suppose $\mathcal{H}$ is infinite-dimensional. Suppose $\{\, e_\alpha : \alpha \in I \,\}$ and $\{\, f_\beta : \beta \in J \,\}$ are orthonormal bases. For all $\alpha \in I$, set

$$B_\alpha = \{\, \beta \in J : \langle e_\alpha, f_\beta\rangle \ne 0 \,\}$$

Then this is countable and non-empty because

$$1 = \|e_\alpha\|^2 = \sum_{\beta\in J} |\langle e_\alpha, f_\beta\rangle|^2$$

Conversely, for all β there is α such that $\langle e_\alpha, f_\beta\rangle \ne 0$ by the same reasoning. So

$$J = \bigcup_{\alpha\in I} B_\alpha$$

Thus

$$|J| \le \sum_{\alpha\in I} |B_\alpha| \le |I|\aleph_0 = |I|$$

Similarly, we have $|I| \le |J|$. So, by Cantor-Bernstein-Schroeder, we have $|I| = |J|$. $\square$ Proposition 83

Definition 84. A *unitary* is a linear map $U\colon H \to K$ of one Hilbert space *onto* another such that $\|Ux\| = \|x\|$.

Remark 85. This implies

$$\langle Ux, Uy\rangle = \langle x, y\rangle$$

for all $x, y \in \mathcal{H}$.

Proof. If $\mathbb{F} = \mathbb{R}$, then

$$\begin{aligned} \langle x \pm y, x \pm y \rangle &= \|x\|^2 \pm 2\langle x, y \rangle + \|y\|^2 \\ \implies \langle x, y \rangle &= \|x+y\|^2 - \|x-y\|^2 \end{aligned}$$

If $\mathbb{F} = \mathbb{C}$, then

$$\langle x, y \rangle = \frac{\|x+y\|^2 - \|x-y\|^2 + \|x+iy\|^2 - i\|x-iy\|^2}{4}$$

☐ Remark 85

Example 86. $L^2(\mathbb{T}, m)$ with

$$\|f\|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(\exp(i\theta))|^2 d\theta$$

So

$$e_n = \exp(in\theta)$$

are orthonormal. Then

$$\overline{\operatorname{span}\{\, e_n : n \in \mathbb{Z} \,\}} \supseteq \overline{\text{trig polynomials}} = \overline{C(\mathbb{T})} = L^2$$

Example 87. $\ell^2(\mathbb{Z})$ has orthonormal basis δ_n. $Uf = \hat{f}$.

$$f \sim \sum \hat{f} \exp(in\theta) \sim (\hat{f}(n) : n \in \mathbb{Z})$$

is a unitary map.

Definition 88. A subset A of a topological space X is *nowhere dense* if $\overline{A}$ has no interior. A subset B of a complete metric space is said to be *of first category* if it is the countable union of nowhere dense sets.

Theorem 89 (Baire category theorem)**.** *If X is a complete metric space and $B \subseteq X$ is of first category, then $X \setminus B$ is dense in X.*

Sketch. Let $U \subseteq X$ be open; suppose $x \in U$. Choose $r > 0$ such that $\overline{b_r(x)} \subseteq U$. It suffices to find $y \in \overline{b_r(x)}$ such that $y \notin B$. Write

$$B = \bigcup_{n=1}^{\infty} A_n$$

with $\overline{A_n}$ has no interior. Find $x_1 \in b_r(x)$ and $r_1 > 0$ such that $r_1 \leq \frac{r}{2}$ and

$$\overline{b_{r_1}(x_1)} \cap \overline{A_1} = \emptyset$$

Recursively find $x_{n+1} \in b_{r_n}(x_n)$ such that

$$\overline{b_{r_{n+1}}(x_{n+1})} \cap \overline{A_{n+1}} = \emptyset$$

and $r_{n+1} \leq \frac{r_n}{2}$. Then $(x_n : n \in \mathbb{N})$ are Cauchy, and thus converge to $x \in X$; then

$$x \in \bigcap_{n \in \mathbb{N}} \overline{b_r(x_n)}$$

and $x \notin B$. ☐ Theorem 89

Corollary 90. *If U_i are dense open subsets of a complete metric space, then*

$$\bigcap_{i=1}^{\infty} U_i$$

is dense.

Theorem 91 (Banach-Steinhaus, or uniform boundedness principle)**.** *Suppose X, Y are Banach spaces; suppose $\mathcal{A} \subseteq \mathcal{B}(X, Y)$. Suppose that for all $x \in X$ we have that*

$$\sup_{A \in \mathcal{A}} \|Ax\| = k_x < \infty$$

Then

$$\sup_{A \in \mathcal{A}} \|A\| < \infty$$

Proof. Let $B_n = \{ x \in X : k_x \leq n \}$.

Claim 92. *B_n is closed.*

Proof. Suppose $(x_k : k \in \mathbb{N})$ in B_n with $x_k \to x$; suppose $A \in \mathcal{A}$. Then

$$\|Ax\| = \lim_{k \to \infty} \|Ax_k\| \leq n$$

So $x \in B_n$, and B_n is closed. □ Claim 92

But

$$X = \bigcup_{n=1}^{\infty} B_n$$

By Baire category theorem, we then have that there is some n_0 such that B_{n_0} has interior; say $\overline{b_r(x_0)} \subseteq B_{n_0}$.

Now, if $x \in X$ with $\|x\| \leq 1$, then $x_0 + rx \in B_{n_0}$; but then

$$\begin{aligned} \|Ax\| &= \left\| \frac{A(x_0 + rx) - Ax_0}{r} \right\| \\ &\leq \frac{1}{r}(\|A(x_0 + rx)\| + \|Ax_0\|) \\ &\leq \frac{2n_0}{r} \end{aligned}$$

So

$$\sup_{\|x\| \leq 1} \sup_{A \in \mathcal{A}} \|Ax\| \leq \frac{2n_0}{r} < \infty$$

□ Theorem 91

Remark 93. We didn't use that Y is a Banach space. Given Y a normed linear space that's not complete, we could always embed Y in its metric closure, which turns out to be a Banach space, and then apply Banach-Steinhaus.

Corollary 94. *Suppose X, Y are Banach spaces; suppose $(T_n : n \in \mathbb{N})$ are in $\mathcal{B}(X, Y)$ such that*

$$\lim_{n \to \infty} T_n x$$

which we define to be Tx, exists for all $x \in X$. Then $T \in \mathcal{B}(X, Y)$.

Proof. Since $T_n x \to Tx$ for all $x \in X$, we have that

$$\sup_{n \geq 1} \|T_n x\| = k_x < \infty$$

By the uniform boundedness principle, we have

$$\sup_{n \geq 1} \|T_n\| = L < \infty$$

Thus

$$\|Tx\| = \lim_{n \to \infty} \|T_n x\| \leq L\|x\|$$

So T is continuous. Also, T is linear, since

$$\begin{aligned} T(ax+y) &= \lim_{n\to\infty} T_n(ax+y) \\ &= \lim_{n\to\infty} (aT_nx + T_ny) \\ &= aTx + Ty \end{aligned}$$

So $T \in \mathcal{B}(X,Y)$. $\square$ Corollary 94

Theorem 95 (Open mapping theorem)**.** *Suppose X, Y are Banach spaces. Suppose $T \in \mathcal{B}(X,Y)$ is surjective. Then T is open. (That is, for open $U \subseteq X$, we have TU is open.)*

Proof. We are given that

$$Y = TX = \bigcup_{n=1}^{\infty} T(b_n(X))$$

By the Baire category theorem, there is $n_0 \in \mathbb{N}$, $x_0 \in X$, and $r > 0$ such that

$$\overline{T(b_{n_0}(X))} \supseteq b_r(x_0)$$

Note then that

$$\begin{aligned} \overline{T(b_1(x))} &\supseteq \overline{T\Big(B_{\frac{1}{2}}(0)\Big)} - \overline{T\Big(B_{\frac{1}{2}}(0)\Big)} \\ &\supseteq b_{\frac{r}{2n_0}}\left(\frac{x_0}{2n_0}\right) - b_{\frac{r}{2n_0}}\left(\frac{x_0}{2n_0}\right) \\ &= b_{\frac{r}{n_0}}(0) \end{aligned}$$

and

$$\begin{aligned} \overline{T(b_1(x))} &= \frac{1}{n_0}\overline{T(b_{n_0}(x))} \\ &\supseteq b_{\frac{r}{n_0}}\left(\frac{x_0}{n_0}\right) \end{aligned}$$

So

$$\overline{T(b_1(X))} \supseteq b_\rho(0)$$

for some $\rho > 0$.

Claim 96. *If $\varepsilon > 0$, then*

$$T(b_{1+\varepsilon}(x)) \supseteq \overline{T(b_1(x))}$$

Proof. Fix $y \in \overline{Tb_1(x)}$. Pick $x_0 \in X$ with $\|x_0\| \leq 1$ and

$$\|Tx_0 - y\| < \frac{\varepsilon\rho}{2}$$

Let $y_0 = Tx_0$; then

$$\|y - y_0\| < \frac{\varepsilon}{2}\rho$$

So

$$\begin{aligned} y - y_0 &\in \overline{T(b_{\frac{\varepsilon}{2}})} \\ &= \frac{\varepsilon}{2}\overline{T(b_1(x))} \\ &\supseteq \frac{\varepsilon}{2} b_\rho(0) \\ &= b_{\frac{\varepsilon\rho}{2}}(0) \end{aligned}$$

Pick $x_1 \in X$ with
$$\|x_1\| < \frac{\varepsilon}{2}$$
such that
$$\|Tx_1 - (y - y_0)\| < \frac{\varepsilon\rho}{4}$$
And again let $y_1 = Tx_1$. Recursively select $x_{n+1} \in X$ with
$$\|x_{n+1}\| < \frac{\varepsilon}{2^{n+1}}$$
and
$$\|y_{n+1} - (y - y_0 - y_1 - \cdots - y_n)\| < \frac{\varepsilon\rho}{2^{n+2}}$$
where $y_{n+1} = Tx_{n+1}$. Let
$$x = \sum_{n=0}^{\infty} x_n$$
This converges because
$$\sum_{n=0}^{\infty}\|x_n\| < 1 + \sum_{n=1}^{\infty}\frac{\varepsilon}{2^n} = 1 + \varepsilon$$
and in particular we also get $\|x\| < 1 + \varepsilon$. Then
$$\begin{aligned} Tx &= \sum_{n=0}^{\infty} Tx_n \\ &= \sum_{n=0}^{\infty} y_n \\ &= \lim_{n\to\infty} y_0 + \cdots + y_n \\ &= y \end{aligned}$$

□ Claim 96

So
$$T(B_{1+\varepsilon}(x)) \supseteq \overline{T(b_1(x))} \supseteq b_\rho(0)$$
So
$$T(b_1(x)) \supseteq b_{\frac{\rho}{1+\varepsilon}}(0)$$
Let $\varepsilon \to 0$. Then
$$T(b_1(x)) \supseteq b_\rho(0)$$
Let U be open; suppose $x \in U$. Then there is $r > 0$ such that $b_r(x) \subseteq U$. Let $y = Tx$. Then
$$\begin{aligned} T(U) &\supseteq T(b_r(x)) \\ &= Tx + T(b_r(0)) \\ &\supseteq Tx + b_{\frac{r}{\rho}}(0) \\ &= b_{\frac{r}{\rho}}(0) \end{aligned}$$
So TU is open. □ Theorem 95

Theorem 97 (Banach isomorphism theorem)**.** *If X, Y are Banach spaces and $T\colon X \to Y$ is a continuous linear bijection then T is an isomorphism. (i.e. T^{-1} is also continuous.)*

Proof. T is surjective, so it is open by the open mapping theorem. T is injective, so T^{-1} is well-defined and linear. If $U \subseteq X$ is open, then $(T^{-1})^{-1}(U) = T(U)$ is open. So T^{-1} is continuous. □ Theorem 97

Corollary 98. *Suppose X, Y are Banach spaces. Suppose $T \in \mathcal{B}(X, Y)$ is surjective. Then we have the following commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "T"] \arrow[dr, "q"'] & & Y \\ & X/\ker T \arrow[ur, "\dot{T}"'] & \end{tikzcd}$$

and in particular we have $\dot{T}$ is an isomorphism $X/\ker(T) \to Y$.

Proof. T is continuous so $\ker(T)$ is a closed subspace. So $X/\ker(T)$ is a Banach space. Define $\dot{T}(\dot{x}) = Tx$; this is well-defined since if $x_1, x_2 \in \dot{x}$, then $x_1 - x_2 \in \ker(T)$, and

$$\begin{aligned} Tx_2 &= Tx_1 + T(x_2 - x_1) \\ &= Tx_1 + 0 \\ &= Tx_1 \end{aligned}$$

Also

$$\begin{aligned} \|\dot{T}\| &= \sup_{\|\dot{x}\| \leq 1} \|\dot{T}\dot{x}\| \\ &= \sup_{\inf_{m \in \ker(T)} \|x+m\| \leq 1} \|Tx\| \\ &= \sup_{\inf_{m \in \ker(T)} \|x+m\| \leq 1} \|T(x+m)\| \\ &\leq \sup_{\inf_{m \in \ker(T)} \|x+m\| \leq 1} \|T\|\|x+m\| \end{aligned}$$

If $\varepsilon > 0$ then there is $x + m \in \dot{x}$ such that

$$\|x + m\| < (1 - \varepsilon) + \varepsilon = 1$$

So this yields $\|\dot{T}\| = \|T\|$. So $\dot{T}$ is continuous and bijective. By the Banach isomorphism theorem, we have that $\dot{T}$ is an isomorphism. □ Corollary 98

Corollary 99. *Suppose X is a Banach space with respect to two different norms $\|\cdot\|_1$ and $\|\cdot\|_2$. If there is a constant C such that*

$$\|x\|_2 \leq C\|x\|_1$$

for all $x \in X$, then there is C' such that

$$\|x_1\| \leq C'\|x\|_2$$

for all $x \in X$.

Proof. Hypothesis says that

$$\mathrm{id}_X \colon (X, \|\cdot\|_1) \to (X, \|\cdot\|_2)$$

is a continuous, linear bijection; so it is an isomorphism. So

$$\|x\|_1 = \|(\mathrm{id}_X)^{-1}x\|_1 \leq \|\mathrm{id}_X^{-1}\|\|x\|_2$$

□ Corollary 99

Corollary 100. *If X is a finite-dimensional vector space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, then any two norms on X are comparable. So the topology with respect to any norm is the usual metric topology in $\mathbb{F}^n$.*

Proof. Let $\|\cdot\|_1$ be some norm on $X = \mathbb{F}^n$. Fix a basis $e_1, \ldots, e_n$. Define the usual norm on X by

$$\left\| \sum_{i=1}^{n} x_i e_i \right\|_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{\frac{1}{2}}$$

Then

$$\begin{aligned}\left\|\sum_{i=1}^{n} x_i e_i\right\|_1 &\leq \sum_{i=1}^{n} |x_i| \|e_i\|_1 \\ &\leq \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} \|e_i\|_1^2\right)^{\frac{1}{2}} \text{ by Cauchy-Schwarz} \\ &= C\left\|\sum_{i=1}^{n} x_i e_i\right\|_2\end{aligned}$$

so $\mathrm{id}\colon (X, \|\cdot\|_2) \to (X, \|\cdot\|_1)$ is a continuous bijection; so it is an isomorphism.

Alternative proof: let $S = \{\, x \in X : \|x\|_2 = 1 \,\}$; then this is compact. So $\mathrm{id}(S)$ is compact in $(X, \|\cdot\|_1)$. But $0 \notin S$; so

$$\inf_{x \in S} \|x\|_1 = r > 0$$

So $\|x\|_1 \geq r\|x\|_2$, and

$$\|x\|_2 \leq \frac{1}{r}\|x\|_1$$

□ Corollary 100

Definition 101. If $T\colon M \subseteq X \to Y$ is linear, the *graph* of T is

$$\mathcal{G}(T) = \{\, (x, Tx) \in X \oplus Y \,\}$$

(Note that $X \oplus Y$ is a Banach space with norm $\|(x, y)\| = \|x\| + \|y\|$ (or $(\|x\| + \|y\|)^{\frac{1}{2}}$, which produces an equivalent norm).) T is called *closed* if $\mathcal{G}(T)$ is closed.

Theorem 102 (Closed graph theorem)**.** *If $T\colon X \to Y$ is linear (and defined on all of X) and T is closed, then T is continuous.*

Proof. We have the following commutative diagram

$$\begin{tikzcd} X \arrow[rr, "T"] & & Y \\ & \mathcal{G}(T) \arrow[ul, "\pi_1"] \arrow[ur, "\pi_2"'] & \end{tikzcd}$$

where π_1 and π_2 are both continuous. So π_1 is injective and surjective to X; so π_1^{-1} is continuous. So $T = \pi_2 \circ \pi_1^{-1}$. □ Theorem 102

Corollary 103. *Suppose $T\colon X \to Y$ is linear; suppose that whenever $(x_n : n \in \mathbb{N}) \to 0$ and $(Tx_n : n \in \mathbb{N})$ converges, we have that $(Tx_n : n \in \mathbb{N}) \to 0$.*

Proof. Suppose $((x_n, Tx_n) : n \in \mathbb{N})$ is in $\mathcal{G}(T)$ and converges to $(x_0, y_0) \in X \oplus Y$. Then $(x_n - x_0 : n \in \mathbb{N}) \to 0$. So

$$T(x_n - x_0) = Tx_n - Tx_0 \to y_0 - Tx_0$$

By hypothesis, we have $y_0 - Tx_0 = 0$, and $y = Tx_0$. So $\mathcal{G}(T)$ is closed. □ Corollary 103

Example 104. Suppose $\mathcal{H}$ is a HIlbert space; suppose $T\colon \mathcal{H} \to \mathcal{H}$ is linear and $\langle Tx, y\rangle = \langle xTy\rangle$ for all $x, y \in \mathcal{H}$.

Claim 105. *T is continuous.*

Proof. Suppose $x_n \to 0$; suppose $Tx_n \to y$. Then

$$\|y\|^2 = \langle y, y\rangle = \lim\langle Tx_n, y\rangle = \lim\langle x_n, Ty\rangle = 0$$

since the $x_n \to 0$. So $y = 0$. By closed graph theorem, we have that T is continuous. □ Claim 105

Example 106. $\mathcal{H} = \ell_2$. Let

$$D((x_n)) = \left(\frac{x_n}{n}\right)_{n\geq 1}$$

$$D = \begin{pmatrix} 1 & & 0 \\ & \frac{1}{2} & \\ & & \frac{1}{3} \\ 0 & & \end{pmatrix}$$

is continuous with $\|D\| = 1$. But $\operatorname{Ran}(D)$ is not closed as

$$\left(1, \frac{1}{2}, \frac{1}{3}, \dots\right) \notin \operatorname{Ran}(D)$$

D is injective and $D \colon \ell_2 \to \operatorname{Ran}(D)$ is bijective. Then

$$\mathcal{G}(D^{-1}) = \{\, (Dx, x) : x \in \ell_2 \,\}$$

is closed because D is continuous. So D^{-1} is closed but not continuous, as $D^{-1}(e_n) = ne_n$ is unbounded.

3.3 Some Fourier series

Definition 107. If $f \in L^1(\mathbb{T})$, define the *Fourier coefficients*

$$\widehat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\exp(-in\theta)d\theta$$

for $n \in \mathbb{Z}$. For $N \geq 0$, define

$$S_N(f) = \sum_{k=-N}^{N} \widehat{f}(k)\exp(ik\theta)$$

Remark 108. The functional $\varphi_n(f) = \widehat{f}(n)$ is continuous on $L^1(\mathbb{T})$, and hence is continuous on $C(\mathbb{T})$ and on $L^p(\mathbb{T})$ for $1 < p \leq \infty$. So the S_N are also continuous on the above.

Recall also that the trigonometric polynomials

$$\left\{ p(\theta) = \sum_{k=-N}^{N} a_k \exp(ik\theta), a_k \in \mathbb{C}, N \geq 1 \right\}$$

are dense in $C(\mathbb{T})$ by the Weierstrass theorem. We also have $C(\mathbb{T})$ is dense in $L^p(\mathbb{T})$ if $1 \leq p < \infty$ by Lusin's theorem. So the trigonometric polynomials are dense in $L^p(\mathbb{T})$ if $1 \leq p < \infty$. (We also have that $f \in L^\infty$ is a bounded pointwise limit of continuous functions.)

Perhaps there is hope, then, that $S_N(f) \to f$ in L^p or $C(\mathbb{T})$: if

$$\|f - p\| < \varepsilon$$

then

$$\|S_N(f) - S_N(p)\| \leq \|S_N\|\|f - p\| < \varepsilon\|S_N\|$$

So

$$\|S_N(f) - f\| \leq \|S_N(f) - S_N(p)\| + \|S_N(p) - p\| + \|p - f\| \leq (\|S_N\| + 1)\|f - p\|$$

for $N \geq \deg(p)$. The problem is that the $\|S_N\|$ could blow up.

Good news: in $L^2(\mathbb{T})$, we have that $\{\exp(in\theta) : n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{T})$, and

$$\widehat{f}(n) = \langle f, \exp(in\theta)\rangle$$

So S_N is the orthogonal projection onto

$$\operatorname{span}\exp(in\theta) : -N \leq n \leq N$$

So $\|S_N\| = 1$ for all $N \geq 0$, and $S_N(f) \to f$ for all $f \in L^2(\mathbb{T})$. If $1 < p < \infty$, then

$$\sup_{N \geq 1} \|S_n\|_{\mathcal{B}(L^p)} < \infty$$

So $S_N(f) \to f$ in L^p.

Not so nice in $L^1(\mathbb{T})$ or $C(\mathbb{T})$. Note, however, that

$$\begin{aligned}
S_N(f)(\theta) &= \sum_{n=-N}^{N} \widehat{f}(k) \exp(ik\theta) \\
&= \sum_{n=-N}^{N} \frac{1}{2\pi} \int_0^{2\pi} f(t) \exp(-ikt) dt \exp(ik\theta) \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(t) \left(\sum_{k=-N}^{N} \exp(ik(\theta - t) \right) dt \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(t) D_N(\theta - t) dt
\end{aligned}$$

where

$$D_N(x) = \sum_{k=-N}^{N} \exp(ikx)$$

We estimate $\|D_N\|_1$:

$$\begin{aligned}
D_N(\theta) &= \sum_{k=-N}^{N} \exp(ik\theta) \\
&= \frac{\exp(i(N+1)\theta) - \exp(-iN\theta)}{\exp(i\theta - 1)} \\
&= \frac{\exp\big(i\big(N+\frac{1}{2}\big)\theta\big) - \exp\big(-i\big(N+\frac{1}{2}\big)\theta\big)}{2i} \frac{2i}{\exp\big(i\frac{\theta}{2}\big) - \exp\big(-i\frac{\theta}{2}\big)} \\
&= \frac{\sin\big(\big(N+\frac{1}{2}\big)\theta\big)}{\sin\big(\frac{1}{2}\theta\big)}
\end{aligned}$$

So $D_N(0) = 2N + 1$; so $\|D_N\|_\infty \to \infty$ as $N \to \infty$. Also

$$\begin{aligned}
\|D_N\|_1 &= \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{\sin\big(\big(N+\frac{1}{2}\big)\theta\big)}{\sin\big(\frac{\theta}{2}\big)} \right| d\theta \\
&\geq \frac{1}{\pi} \int_0^{\pi} \left| \frac{\sin\big(\big(N+\frac{1}{2}\big)\theta\big)}{\frac{\theta}{2}} \right| d\theta \\
&= \frac{2}{\pi} \int_0^{(N+\frac{1}{2})\pi} \frac{|\sin(x)|}{x} dx \text{ (using the substitution } x = \left(N + \frac{1}{2}\right)\theta) \\
&\geq \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{\sin(x)}{x} dx + \sum_{k=1}^{2N} \frac{2}{\pi} \int_{k\frac{\pi}{2}}^{(k+1)\frac{\pi}{2}} \frac{|\sin(x)|}{(k+1)\frac{\pi}{2}} dx \\
&\geq \frac{2}{\pi} \frac{\pi}{4} + \sum_{k=1}^{2N} \frac{2}{\pi} \frac{2}{\pi} \frac{1}{k+1} \int_0^{\frac{\pi}{2}} \sin(x) dx \\
&= \frac{1}{2} + \frac{4}{\pi^2} \sum_{k=2}^{2N+1} \frac{1}{k} \\
&\approx \frac{4}{\pi^2} \log(N) \\
&\to \infty
\end{aligned}$$

Theorem 109. *For $\theta_0 \in [0, 2\pi]$, we have that*

$$\{ f \in C(\mathbb{T}) : S_N(f)(\theta_0) \to f(\theta_0) \}$$

is of first category.

Proof. If $S_N(f)(\theta_0) \to f(\theta_0)$, then

$$\{ S_N(f)(\theta_0) : N \geq 1 \}$$

is bounded. Consider the functional

$$\begin{aligned} \psi_N(f) &= S_N(f)(\theta_0) \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(t) D_N(\theta_0 - t) dt \end{aligned}$$

We can pick $f_\varepsilon \in C(\mathbb{T})$ such that

$$f_\varepsilon(t) = \begin{cases} 1 & D_N(\theta_0 - t) \geq \varepsilon \\ -1 & D_N(\theta_0 - t) \leq -\varepsilon \\ \text{piecewise linear} & \text{else} \end{cases}$$

But

$$|\psi_N(f_\varepsilon)| \to \|D_N\| \approx \frac{4}{\pi^2} \log(N) \to \infty$$

But $\|f_\varepsilon\|_\infty = 1$. So

$$\sup_{N \geq 1} \|\psi_N\| = \infty$$

By the uniform boundedness principle, there is $f \in C(\mathbb{T})$ such that $|\psi_N(f)| \to \infty$. In fact, from the proof, we have that

$$\{ f : S_N(f)(\theta_0) \text{ is bounded} \}$$

is of first category. □ Theorem 109

Corollary 110.

$$\{ f \in C(\mathbb{T}) : S_N(f)(\theta) \text{ is bounded for some } \theta \in \mathbb{Q} \cap [0, 2\pi] \}$$

is of first category.

Theorem 111 (Carleson, 1962)**.** *If $f \in L^p(\mathbb{T})$ for $p > 1$ (which then contains $C(\mathbb{T})$) then $S_N(f) \to f$ almost everywhere.*

Proposition 112 (Kolmogorov, before 1960)**.** *There is $f \in L^1(\mathbb{T})$ such that $S_N(f)$ diverges almost everywhere.*

Theorem 113. *The map*

$$\begin{aligned} \Lambda \colon L^1(\mathbb{T}) &\to c_0(\mathbb{Z}) \\ f &\mapsto \widehat{f} \end{aligned}$$

is injective and bounded but not surjective.

Proof. Well,

$$\begin{aligned} \left| \widehat{f}(n) \right| &= \left| \frac{1}{2\pi} \int_0^{2\pi} f(\exp(i\theta)) \exp(-in\theta) d\theta \right| \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} |f(\exp(i\theta))| d\theta \\ &= \|f\|_1 \end{aligned}$$

So $\|\Lambda f\| = \sup |\widehat{f}(n)| \leq \|f\|_1$.

Lemma 114 (Riemann-Lebesgue lemma)**.** *If $f \in L^1(\mathbb{T})$, then $|\widehat{f}(n)| \to 0$ as $n \to \infty$.*

Proof. If $\varepsilon > 0$, pick p a trigonometric polynomial such that $\|f - p\|_1 < \varepsilon$. Then

$$\begin{aligned} \left|\widehat{f}(n)\right| &= \left|\widehat{(f-p)}(n) + \widehat{p}(n)\right| \\ &= \left|\widehat{(f-p)}(n)\right| \\ &\le \|f-p\|_1 \\ &< \varepsilon \end{aligned}$$

for $|n| > \deg(p)$. ☐ Lemma 114

So $\Lambda f \in c_0(\mathbb{Z})$.

Claim 115. *Λ is injective.*

Proof. Suppose $f \in L^1(\mathbb{T})$; suppose $\Lambda f = 0$. Then

$$\frac{1}{2\pi}\int_0^{2\pi} f(\theta)\exp(-in\theta d\theta = 0$$

for all $n \in \mathbb{Z}$. So

$$\frac{1}{2\pi}\int_0^{2\pi} f(\theta)p(\theta)d\theta = 0$$

for all trigonometric polynomials. But for $g \in C(\mathbb{T})$, there exist trigonometric polynomials $p_n \to g$ uniformly with $\|p_n\|_\infty \le \|g\|_\infty$. So

$$0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)p_n(\theta)d\theta \to \frac{1}{2\pi}\int_0^{2\pi} f(\theta)g(\theta)d\theta$$

by Lebesgue dominated convergence theorem. Find bounded g_n with

$$g_n \to \frac{\overline{f(\theta)}}{|f(\theta)|}$$

almost everywhere. Then

$$0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)g_n(\theta)d\theta \to \frac{1}{2\pi}\int_0^{2\pi} |f| d\theta$$

So $f = 0$ almost everywhere. ☐ Claim 115

Claim 116. *Λ is not surjective.*

Proof. If Λ were surjective, then by the Banach isomorphism theorem Λ would be an isomorphism. So for all $f \in L^1(\mathbb{T})$ we would have

$$\|f\|_1 \le C\|\Lambda f\|_\infty$$

But $\|D_N\|_1 \approx \frac{4}{\pi^2}\log(N)$ and $\|\Lambda D_n\|_\infty = 1$, contradicting the above. ☐ Claim 116

☐ Theorem 113

We can do better using the Cesàro means (via Fejér's theorem) or

$$f(r\exp(i\theta)) = \sum_{k=-\infty}^{\infty} r^{|k|}\widehat{f}(k)\exp(ik\theta)$$

3.4 Hahn-Banach theorems

Definition 117. Suppose X is a vector space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Suppose $f\colon X \to \mathbb{F}$ linear is a functional. A function $p\colon X \to \mathbb{R}$ is *sublinear* if

1. $p(x+y) \leq p(x) + p(y)$
2. $p(tx) = tp(x)$ if $t \geq 0$

Example 118.

1. If X is normed, then $p(x) = \|x\|$ is sublinear.
2. If X has a topology and $U \in \mathcal{O}(0)$, then
$$\bigcup_{k\geq 1} kU = X$$
If we further have that U is convex, we can define the *Minkowski functional* by
$$p_U(x) = \inf\{t > 0 : x \in tU\}$$
It is easily seen that $p_U(sx) = sp_U(x)$ for $s > 0$. If $p_U(x) = s$ and $p_U(y) = t$, then $x \in s'U$ if $s' > s$ and $y \in t'U$ if $t' > t$. So
$$\frac{1}{s'}x, \frac{1}{t'}y \in U$$
By convexity of U, we then have that
$$\frac{x+y}{s'+t'} = \frac{s'}{s'+t'}\frac{1}{s'}x + \frac{t'}{s'+t'}\frac{1}{t'}y \in U$$
So $x + y \in (s'+t')U$, and $p_U(x+y) \leq s' + t'$. Letting $s' \to s$, $t' \to t$, sublinearity falls out.

Theorem 119. *Suppose M_0 is a real vector subspace of X (where X is a real vector space). Suppose p is a sublinear functional on X. Suppose $f_0\colon M_0 \to \mathbb{R}$ is a linear functional. Suppose $f_0(m) \leq p(m)$ for all $m \in M_0$. Then there is $f\colon X \to \mathbb{R}$ linear such that*

1. $f \restriction M_0 = f_0$
2. $f(x) \leq p(x)$ *for all* $x \in X$.

Proof. Extending by 1 dimension, if $M_0 \neq X$, pick $x \in X \setminus M_0$. Let $M = M_0 + \mathbb{R}x$; try to extend the definition of f_0 to $f\colon M \to \mathbb{R}$. In order to set $f(x) = a \in \mathbb{R}$, we need
$$f(m+tx) \leq p(m+tx)$$
If $t > 0$, we get $f(m) + tf(x) \leq p(m+tx)$; if $t < 0$ we get $f(m) - |t|f(x) \leq p(m - |t|x)$.

Case 1. Suppose $t > 0$. Then we need
$$tf(x) \leq p(m+tx) - f(m)$$
$$a = f(x) \leq \frac{p(m+tx) - f(m)}{t}$$

Case 2. Suppose $t < 0$. Then we need
$$f(m) - p(m - |t|x) \leq |t|f(x) = |t|a$$
so
$$\frac{f(m) - p(m-|t|x)}{|t|} \leq a$$

Conversely, if we can find a satisfying the above, then we define $f(m+tx) = f(m) + ta$ to get the desired extension. We then need

$$\sup_{s \geq 0, m \in M_0} \frac{f(m) - p(m - sx)}{s} \leq a \leq \inf_{t \geq 0, m \in M_0} \frac{p(m+tx) - f(m)}{t}$$

If $m' = \frac{m}{s}$, then

$$\begin{aligned} \text{LHS} &= \sup_{m' \in M_0} (f(m') - p(m' - x)) \\ \text{RHS} &= \inf_{m' \in M_0} p(m' + x) - f(m') \end{aligned}$$

Claim 120. LHS $\leq$ RHS.

Proof. Otherwise there is $m_1, m_2 \in M$ such that

$$p(m_2 + x) - f(m_2) < f(m_1) - p(m_1 - x)$$

and

$$p(m_1 + m_2) \leq p(m_2 + x) + p(m_1 - x) < f(m_1 + m_2) \leq p(m_1 + m_2)$$

a contradiction. □ Claim 120

So we can extend (f_0, M_0) to (f, M) by choosing any $a \in [\text{LHS}, \text{RHS}]$.
We now use Zorn's lemma. Consider

$$\mathcal{E} = \{ (M, f) : M_0 \subseteq M \text{ a subspace}, f \colon M \to \mathbb{R} \text{ linear}, f \restriction M_0 = f_0, f(x) \leq p(x) \text{ for all } x \in M \}$$

We can equip this with the partial order $(M_1, f_1) \leq (M_2, f_2)$ if $M_1 \subseteq M_2$ and $f_2 \restriction M_1 = f_1$. Suppose now that $\mathcal{C} = \{ (M_\alpha, f_\alpha) : \alpha \in I \}$ is a chain in $\mathcal{E}$ with I a total order and $\alpha < \beta$ in I implies $(M_\alpha, f_\alpha) \leq (M_\beta, f_\beta)$. Let

$$M = \bigcup_{\alpha \in I} M_\alpha$$

Then M is a vector space containing all of the M_α. Let

$$f = \bigcup_{\alpha \in I} f_\alpha$$

Then f is linear and $f \restriction M_\alpha = f_\alpha$. So if $x \in M$, then there is $\alpha \in I$ such that $x \in M_a$; then $f(x) = f_\alpha(x) \leq p(x)$. Also $M_0 \subseteq M_\alpha \subseteq M$, so $f \restriction M_0 = f_\alpha \restriction M_0 = f_0$. So $(M, f) \in \mathcal{E}$ is an upper bound of $\mathcal{C}$.

So, by Zorn's lemma, we have that $\mathcal{E}$ has a maximal element $(\widetilde{M}, \widetilde{f})$.

Claim 121. $\widetilde{M} = X$.

Proof. Otherwise there is $x \in X \setminus \widetilde{M}$. Let $M_1 = \widetilde{M} + \mathbb{R}x$. By the first part of the proof, we can extend $\widetilde{f}$ to f_1 on M_1 with $(\widetilde{M}, \widetilde{f}) < (M_1, f_1)$, contradicting maximality. □ Claim 121

□ Theorem 119

Theorem 122 (Hahn-Banach theorem)**.** *Suppose X is a Banach space, $M_0 \subseteq X$ is a subspace (not neceessarily closed). Suppose $f_0 \in M_0^*$ is a bounded linear functional on M_0. Then there is $f \in X^*$ such that $f \restriction M_0 = f_0$ and $\|f\| = \|f_0\|$.*

Proof.

Case 1. Suppose $\mathbb{F} = \mathbb{R}$. Define $p(x) = \|f_0\|\|x\|$; thus is positive-homogeneous and satisfies the triangle inequality, so is sublinear. Also $f_0(m) \leq \|f_0\|\|m\| = p(m)$ for all $m \in M_0$. So, by the previous theorem, there is a linear functional $f \in X^*$ such that $f \restriction M_0 = f_0$ and $f(x) \leq p(x) = \|f_0\|\|x\|$ for all $x \in X$. Then

$$-f(x) = f(-x) \leq p(-x) = \|f_0\|\|x\|$$

so

$$-\|f_0\|\|x\| \leq f(x) \leq \|f_0\|\|x\|$$

i.e. $|f(x)| \leq \|f_0\|\|x\|$. So $\|f\| = \|f_0\|$.

Case 2. Suppose $\mathbb{F} = \mathbb{C}$. Think of X as a vector space over $\mathbb{R}$. Let $g_0(m) = \mathrm{Re}(f_0(m))$ for $m \in M_0$. Then

$$g_0(m) \leq |f_0(m)| \leq \|f_0\|\|m\|$$

By the first case, we can extend g_0 to a continuous real linear functional $g\colon X \to \mathbb{R}$ such that $g \restriction M_0 = g_0$ and $\|g\| \leq \|g_0\| \leq \|f_0\|$.

Define $f(x) = g(x) + ig(-ix)$. Then f is continuous and $\mathbb{R}$-linear. Also,

$$\begin{aligned} f(ix) &= g(ix) + ig(-i(ix)) \\ &= i(g(x) + (-i)g(ix)) \\ &= i(g(x) + ig(-ix)) \\ &= if(x) \end{aligned}$$

So f is $\mathbb{C}$-linear. Also, if $m \in M_0$, then

$$\begin{aligned} f(m) &= g(m) + ig(-im) \\ &= g_0(m) + ig_0(-im) \\ &= \mathrm{Re}(f_0(m)) + i\,\mathrm{Re}(f_0(-im)) \\ &= f_0(m) \end{aligned}$$

since if $f_0(m) = a + ib$, then $f_0(-im) = -i(a+ib) = b - ia$. Finally, if $x \in X$, then $f(x) = \exp(i\theta)|f(x)|$; so

$$\begin{aligned} |f(x)| &= f(\exp(-i\theta)x) \\ &= \mathrm{Re}(f(\exp(i\theta)x)) \\ &= g(\exp(-i\theta)x) \\ &\leq \|g_0\|\|\exp(-i\theta)x\| \\ &\leq \|f_0\|\|x\| \end{aligned}$$

So $\|f\| \leq \|f_0\|$.

☐ Theorem 122

Corollary 123. *If X is a Banach space with $x \in X$, then there is $f \in X^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$.*

Proof. Define f_0 on $\mathbb{F}x$ by $f_0(\lambda x) = \lambda\|x\|$. Then

$$\|f_0\| = \sup_{\lambda \in \mathbb{F}} \frac{|\lambda\|x\||}{\|\lambda x\|} = 1$$

We can then extend by the Hahn-Banach theorem. ☐ Corollary 123

Corollary 124. *If $x \in X$ then*

$$\|x\| = \sup_{f \in X^*, \|f\| \leq 1} |f(x)|$$

Proof. Well, $|f(x)| \leq \|f\|\|x\| \leq \|x\|$. But by the corollary there is f with $\|f\| = 1$ and $f(x) = \|x\|$. □ Corollary 124

Corollary 125. *X^* separates points of X.*

Proof. Suppose $x \neq y$. Then there is $f \in X^*$ such that $f(x) - f(y) = f(x-y) \neq 0$. So $f(x) \neq f(y)$. □ Corollary 125

Corollary 126. *Suppose X is a Banach space with $M \subseteq X$ a closed subspace. Suppose $x \notin M$. Then there is $f \in X^*$ with $\|f\| = 1$ such that $r \restriction M = 0$ and $f(x) = \operatorname{dist}(x, M)$.*

Proof. Let $q\colon X \to X/M$ be the quotient map. Then $q(x) = \dot{x} \neq 0$. Then

$$\|\dot{x}\| = \inf_{m \in M} \|x - m\| = \operatorname{dist}(x, M)$$

By one of the previous corollaries, there is $g \in (X/M)^*$ such that $\|g\| = 1$ and $g(\dot{x}) = \|\dot{x}\|$. Let $f = g \circ q$; then $\|f\| \leq \|g\|\|q\| = 1 \cdot 1 = 1$, and

$$f(x) = g(\dot{x}) = \|\dot{x}\| = \operatorname{dist}(x, M)$$

Furthermore, for $m \in M$, we have $f(m) = g(\dot{m}) = g(0) = 0$. □ Corollary 126

If X is a Banach space, there is a natural map $X \to X^{**}$ by $x \mapsto \widehat{x}$ where $\widehat{x}(f) = f(x)$.

Proposition 127. *The natural map $X \to X^{**}$ is isometric.*

Proof. Well

$$\begin{aligned} \|\widehat{x}\| &= \sup_{f \in X^*, \|f\| \leq 1} |\widehat{x}(f)| \\ &= \sup_{f \in X^*, \|f\| \leq 1} |f(x)| \\ &= \|x\| \end{aligned}$$

by a previous corollary. □ Proposition 127

Definition 128. X is *reflexive* if $X = X^{**}$; i.e. the natural map above is surjective.

We get chains

$$X \subseteq X^{**} \subseteq X^{****} \subseteq X^{******} \subseteq \ldots$$

and

$$X^* \subseteq X^{***} \subseteq X^{*****} \subseteq \ldots$$

Proposition 129. *If $X \neq X^{**}$, then $X^* \neq X^{***}$.*

Proof. If $X \neq X^{**}$, then there is $y \in X^{**} \setminus X$. By a previous corollary, there is $f \in X^{***}$ such that $f \restriction X = 0$ and $f(y) \neq 0$. But if $g \in X^*$ and $g \restriction X = 0$, then $g = 0$, and $\widehat{g} = 0$. So $f \neq \widehat{g}$ for any $g \in X^*$. So $X^* \neq X^{***}$. □ Proposition 129

Example 130.

1. Suppose $\mathcal{H}$ is a Hilbert space. Then $\mathcal{H}^*$ is $\overline{\mathcal{H}}$, and $\mathcal{H}^{**}$ is $\overline{\overline{\mathcal{H}}} = \mathcal{H}$. So $\mathcal{H}$ is reflexive.

2. Suppose $1 < p < \infty$. Then $\ell_p^* = \ell_q$ for $\frac{1}{p} + \frac{1}{q} = 1$. Then $\ell_p^{**} = \ell_q^* = \ell_p$. So ℓ_p is reflexive. Similarly for $L^p(\mu)$ for $1 < p < \infty$.

3. $c_0^* = \ell_1$ and $\ell_1^* = \ell_\infty$; so c_0 is not reflexive.

4. $L^1(0,1)^* = L^\infty(0,1)$ which is not separable; so $L^\infty(0,1)^*$ is not separable; So $L^1(0,1)$ is not reflexive.

5. $C[0,1]^* = M([0,1]) \supseteq L^1(0,1)$ (where $M([0,1])$ is the set of finite regular complex Borel measures). So $C[0,1]^*$ is not reflexive.

6. If $\dim(X) < \infty$ then $\dim(X^*) = \dim(X)$.

Example 131 (Banach limits). Weant a map L which takes a bounded sequence $x = (x_n : n < \omega)$ of real numbers and satisfies

1. $\liminf x_n \leq L(x) \leq \limsup x_n$

2. $L(x) = L(Sx)$ where $Sx = (x_{n+1} : n < \omega)$. (Translation-invariance.)

This is called a *Banach limit*. So we're looking for a continuous linear functional L on $\ell_{\infty,\mathbb{R}}$ of norm 1. Let $M = \operatorname{span} x - Sx : x \in \ell_{\infty,\mathbb{R}}$. We need

1. $L \restriction M = 0$

2. If $u = (1 : n < \omega)$, then we need $L(u) = 1$.

Claim 132. $\operatorname{dist}(u, M) = 1$.

Proof. Well, $\operatorname{dist}(u, M) \leq \|u - 0\| = 1$. Suppose $x \in \ell_{\infty,\mathbb{R}}$ satisfies $\|u - (x - Sx)\| = 1 - \varepsilon < 1$. Write $x = (x_n : n < \omega)$; then $Sx = (x_{n+1} : n < \omega)$. Then $u - (x - Sx) = (1 - (x_n - x_{n+1}) : n < \omega)$. So $1 - (x_n - x_{n+1}) \leq 1 - \varepsilon$; so

$$x_{n+1} \leq x_n - \varepsilon \leq x_{n-1} - 2\varepsilon \leq \cdots \leq x_1 - n\varepsilon$$

So $(x_n : n \in \mathbb{N}) \to -\infty$, contradicting boundedness. $\square$ Claim 132

By one of the corollaries, we then have $L \in \ell^*_{\infty,\mathbb{R}}$ with $\|L\| = 1$ such that $L \restriction M = 0$ and $L(u) = 1$. Thus, for $x \in \ell_{\infty,\mathbb{R}}$ we have $L(x) - L(Sx) = L(x - Sx) = 0$; so L is translation-invariant.

Claim 133. *If $x \in c_0$, then $L(x) = 0$.*

Proof. Say $x \in c_0$. Then $S_n x = (x_{n+i} : i \in \mathbb{N})$; so $S^n x \to 0$ in ℓ_∞. But

$$x - S^n x = (x - Sx) + (Sx - S^2 x) + \cdots + (S^{n-1}x - S^n x)$$

and each summand is in M; so $L(x) = L(S^n x) \to L(0) = 0$. So $L(x) = 0$. $\square$ Claim 133

Take $x \in \ell_{\infty,\mathbb{R}}$; let

$$\alpha = \liminf x_n$$
$$\beta = \limsup x_n$$

Write $x = y + z$ with $\alpha \leq y_n \leq \beta$, $z \in c_0$. Then $L(x) = L(y) + L(z) = L(y)$. Let

$$w_n = y_n - \frac{\alpha + \beta}{2} \in \left[-\frac{\beta - \alpha}{2}, \frac{\beta - \alpha}{2}\right]$$

Let $w = (w_n; n \in \mathbb{N})$. Then

$$y = \left(\frac{\alpha + \beta}{2}\right) u + w$$

So

$$|L(w)| \leq \|w\| \leq \frac{\beta - \alpha}{2}$$

So

$$-\frac{\beta - \alpha}{2} \leq L(w) \leq \frac{\beta - \alpha}{2}$$

So

$$
\begin{aligned}
\alpha &= -\frac{\beta-\alpha}{2} + \frac{\beta-\alpha}{2} \\
&\le L(y) \\
&= L(w) + \left(\frac{\alpha+\beta}{2}\right) L(u) \\
&\le \frac{\beta-\alpha}{2} + \frac{\alpha+\beta}{2} \\
&= \beta
\end{aligned}
$$

So we have the desired properties.

Remark 134. We can extend L to $\widetilde{L}$ on ℓ_∞ by

$$\widetilde{L}(x) = L(\mathrm{Re}(x)) + iL(\mathrm{Im}(x))$$

This is translation-invariant, and if

$$\lim_{n\to\infty} x_n = x_\infty$$

then $\widetilde{L}(x) = x_\infty$.

4 LCTVSs and weak topologies

Definition 135. A *seminorm* on a vector space V is a map $p\colon V \to [0,\infty)$ such that

1. $p(tv) = |t|p(v)$ for all $t \in \mathbb{F}$, all $v \in V$
2. $p(v+w) \le p(v) + p(w)$ for all $v, w \in V$ (triangle inequality)

Remark 136. This is not necessarily a norm because $p(v) = 0 \not\Longrightarrow v = 0$ in general.

Definition 137. A *locally convex topological vector space* (LCTVS) is a vector space X with a family $\mathcal{P} = \{\, p_\alpha : \alpha \in I \,\}$ of seminorms such that if $x \in X$ and $p_\alpha(x) = 0$ for all $\alpha \in I$, then $x = 0$. Put a topology on X determined by a subbase

$$U(x_0, r, p_\alpha) = \{\, x \in X : p_\alpha(x - x_0) < r \,\}$$

Remark 138. $U(x_0, r, p_\alpha)$ is convex because of the triangle inequality: if $x, y \in U(x_0, r, p_\alpha)$ and $0 < t < 1$, then

$$p_\alpha(tx + (1-t)y) \le p_\alpha(tx) + p_\alpha((1-t)y) < |t|r + |1-t|r = r$$

and $tx + (1-t)y \in U(x_0, r, p_\alpha)$.

Remark 139. We have some translation-invariance: $U(x_0, r, p_\alpha) = x_0 + U(0, r, p_\alpha)$, and U is an open neighbourhood of 0 if and only if $x_0 + U$ is an open neighbourhood of x_0.

Theorem 140. *Suppose X is a LCTVS.*

1. *A neighbourhood base at 0 is given by the sets*

   $$U_{F,r} = \{\, x \in X : p_\alpha(x) < r \text{ for all } p_\alpha \in F \,\}$$

   where $F \subseteq_{\mathrm{fin}} \mathcal{P}$ and and $r > 0$.

2. *X is Hausdorff.*
3. *Addition and scalar multiplication are continuous.*
4. *A net $(x_\beta : \beta \in B)$ converges to x_0 if and only if $p_\alpha(x - x_\beta) \to 0$ for all $p_\alpha \in \mathcal{P}$.*

Proof.

1. Well,

$$U_{F,r} = \bigcap_{p_\alpha \in F} U(0, r, p_\alpha)$$

is open and contains 0. Suppose $F \subseteq_{\text{fin}} \mathcal{P}$ and

$$U = \bigcap_{\alpha \in F} U(x_\alpha, r_\alpha, p_\alpha)$$

is a basic open neighbourhood of 0. Well, $0 \in U(x_\alpha, r_\alpha, p_\alpha)$, so $p_\alpha(x_\alpha - 0) < r_\alpha$. Let

$$r = \min_{\alpha \in F}(r_\alpha - p_\alpha(x_\alpha))$$

Claim 141. $U_{F,r} \subseteq U$.

Proof. If $x \in U_{F,r}$, then $p_\alpha(x) < r$. So

$$p_\alpha(x_\alpha - x) \leq p_\alpha(x_\alpha - 0) + p_\alpha(0 - x) < p(x_\alpha) + r \leq p_\alpha(x_\alpha) + r_\alpha - p_\alpha(x_\alpha) = r_\alpha$$

So

$$x \in \bigcap_{\alpha \in F} U(x_\alpha, r_\alpha, p_\alpha) = U$$

and $U_{F,r} \subseteq U$. □ Claim 141

2. If $x \neq y$ in X, then there is α such that $p_\alpha(x - y) = r > 0$. Then

$$U\Big(x, \frac{r}{2}, p_\alpha\Big) \cap U\Big(y, \frac{r}{2}, p_\alpha\Big)$$

3. We do addition; scalar multiplication is similar. Let $A\colon X \times X \to X$ by $A(x, y) = x + y$. Let U be open in X. We need to show that $A^{-1}(U)$ is open in the product topology. Suppose $(x_0, y_0) \in A^{-1}(U)$; then $x_0 + y_0 \in U$. But U is open; so there is $F \subseteq_{\text{fin}} \mathcal{P}$ and $r > 0$ such that

$$(x_0, y_0) + U_{F,r} = \bigcap_{p_\alpha \in F} U(x_0 + y_0, r, p_\alpha) \subseteq U$$

Claim 142.

$$\big(x_0 + U_{F,\frac{r}{2}}\big) \times \big(y_0 + U_{F,\frac{r}{2}}\big) \subseteq A^{-1}(U)$$

Proof. Suppose

$$p_\alpha(x - x_0) < \frac{r}{2}$$
$$p_\alpha(y - y_0) < \frac{r}{2}$$

for all $p_\alpha \in F$. Then

$$p_\alpha((x + y) - (x_0 + y_0)) \leq p_\alpha(x - x_0) + p_\alpha(y - y_0) < \frac{r}{2} + \frac{r}{2} = r$$

□ Claim 142

4.

$$\begin{aligned} x_\beta \to x &\iff (\forall F \subseteq_{\text{fin}} \mathcal{P})(\forall r > 0)(\exists \beta_0)(\forall \beta \geq \beta_0)(x_\beta \in x + U_{F,r}) \\ &\iff \forall F \forall r \exists \beta_0 (\forall \beta \geq \beta_0)(\forall p_\alpha \in F)(p_\alpha(x - x_\beta) < r) \\ &\iff \forall p_\alpha (p_\alpha(x - x_\beta) \to 0) \end{aligned}$$

☐ Theorem 140

Example 143.

1. $(X, \|\cdot\|)$ a normed vector space.
2. Let X be a normed vector space. Let Y be a vector subspace of X^* (not necessarily closed). Suppose that for all $x \neq 0$ in X, there is $\varphi \in Y$ with $\varphi(x) \neq 0$. For $\varphi \in Y$, define a seminorm $p_\varphi(x) = |\varphi(x)|$. τ_y is the locally convex topology generated by $\{ p_\varphi : \varphi \in Y \}$. (X, τ_Y) is thus a LCTVS.

   In particular, if X is a Banach space, then (X, τ_{X^*}) is the *weak topology* on X. We write $x_\alpha \xrightarrow{w} x$ if and only if $\varphi(x_\alpha) \to \varphi(x)$ for all $\varphi \in X^*$.

   If $X = Y^*$ for a Banach space Y, then (Y^*, τ_Y) is the weak-* topology on Y^* and $f_\alpha \xrightarrow{w^*} f$ in Y^* if and only $f_\alpha(y) \to f(y)$ for all $y \in Y$.

   Remark 144.

   $$\begin{aligned} U_{F,r} &= \{ x : |p_\varphi(x)| < r \text{ for } \varphi \in F \} \\ &= \left\{ x : \left| \frac{1}{r}\varphi(x) \right| < 1, \frac{1}{r}\varphi \in \frac{1}{r}F \right\} \\ &= U_{\frac{1}{r}F,1} \end{aligned}$$

   Proposition 145. *Suppose Z a LCTVS; suppose $T \colon Z \to (X, \tau_Y)$ is linear. Then T is continuous if and only if $\varphi \circ T \colon Z \to \mathbb{F}$ is continuous for all $\varphi \in Y$.*

   Proof.

   ($\implies$) Note that τ_Y is the weakest topology that makes all $\varphi \in Y$ continuous on (X, τ_Y). Then $\varphi \circ T$ is continuous as the composition of continuous functions.

   ($\impliedby$) T is continuous if and only if $T^{-1}(x + U_{F,r})$ is open for all $F \subseteq_{\text{fin}} \mathcal{P}$, all $r > 0$. But

   $$\begin{aligned} T^{-1}(x + U_{F,r}) &= T^{-1}(x) + T^{-1}(U_{F,r}) \\ &= T^{-1}(x) + \bigcap_{\varphi \in F} (\varphi \circ T)^{-1}(b_r(0)) \end{aligned}$$

   which is open because all $\varphi \circ T$ is continuous.

   ☐ Proposition 145

3. Suppose X is a Banach space. Then we have the following topologies on $\mathcal{B}(X)$:

   Weak operator topology In which $T_\alpha \xrightarrow{\text{WOT}} T$ if and only if $\varphi(T_\alpha x) \to \varphi(Tx)$ for all $x \in X$ all $\varphi \in X^*$. For each $x \in X$, each $\varphi \in X^*$, define $\Psi_{x,\varphi} \in \mathcal{B}(X)^*$ by $\Psi_{x,\varphi}(T) = \varphi(Tx)$.

   $$Y = \operatorname{span}\{ \Psi_{x,\varphi} : x \in X, \varphi \in X^* \} \subseteq \mathcal{B}(X)^*$$

   (where the span is the algebraic span, not the closed span.) Note that this is not closed. $(\mathcal{B}(X), \text{WOT}) = (X, \tau_Y)$.

   Strong operator topology For $x \in X$, define $p_x(T) = \|Tx\|$. Then $\{ p_x : x \in X \}$ determines the strong operator topology. We have $T_\alpha \xrightarrow{\text{SOT}} T$ if and only if $T_\alpha x \to Tx$ for all $x \in X$; so this is the topology of pointwise convergence.

Theorem 146. *Suppose X a LCTVS; suppose $f \colon X \to \mathbb{F}$ is linear. The following are equivalent:*

1. *f is continuous.*
2. *f is continuous at 0.*

3. $\ker(f)$ *is closed.*
4. *There is* $F = \{ p_{\alpha_1}, \dots, p_{\alpha_n} \} \subseteq_{\text{fin}} \mathcal{P}$ *and* $C < \infty$ *such that*

$$|f(x)| \le C \sum_{i=1}^{n} p_{\alpha_i}(x)$$

Proof.

(1) $\implies$ **(2)** Trivial.

(2) $\implies$ **(3)** $f^{-1}(0) = X \setminus f^{-1}(\mathbb{F} \setminus \{ 0 \})$. But $\mathbb{F} \setminus \{ 0 \}$ is open. So $f^{-1}(0) = \ker(f)$ is closed.

(3) $\implies$ **(4)** Without loss of generality $f \ne 0$. Pick $x_0 \in X$ such that $f(x_0) = 1$. Pick $F \subseteq_{\text{fin}} \mathcal{P}$ and $r > 0$ such that

$$(x_0 + U_{F,r}) \cap \ker(f) = \emptyset$$

(Possible since $\ker(f)$ is closed.) Then

$$0 \notin f(x_0 + U_{F,r}) = 1 + f(U_{F,r})$$

So $-1 \notin f(U_{F,r})$. Note, though, that

$$U_{F,r}\{ x : p_\alpha(x) < r \text{ for all } \alpha \in F \}$$

is *balanced*; i.e. if $x \in U_{F,r}$ and $\lambda \in \mathbb{F}$ satisfies $|\lambda| \le 1$, then $\lambda x \in U_{F,r}$ (since $p(\lambda x) = |\lambda| p(x) \le p(x) < r$). So $f(U_{F,r})$ is balanced in $\mathbb{F}$. So $f(U_{F,r})$ is convex and disjoint from $\{ \lambda : |\lambda| = 1 \}$. So $f(U_{F,r}) \subseteq \mathbb{D} = \{ \lambda : |\lambda| < 1 \}$. Thus if $kp_\alpha(x) < r$ for all $p_\alpha \in F$ then $|f(x)| < 1$. In particular, if

$$\sum_F p_\alpha(x) < r$$

then $p_\alpha(x) < r$ for all $p_\alpha \in F$. So $|f(x)| < 1$. So

$$|f(x)| \le \frac{1}{r} \sum p_\alpha(x)$$

(4) $\implies$ **(1)** Suppose $x_\beta \to x$. Then $p(x_\beta - x) \to 0$ for all α. So

$$|f(x_\beta) - f(x)| = |f(x_\beta - x)| \le C \sum_F p_\alpha(x_\beta - x) \to 0$$

So $f(x_\beta) \to f(x)$. So f is continuous.

□ Theorem 146

Corollary 147. *Suppose* $f\colon (X, \tau_Y) \to \mathbb{F}$ *is a linear functional. Then* f *is continuous if and only if* $f \in Y$.

Proof.

($\impliedby$) Trivial.

($\implies$) Suppose f is continuous on (X, τ_Y). Then by the theorem, there are $f_1, \dots, f_n \in Y$ such that

$$|f(x)| \le C \sum_{i=1}^{n} |f_i(x)|$$

In particular, if

$$x \in \bigcap_{i=1}^{n} \ker(f_i)$$

then RHS $= 0$. So $f(x) = 0$. So

$$\bigcap_{i=1}^{n} \ker(f_i) \subseteq \ker(f)$$

Lemma 148. *Suppose X is a vector space. Suppose $f_1, \dots, f_n$ are linear functionals (not necessarily continuous). Suppose*

$$\ker(f) \supseteq \bigcap_{i=1}^{n} \ker(f_i)$$

Then $f \in \operatorname{span}\{ f_1, \dots, f_n \}$.

Proof. We have

$$\begin{tikzcd} X \arrow[rr, "q"] \arrow[dr, "f"'] & & X/\bigcap_{i=1}^{n} \ker(f_i) \arrow[dl, "\exists!\widetilde{f}"] \\ & X/\ker(f) & \end{tikzcd}$$

But we can identify

$$F\colon X/\bigcap_{i=1}^{n} \ker(f_i) \cong \{ (f_1(x), \dots, f_n(x)) : x \in X \} \subseteq \mathbb{F}^n$$

Then

$$\ker(F) = \bigcap_{i=1}^{n} \ker(f_i)$$

We can extend $\widetilde{f}$ to a linear functional $\widetilde{\widetilde{f}}$ on $\mathbb{F}$:

$$\widetilde{\widetilde{f}}((v_1, \dots, v_n)) = \sum a_i v_i$$

Then

$$f(x) = \widetilde{f} \circ q(x) = \widetilde{\widetilde{f}}(F(x)) = \sum_{i=1}^{n} a_i f_i(x) \in \operatorname{span}\{ f_1, \dots, f_n \}$$

□ Lemma 148

So

$$f = \sum_{i=1}^{n} a_i f_i \in Y$$

□ Corollary 147

Remark 149. If we start with $Y \subseteq X^*$ which is not closed, then

$$(X, \tau_Y)^* = Y$$

is not a Banach space.

Lemma 150. *Suppose X is a Banach space; suppose Y is a closed subspace of X^* which norms X. i.e.*

$$\|x\| = \sup_{\substack{f \in Y \\ \|f\| \le 1}} |f(x)|$$

Then if a sequence $(x_n : n \in \mathbb{N})$ converges in (X, τ_Y) then

$$\sup_{n \in \mathbb{N}} \|x_n\| < \infty$$

Proof. For $x \in X$, define $\widehat{x} \in Y^*$ by $\widehat{x}(f) = f(x)$. Since Y is closed, it is a Banach space. But $(x_n : n \in \mathbb{N}) \xrightarrow{\tau_Y} x$ says that $\widehat{x_n}(f) \to \widehat{x}(f)$ for all $f \in Y$; so $\{ \widehat{x_n}(f) : n \geq 1 \}$ is bounded for all f. By the uniform boundedness principle, we have that

$$\sup_{n \in \mathbb{N}} \|\widehat{x_n}\| < \infty$$

Since Y norms X, we get $\|\widehat{x_n}\| = \|x_n\|$. So

$$\sup_{n \geq 1} \|x_n\| < \infty$$

□ Lemma 150

Example 151. Let (ℓ_1, τ_{c_0}) be ℓ_1 with the weak-* topology from $\ell_1 = c_0^*$. Suppose $x_n = (x_{ni} : i \in \mathbb{N}) \in \ell^1$. Suppose $x_n \xrightarrow{w^*} x = (x_i : i \in \mathbb{N})$. Then $e_i(x_n) = \langle x_n, e_i \rangle = x_{ni} \to \langle x, e_i \rangle = x_i$ (where $e_i \in c_0$). By lemma, we have

$$\sup_{n \in \mathbb{N}} \|x_n\|_1 = M < \infty$$

Conversely, suppose the above two statements hold. Suppose $y = (y_1, y_2, \dots) \in c_0$; suppose $\varepsilon > 0$. Pick N such that $|y_i| < \varepsilon$ if $i > N$. Then

$$\begin{aligned} |\langle x_n, y \rangle - \langle x, y \rangle| &= \left| \sum_{i=1}^{\infty} (x_{ni} - x_i) y_i \right| \\ &= \sum_{i=1}^{N} (x_{ni} - x_i) y_i + \sum_{i=N+1}^{\infty} (x_{ni} - x_i) y_i \\ &\leq N \|y\| \max_{1 \leq i \leq n} |x_{n_i} - x_i| + \sum_{i \geq N} |x_{n_i} - x_i| \varepsilon \end{aligned}$$

Pick N_2 such that $n \geq N_2$ implies

$$|x_{n_i} - x_i| < \frac{\varepsilon}{N \|y\|}$$

for $1 \leq i \leq N$. Ten

$$|\langle x_n, y \rangle - \langle x, y \rangle| < N \|y\| \frac{\varepsilon}{N \|y\|} + \|x_n - x\|_1 \varepsilon \leq \varepsilon + (\|x_n\|_1 + \|x\|_1) \varepsilon \leq (1 + 2M) \varepsilon$$

So $\langle x_n, y \rangle \to \langle x, y \rangle$ for $y \in c_0$.

On the other hand, there is an unbounded net converging to 0 in (ℓ_1, τ_{c_0}). Let Λ be the set of finite subsets of c_0 ordered by set inclusion. Then, if $F \in \Lambda$, we have

$$\bigcap_{y \in F} \ker(y)$$

is a closed subspaces of ℓ_1 of finite codimension. By axiom of choice, we can pick

$$x_F \in \bigcap_{y \in F} \ker(y)$$

such that $\|x_F\| = |F|$.

Claim 152. *$(x_F : F \in \Lambda) \to 0$ in τ_{c_0}.*

Proof. Take $y \in c_0$. If $F \geq \{ y_0 \}$, then $\langle x_F, y \rangle = 0 \to 0$. □ Claim 152

4.1 Geometric Hahn-Banach theorem

Given convex, disjoint A and B, we wish to find some linear functional f separating them; i.e.

$$A \subseteq \{\, x \in X : \operatorname{Re}(f(x)) \le a \,\}$$
$$B \subseteq \{\, x \in X : \operatorname{Re}(f(x)) > a \,\}$$

We also want f to be continuous; we then need a topological condition on A and B.

Definition 153. A *hyperplane* is a set $H = \{\, x \in X : \operatorname{Re}(f(x)) = a \,\}$ where X is a LCTVS and f is a linear functional. We are interested in *closed hyperplanes*, in which we require that f be continuous.

Lemma 154. *Suppose X is a LCTVS. Suppose U is open and convex with $0 \in U$. Recall the Minkowski functional*

$$p_U(x) = \inf\{\, r > 0 : x \in rU \,\}$$

Then p_U is continuous and $\{\, x \in X : p_U(x) < 1 \,\} = U$.

Proof. Since $0 \in U$ and U is convex, if $0 < r < s$, then $rU \subseteq sU$. Suppose $p_U(x) = r < 1$, then for $r < s < 1$, we have $x \in sU \subseteq U$. Conversely, if $x \in U$ there is $\varepsilon > 0$ such that $(1+\varepsilon)x \in U$ (since $t \mapsto tx$ is continuous and $1x \in U$ and U is open; thus $\{\, t \in \mathbb{R} : tx \in U \,\}$ is open in $\mathbb{R}$, and thus contains $(1-\varepsilon, 1+\varepsilon)$ for some $\varepsilon > 0$). So

$$x \in \frac{1}{1+\varepsilon} U$$

So $p_U(x) \le \frac{1}{1+\varepsilon} < 1$. So $\{\, x \in X : p_U(x) \le 1 \,\} = U$.

To see continuity, suppose $x_0 \in X$; suppose $p_U(x_0) = r_0 \in V \subseteq \mathbb{R}$ where V is open. Then $p_U(x_0) \in (r_0 - \varepsilon, r_0 + \varepsilon) \subseteq V$ for some $\varepsilon > 0$. Then

$$p_U^{-1}(V) \supseteq p_U^{-1}(r_0 - \varepsilon, r_0 + \varepsilon)$$

If $x \in x_0 + \frac{\varepsilon}{2} U$, then

$$p_U(x) \le p_U(x_0) + p_U(x - x_0) < p_U(x_0) + \frac{\varepsilon}{2}$$

and

$$p_U(x) \ge p_U(x_0) - p_U(x - x_0) > p_U(x_0) - \frac{\varepsilon}{2}$$

□ Lemma 154

Theorem 155 (Hyperplane theorem)**.** *Suppose X a LCTVS; suppose $U \subseteq X$ is open and convex with $0 \notin U$. Then there is $f \in X^*$ such that $\operatorname{Re}(f(x)) > 0$ for all $x \in U$. i.e. U is disjoint from the closed hyperplane $H = \{\, x \in X : \operatorname{Re}(f(x)) = 0 \,\}$*

Proof.

Case 1. Suppose $\mathbb{F} = \mathbb{R}$. Pick $x_0 \in U$. Define $V = x_0 - U$. Then V is open and convex with $0 \in V$. So p_V is a continuous sublinear functional. Define f_0 on $\mathbb{R}x_0$ by $f_0(tx_0) = t$ for $t \in \mathbb{R}$. Now, $x_0 \notin V$; so, by our lemma, we have $p_V(x_0) \ge 1$. Then if $t \ge 0$, we have $f_0(tx_0) = t \le tp_V(x_0) = p_V(tx_0)$; if $t < 0$, then $f_0(tx_0) = t < 0 \le p_V(tx_0)$. So $f_0 \le p_V$ on $\mathbb{R}x_0$. Thus there is linear $f \colon X \to \mathbb{R}$ such that $f(x) \le p_V(x)$ for all $x \in X$.

Claim 156. *f is continuous.*

Proof. It suffices to check continuity at 0. But

$$f^{-1}(b_r(0)) = \{\, x \in X : -r < f(x) < r \,\} = \{\, x \in X : f(x) < r \,\} \cap \{\, x \in X : f(-x) < r \,\}$$

But $V \subseteq \{\, x \in X : f(x) < r \,\}$ and $-V \subseteq \{\, x \in X : f(-x) < r \,\}$. So

$$f^{-1}(b_r(0)) \supseteq rV \cap (-rV)$$

which is an open neighbourhood of 0. □ Claim 156

Now, if $x \in U$ then $x_0 - x \in V$; so $1 - f(x) = f(x_0 - x) \leq p_V(x_0 - x) < 1$ and $f(x) > 0$.

Case 2. Suppose $\mathbb{F} = \mathbb{C}$. Consider X as a real LCTVS. Find $f \colon X \to \mathbb{R}$ that is $\mathbb{R}$-linear and continuous such that $f(x) > 0$ for all $x \in U$. Define $g(x) = f(x) + if(-ix)$. As before, we have that g is $\mathbb{C}$-linear and continuous, and $\operatorname{Re}(g(x)) = f(x)$. Thus if $x \in U$, we have $\operatorname{Re}(g(x)) = f(x) > 0$.

□ Theorem 155

We point out some special cases:

Corollary 157. *Suppose X is a Banach space; suppose $U \subseteq X$ is open and convex with $0 \notin U$. Then there is $f \in X^*$ such that $\operatorname{Re}(f(x)) > 0$ for all $x \in U$.*

Corollary 158. *Suppose X is a Banach space; suppose Y is a vector subspace of X^* that separates points. Suppose $U \subseteq X$ is convex and τ_Y-open with $0 \notin U$. Then there is $f \in Y$ such that $\operatorname{Re}(f(x)) > 0$ for all $x \in U$.*

Proof. A linear $f \colon X \to \mathbb{F}$ is τ_Y-continuous if and only if $f \in Y$. □ Corollary 158

Theorem 159 (Separation theorem)**.** *Suppose X is a LCTVS; suppose A and B are disjoint convex subsets of X with A open. Then there is $f \in X^*$ and $d \in \mathbb{R}$ such that $\operatorname{Re}(f(b)) \leq d < \operatorname{Re}(f(a))$ for all $b \in B$, all $a \in A$.*

Proof. Let

$$C = A - B = \bigcup_{b \in B} (A - b)$$

Then C is open as the union of the open $A - b$. Also, $0 \notin C$ since $A \cap B = \emptyset$. Finally, C is convex because

$$t(a_1 - b_1) + (1-t)(a_2 - b_2) = (ta_1 + (1-t)a_2) - (tb_1 + (1-t)b_2) \in A - B$$

By the hyperplane theorem, there is $f \in X^*$ such that $\operatorname{Re}(f(a)) - \operatorname{Re}(f(b)) = \operatorname{Re}(f(c)) > 0$ for all $c \in C$. So $\operatorname{Re}(f(b)) < \operatorname{Re}(f(a))$ for all $a \in A$, all $b \in B$. Then

$$\sup_{b \in B} \operatorname{Re}(f(b)) = d \leq \inf_{a \in A} \operatorname{Re}(f(a))$$

But A is open; so $\operatorname{Re}(f(A))$ is open in $[c, \infty)$. So $\operatorname{Re}(f(A)) \subseteq (c, \infty)$. □ Theorem 159

Corollary 160. *If A, B are both open and convex with $A \cap B = \emptyset$, then there is $f \in X^*$ and $d \in \mathbb{R}$ such that $\operatorname{Re}(f(b)) < d < \operatorname{Re}(f(a))$ for all $a \in A$ and all $b \in B$.*

Example 161. Let $X = (\ell_1, \tau_{c_0})$. Let

$$A = \left\{ x \in \ell_1 : \sum_{i=1}^{\infty} x_i = 0 \right\}$$

Then A is a norm-closed linear subspace and

$$\delta_1 = (1, 0, 0, \dots) \notin A$$

If $0 \neq y \in c_0$ with $y = (y_1, y_2, \dots)$, then we have $y_{n_0} \neq 0$ for some $n_0 \in \mathbb{N}$. Then if $\lambda \in \mathbb{C}$, we have

$$x_m = \frac{\lambda}{y_{n_0}} (\delta_{n_0} - \delta_m) \in A$$

Then

$$y(x_m) = \frac{\lambda}{y_{n_0}} (y_{n_0} - y_m) \xrightarrow{m \to \infty} \lambda$$

So $y(A) = \mathbb{C} \ni y(\delta_1)$. So we have no hope of separation.

What went wrong? A is not τ_{c_0}-closed, because $x_m = \delta_1 - \delta_m \in A$ with

$$\sup_{m\geq 0} \|x_m\| = 2$$

and x_{m_i} has limit

$$= \begin{cases} 1 & i = 1 \\ 0 & \text{else} \end{cases}$$

So $x_m \xrightarrow{\tau_{c_0}} \delta_1 \notin A$.

Lemma 162. *Suppose X is a LCTVS. Suppose $K \subseteq X$ is compact and $V \supseteq K$ is open. Then there is open $U \ni 0$ such that $K + U \subseteq V$.*

Proof. For each $x \in K$, there is a finite set F_x of seminorms and $r_x > 0$ such that $U(F_x, r_x) + x \subseteq V$. (Recall that

$$\begin{aligned} U(F_x, r_x)(x) &= \{ y : p(y - x) < r \text{ for all } p \in F_x \} \\ &= U(F_x, r_x) \\ U(F_x, r_x) &= \{ y : p(y) < r_x \text{ for all } p \in F_x \} \end{aligned}$$

are the basic open sets.) Then $\{ x + U(F_x, \frac{r_x}{2}) : x \in K \}$ is an open cover of K; so there is a finite subcover

$$K \subseteq \bigcup_{i=1}^{m} \left\{ x_i + U\left(F_{x_i}, \frac{r_{x_i}}{2}\right) \right\}$$

Let

$$\begin{aligned} F &= \bigcup_{i=1}^{n} F_{x_i} \\ r &= \min_{1\leq i\leq n} r_i \end{aligned}$$

Suppose $y \in K$. Then there is i_0 such that $y \in x_{i_0} + U(F_{i_0}, \frac{r_{i_0}}{2}$. Let $U = U(F, \frac{r}{2})$; then $U \subseteq U(F_i, r_{\frac{i}{2}})$ for all i. So

$$\begin{aligned} y + U &\subseteq x_{i_0} + U(F_{i_0}, \frac{r_{i_0}}{2}) + U(F, \frac{r}{2}) \\ &\subseteq x_{i_0} + U(F_{i_0}, r_{i_0}) \\ &\subseteq V \end{aligned}$$

So $K + U \subseteq V$. □ Lemma 162

Corollary 163. *Suppose X is a LCTVS; suppose $A, B \subseteq X$ are closed and convex, B is compact, and $A \cap B = \emptyset$. Then there is $f \in X^*$ such that*

$$\sup_{a\in A} \mathrm{Re}(f(a)) = \alpha < \beta = \inf_{b\in B} \mathrm{Re}(f(b))$$

Proof. Well, A^c is open, and $B \subseteq A^c$. We may thus pick U such that $B + U \subseteq A^c$. Thus $(B + U) \cap A = \emptyset$. Note that $B + U$ is convex as

$$t(b_1 + u_1) + (1 - t)(b_2 + u_2) = (tb_1 + (1 - t)b_2) + (tu_1 + (1 - t)u_2)$$

Thus, by the separation theorem, there is $f \in X$ and α such that

$$\sup_{a\in A} \mathrm{Re}(f(x)) = \alpha < \mathrm{Re}(b + u)$$

for all $b \in B$ and all $u \in U$. Since B is compact, this yields that

$$\inf_{b\in B} \mathrm{Re}(f(b)) = \beta > \alpha$$

□ Corollary 163

Definition 164. Suppose X is a LCTVS, $f \in X^*$, and $\alpha \in \mathbb{R}$. Then

$$H_{f,\alpha} = \{ y : \mathrm{Re}(f(y)) \le \alpha \}$$

is called a *closed half-space.*

Corollary 165. *Suppose X is a LCTVS; suppose $A \subseteq X$. Then*

$$\overline{\mathrm{conv}(A)} = \bigcap_{H_{f,\alpha} \supseteq A} H_{f,\alpha}$$

(where $\mathrm{conv}(A)$ is the convex hull of A: the intersection of all convex sets containing A).

Proof. The RHS is closed and convex and contains A; so

$$\overline{\mathrm{conv}(A)} \subseteq \bigcap_{H_{f,\alpha} \supseteq A} H_{f,\alpha}$$

Let $x \notin \overline{\mathrm{conv}(A)}$. Apply the last result with $\widetilde{A} = \overline{\mathrm{conv}(A)}$ and $\widetilde{B} = \{ x \}$, which is compact. Then there is $f_0 \in X^*$ such that

$$\sup_{a \in A} \mathrm{Re}(f_0(a)) \le \alpha_0 < \beta = \mathrm{Re}(f_0(x))$$

Thus $A \subseteq H_{f_0,\alpha}$ and $x \notin H_{f,\alpha}$.oSo

$$x \notin \bigcap_{H_{f,\alpha} \supseteq A} H_{f,\alpha}$$

So

$$\bigcap_{H_{f,\alpha} \supseteq A} \subseteq \overline{\mathrm{conv}(A)}$$

□ Corollary 165

Corollary 166. *Suppose X is a LCTVS. Then X^* separates points.*

Proof. Suppose $x_0, x_1 \in X$ have $x_0 \neq x_1$. Let

$$A = \{ x_0 \}$$
$$B = \{ x_1 \}$$

Then there is $H_{f,\alpha} \supseteq A$ such that $x_1 \notin H_{f,\alpha}$. By a previous corollary, we get that there is $f \in X^*$ such that $\mathrm{Re}(f(x_0)) = \alpha \neq \mathrm{Re}(f(x_1))$. □ Corollary 166

Proposition 167. *Suppose X is a normed linear space.*

1. *Every norm-closed convex set is weakly closed.*
2. *Every norm-closed ball in X^* is weak-* closed.*

Proof.

1. Suppose $C \subseteq X$ is norm-closed and convex. Then

$$C = \bigcap_{H_{f,\alpha} \supseteq C} H_{f,\alpha}$$

But each $H_{f,\alpha}$ is weakly closed. So C is weakly closed as the intersection of weakly closed sets.

2. Suppose $f_0 \in X^*$. Then

$$\begin{aligned}\overline{b_{r_0}(f_0)} &= \{ y : \|y - f_0\| \le r \} \\ &= \{ y : |\widehat{x}(y - f_0)| \le r \text{ for all } x \in X, \|x\| \le 1 \} \\ &= \bigcap_{\|x\| \le 1} \{ y : |(y - f_0)(x)| \le r \}\end{aligned}$$

But $\{ y : |(y - f_0)(x)| \le r \} = \widehat{x}^{-1}(\overline{\mathbb{D}_r})$ is closed. So $\overline{b_{r_0}(f_0)}$ is closed as the intersection of closed sets.

$\square$ Proposition 167

Example 168. Let

$$A = \{ x \in \ell^1 : \sum_{i=1}^{\infty} x_i = 0 \}$$

Then A is not weak-*-closed. Last time we showed that if $f \in c_0$ then $f(A) = \mathbb{C}$. So $\overline{\operatorname{conv}(A)} = \ell^1$ (where the closure is taken in the weak-* topology).

Theorem 169 (Goldstine's theorem). *Suppose X is a normed linear space. Then $b_1(X)$ is weak-* dense in $b_1(X^{**})$. i.e. The weak-* closure*

$$\overline{\{ \widehat{x} : x \in X, \|x\| \le 1 \}} = b_1(X^{**})$$

Proof. Suppose not. Then there is $x^{**} \in b_1(X^{**})$ and $x^{**} \notin A$ where A is the weak-* closure $A = \overline{b_1(X)}$. Then $\{ x^{**} \} = B$ is compact and convex and A is a convex, weak-*-closed set. So there is f that is weak-* continuous (i.e. $f \in X^*$) such that

$$\sup_{a \in A} \operatorname{Re}(f(a)) = \alpha < \operatorname{Re}(f(x^{**}))$$

Then in particular we have

$$\|f\| = \sup_{x \in b_1(X)} \operatorname{Re}(f(x)) \le \alpha$$

Since $\|f\| \le \alpha$ and $x^{**} \in b_1(X^{**})$. So $|\langle f, x^{**} \rangle| \le \|x^{**}\| \|f\| \le \alpha$. So $\operatorname{Re}(f(x^{**})) \le \alpha$, a contradiction.

$\square$ Theorem 169

Hence if $\psi \in b_1(X^{**})$ then there is a net $(x_\lambda : \lambda \in \Lambda)$ in X converging to ψ in the weak-* topology; i.e. $(f(x_\lambda) : \lambda \in \Lambda) \to \psi(f)$ for all $f \in X^*$.

Theorem 170 (Banach-Alaoglu). *Suppose X is a Banach space. Then the closed unit ball of X^* is weak-*-compact.*

Proof. For each $x \in X$, let $\mathbb{D}_x = \{ z \in \mathbb{C} : |z| \le \|x\| \}$. Let

$$\mathcal{D} = \prod_{x \in X} \mathbb{D}_x$$

which is compact by Tychonoff's theorem. Define $\Phi \colon (b_1(X^*), \tau \text{ the weak-* topology}) \to \mathcal{D}$ by $\Phi(f) = (f(x) : x \in X)$.

1. Φ is injective, since $\Phi(f) = \Phi(g)$ if and only if $f(x) = g(x)$ for all x; i.e. $f = g$.
2. Φ is continuous: a basic open set in $\mathcal{D}$ is given by

$$U = \{ d \in \mathcal{D} : d(x_i) \in U_i \}$$

for $x_1, \ldots, x_n \in X$ and U_i open in $\mathbb{D}_{x_i}$. We need to show $\Phi^{-1}(U)$ is open in $(b_1(X^*), \tau)$. But

$$\begin{aligned}\Phi^{-1}(U) &= \{ f : f(x_i) \in U_i \text{ for all } 1 \le i \le n \} \\ &= \{ f : \widehat{x_i}(f) \in U_i \text{ for all } 1 \le i \le n \} \\ &= \bigcap_{i=1}^{n} \widehat{x_i}^{-1}(U_i)\end{aligned}$$

which is open as the intersection of open sets.

3. $\Phi(b_1(X^*)) \subseteq \mathcal{D}$ is closed. To see this, we use nets. Take a net $(f_\alpha : \alpha \in \Lambda)$ in $b_1(X^*)$ such that $(\Phi(f_\alpha) : \alpha \in \Lambda) \to d \in \mathcal{D}$. Define $f\colon X \to \mathbb{C}$ by
$$f(x) = d_x = \lim_{\alpha \in \Lambda} f_\alpha(x)$$
It's easy to see that f is linear. For all x, we have
$$f(x) = \lim_{\alpha \in \Lambda} f_\alpha(x)$$
and $f_\alpha(x) \in \mathbb{D}_x$. So $|f_\alpha(x)| \le \|x\|$; so $|f(x)| \le \|x\|$, and $\|f\| \le 1$. So $f \in b_1(X^*)$. So $\Phi(b_1(X^*))$ is compact.

4. $\Phi\colon (b_1(X^*), \tau) \to \Phi(b_1(X^*))$, where the latter is compact; we now show that $\Phi^{-1}\colon \Phi(b_1(X^*)) \to (b_1(X^*), \tau)$ is continuous. We use nets: let $(d_\alpha : \alpha \in \Lambda)$ be a net in $\Phi(b_1(X^*))$ converging to $d \in \Phi(b_1(X^*))$. For each α, we can find $f_\alpha \in b_1(X^*)$ such that $\Phi(f_\alpha) = d_\alpha$; likewise, we can find $f \in b_1(X^*)$ such that $d = \Phi(f)$. We wish to show that $(f_\alpha : \alpha \in \Lambda) \xrightarrow{\text{wk}^*} f$. But $(d_\alpha : \alpha \in \Lambda) \to d$ in $\mathcal{D}$
$$\begin{aligned}(d_\alpha : \alpha \in \Lambda) \to d \text{ in } \mathcal{D} &\iff (d_\alpha(x) : \alpha \in \Lambda) \to d(x) \text{ for all } x \in X\\ &\iff (f_\alpha(x) : \alpha \in \Lambda) \to f(x) \text{ for all } x \in X\\ &\iff (f_\alpha : \alpha \in \Lambda) \xrightarrow{\text{wk}^*} f\end{aligned}$$

□ Theorem 170

Corollary 171. *If X is a reflexive Banach space then norm-closed, bounded, convex sets in X are weakly compact.*

Proof. Suppose X is reflexive; i.e. $\widehat{X} = X^{**}$. Suppose $A \subseteq X$ is norm-closed, bounded, and convex. Then Proposition 167 yields that A is weakly closed. By the Banach-Alaoglu, we have that $b_1(X^{**}) = \{\, \widehat{x} : \|\widehat{x} \le 1\| \,\}$ is weak-*-compact. So $\{\, \widehat{x} : \|\widehat{x} \le r\| \,\}$ is weak-*-compact. Pick r big enough so that $A \subseteq \{\, \widehat{x} : \|\widehat{x}\| \le r \,\}$. But A is then a weak-*-closed subset of a weak-*-compact set. So A is weak-*-compact. □ Corollary 171

Corollary 172. *Suppose X is a Banach space. Then $b_1(X)$ is weakly compact if and only if X is reflexive.*

Proof.

($\Longleftarrow$) Suppose $\widehat{X} = X^{**}$. Then
$$\begin{aligned}(b_1(X), \text{weak}) &= (b_1(\widehat{X}), \text{weak-*})\\ &= (b_1(X^{**}), \text{weak-*})\end{aligned}$$
which is compact by Banach-Alaoglu.

($\Longrightarrow$) Suppose $b_1(X)$ is weakly compact. By Goldstine, we have that $b_1(\widehat{X})$ is weak-*-dense in $b_1(X^{**})$. But $(b_1(\widehat{X}), \text{weak-*}) = (b_1(X), \text{weak})$ is compact, and thus closed in $b_1(X^{**})$. So $b_1(\widehat{X}) = b_1(X^{**})$.

□ Corollary 172

Definition 173. Suppose V is a vector space; suppose $K \subseteq V$ is *convex*. We say that $F \subseteq K$ is a *face* if for all $x, y \in K$ and all $0 < t < 1$ such that $tx + (1-t)y \in F$, we have $x, y \in F$. We say $x \in K$ is an *extreme point* if $\{\, x \,\}$ is a face.

Example 174.

1. Consider $K = \{\, (x,y) \in \mathbb{R}^2 : \|(x,y)\| \le 1, y \ge 0 \,\}$. The x-axis is a face, and the extreme points are all of the boundary besides the interior of the x-axis.

2. In $(\mathbb{R}^2, \|\cdot\|_\infty)$, let $K = b_1$. We then get a square each of whose sides is a face, and whose extreme points are $\{\, (\pm_1 1, \pm_2 1) \,\}$.

3. Similarly with $(\mathbb{R}^2, \|\cdot, \infty\|)$.

4. We say $(X, \|\cdot\|)$ is *strictly convex* if whenever

$$1 = \|x\| = \|y\| = \left\| \frac{x+y}{2} \right\|$$

we have $x = y$. If $(X, \|\cdot\|)$ is strictly convex, then the extreme points of $b_1(X)$ are given by $\operatorname{Ext}(b_1(X)) = \{ x : \|x\| = 1 \}$.

Proof. If $\|x\| = 1$ and $x = ty + (1-t)z$ for $0 < t < 1$ and $z \in b_1(X)$, then $1 = \|x\| \leq t\|y\| + (1-t)\|z\|$. So $\|y\| = \|z\| = 1$. Then we can find z' such that

$$\frac{1}{2}y + \frac{1}{2}z' = x$$

So $y = z' = x$. So $z = x$. □

5. Recall from the proof of Minkowski that if $1 < p < \infty$ and

$$\frac{|a|^p + |b|^p}{2} = \left| \frac{a+b}{2} \right|^p$$

then $a = b$. Consider now L^p for $1 < p < \infty$. Suppose $\|f\|_p = \|g\|_p = 1$ and

$$\left\| \frac{f+g}{2} \right\|_p = 1$$

Then

$$\int_0^1 \frac{|f|^p + |g|^p}{2} = \int_0^1 \left| \frac{f+g}{2} \right|^p$$

so

$$\frac{|f(x)|^p + |g(x)|^p}{2} = \left| \frac{f(x)+g(x)}{2} \right|^p$$

almost everywhere, and $f(x) = g(x)$ almost everywhere. So $f = g$ in L^p. So $\operatorname{Ext}(b_1(L^p)) =$ is the unit sphere, for $1 < p < \infty$.

6. $\operatorname{Ext}(b(L^1)) = \emptyset$.

Proof. Let $\|f\|_1 = 1$. Then

$$\int_0^1 |f(t)| dt = 1$$

Let

$$g(s) = \int_0^s |f(t)| dt$$

We know that g is continuous and $g(0) = 0$ and $g(1) = 1$. Thus there exists s_0 such that $g(s_0) = \frac{1}{2}$. Let

$$f_1(t) = 2f(t)\chi_{[0,s_0)}(t)$$
$$f_2(t) = 2f(t)\chi_{[s_0,1]}(t)$$

Then

$$\int_0^1 |f_1(t)| dt = \int_0^{s_0} 2|f(t)| dt = 1$$

and

$$\int_0^1 |f_2(t)| dt = \int_{s_0}^1 2|f(t)| dt = 1$$

So $\|f_1\|_1 = \|f_2\|_1$ and

$$\frac{f_1 + f_2}{2} = f$$

But neither is equal to f. □

7. $\operatorname{Ext}(b_1(c_0)) = \emptyset$.

Proof. Let $x = (x_1, x_2, \dots,) \in c_0$ with $\|x\| = 1$. Because

$$\lim_{n\to\infty} x_n = 0$$

we may pick n_0 such that $|x_{n_0}| < \frac{1}{2}$. Let

$$\begin{aligned} y &= (x_1, \dots, x_{n_0-1}, x_{n_0} + \varepsilon, x_{n_0+1}, \dots) \\ z &= (x_1, \dots, x_{n_0-1}, x_{n_0} - \varepsilon, x_{n_0+1}, \dots) \end{aligned}$$

for small ε. Then $\|x\| = \|y\| = \|z\| = 1$ and

$$\frac{y+z}{2} = x$$

but $y \neq x$ and $z \neq x$. □

Lemma 175. *Suppose X is a LCTVS and $K \subseteq X$ is compact and convex. Suppose $f \in X^*$ such that $\sup\{\operatorname{Re}(f(x)) : x \in K\} = \alpha$. Then $F = \{x \in K : \operatorname{Re}(f(x)) = \alpha\}$ is a face.*

Proof. Suppose $x = ty + (1-t)z$ for $y, z \in K$. Then

$$\alpha = t\operatorname{Re}(f(y)) + (1-t)\operatorname{Re}(f(z))$$

So $\operatorname{Re}(f(y)) = \operatorname{Re}(f(z)) = \alpha$. So $y, z \in F$. □ Lemma 175

Lemma 176. *Suppose X is a LCTVS and $K \subseteq X$ is compact and convex. Suppose $F \subseteq K$ is a closed face. Then $F \cap \operatorname{Ext}(K) \neq \emptyset$.*

Proof. Let $\mathcal{F}$ be the collection of all non-empty closed faces $\widetilde{F} \subseteq F$ ordered by $\supseteq$. Suppose $\mathcal{C} = \{F_\alpha\}$ is a chain in $\mathcal{F}$. Each F_α is a closed subset of K, which is compact. So each F_α is compact. So

$$\bigcap_\alpha F_\alpha \neq \emptyset$$

by the finite intersection property, and this is an upper bound of $\mathcal{C}$. By Zorn's lemma there is $\widetilde{F} \subseteq F$ such taht $\widetilde{F}$ is a minimal closed, non-empty face. Suppose for contradiction that $\widetilde{F}$ had more than one point $x_0, y_0 \in F$. By separation, we would then have that there is $f \in X^*$ such that $\operatorname{Re}(f(y_0)) < \operatorname{Re}(f(x_0))$. Let

$$F_1 = \{x \in \widetilde{F} : \operatorname{Re}(f(x)) = \sup \text{ over } \widetilde{F}\}$$

This is a face by the previous lemma, but $y_0 \notin F_1$, contradicting the minimality of $\widetilde{F}$. So $\widetilde{F} = \{\widetilde{x}\}$ is an extreme point. □ Lemma 176

Theorem 177 (Krein-Milman)**.** *Suppose X is a LCTVS. Suppose $\emptyset \neq K \subseteq X$ is compact and convex. Then $\overline{\operatorname{conv}(\operatorname{Ext}(K))} = K$.*

Proof.

($\subseteq$) Clear.

($\supseteq$) Suppose $x_0 \in K$ but $x_0 \notin \overline{\operatorname{conv}(\operatorname{Ext}(K))}$. Separate by $f \in X^*$. Then

$$\sup\{\operatorname{Re}(f(x)) : x \in \overline{\operatorname{conv}(\operatorname{Ext}(K))}\} = \alpha < \operatorname{Re}(f(x_0)) < \beta = \sup\{\operatorname{Re}(f(x)) : x \in K\}$$

But $F = \{x \in K : \operatorname{Re}(f(x)) = \beta\}$ is a face and there is extreme $y \in F$. Then $\operatorname{Re}(f(y)) = \beta > \alpha$, a contradiction.

□ Theorem 177

Corollary 178 (Krein-Milman)**.** *The unit ball of X^* is the weak-*-closed convex hull of its extreme points.*

Proof. $b_1(X^*) = K$ is closed, convex, and compact in the weak-* topology. The theorem the yields that we are done. □ Corollary 178

Corollary 179. *Suppose X is a Banach space. If $\operatorname{Ext}(b_1(X)) = \emptyset$, then there does not exist Y such that $X = Y^*$.*

Thus c_0 and L^1 cannot be the dual of any Banach space.

Example 180. $(C([0,1]), \|\cdot\|_\infty)^*$ can be identified with the regular bounded Borel measures.

$$\operatorname{Ext}(b_1(M[0,1])) = \{ \exp(i\theta)\delta_x : x \in [0,1] \}$$

(where $\delta_x(f) = f(x)$). Hence by Krein-Milman we have if $\mu([0,1]) = 1$ then μ is a weak-* limit of convex combinations of $\exp(i\theta)\delta_x$.

Example 181. We exhibit a compact convex set C in $\mathbb{R}^3$ such that $\operatorname{Ext}(C)$ is not closed. Let C_0 be the circle in $\mathbb{R}^3$ with $(0,0,0)$ and $(0,0,1)$ diametrically opposite. Let L be the line segment $(-1,0,0)$ to $(1,0,0)$ Let C be the convex hull of $C_0 \cup L$. Then $\operatorname{Ext}(C) = \{ (\pm 1, 0, 0) \} \cup C_0 \setminus \{ (0,0,0) \}$.

Theorem 182 (Stone-Weierstrass)**.** *Suppose A is a closed subalgebra of $C_{\mathbb{R}}(X)$ where X is compact and Hausdorff. Suppose A separates points; i.e. suppose that for all $x, y \in X$ with $x \neq y$ there is $f \in A$ such that $f(x) \neq f(y)$. Suppose there is $g \in A$ such that $g(x) > 0$ for all $x \in X$. Then $A = C_{\mathbb{R}}(X)$.*

Proof. Suppose for contradiction that $A \subsetneqq C_{\mathbb{R}}(X)$. Then by Hahn-Banach we have that there is $\varphi \in C_{\mathbb{R}}(X)^*$ such that $\varphi \restriction A = 0$ and $\varphi \neq 0$. Let

$$K = b_1(C_{\mathbb{R}}(X)^*) \cap A^\perp = \{ \varphi \in C_{\mathbb{R}}(X)^* : \varphi \restriction A = 0, \|\varphi\| \leq 1 \}$$

Then K is a bounded, convex set in $C_{\mathbb{R}}(X)^*$; so, by the Krein-Milman theorem, we have that $b_1(C_{\mathbb{R}}(X)^*)$ is weak-*-compact. But $A^\perp$ is weak-*-closed, since if $(\varphi_\lambda : \lambda \in \Lambda)$ is a net in $C_{\mathbb{R}}(X)^*$ converging to φ in the weak-* topology and $a \in A$, then

$$\varphi(a) = \lim_{\lambda \in \Lambda} \varphi_\lambda(a) = 0$$

So K is weak-*-compact. Again by Krein-Milman, we then have that K has an extreme point ψ. But then $\psi \neq 0$ since $\pm\frac{\varphi}{\|\varphi\|} \in K$, and

$$0 = \frac{1}{2}\left(\frac{\varphi}{\|\varphi\|} + \frac{-\varphi}{\|\varphi\|} \right)$$

So 0 is not extreme, and $\psi \neq 0$.

Now, by the Riesz representation theorem there is a finite real regular Borel measure μ such that

$$\psi(f) = \int f d\mu$$

for all $f \in C_{\mathbb{R}}(X)$.

Claim 183. $\operatorname{supp}(\mu) = \{ x_0 \}$. *(Here* $\operatorname{supp}(\mu) = \{ x \in X : |\mu|(U) > 0$ *for all* $U \in \mathcal{O}(x) \}$.*)*

Proof. If $\operatorname{supp}(\mu) = Y$ is not a single point, then there is $f \in A$ such that $f \restriction Y$ is non-constant. Without loss of generality we may assume that $0 < f(x) < 1$ for all $x \in X$; otherwise we add a multiple of g to f to make it everywhere positive and then scale to get that it's less than 1; i.e. find c, d such that $0 < (\frac{cg+f}{d})(x) < 1$ for all $x \in X$. (Note that $\frac{cg+f}{d}$ is still non-constant on Y.)

Let $\mu_1 = f\mu$; let $\mu_2 = (1-f)\mu$. Then for any $h \in C_{\mathbb{R}}(X)$ we have

$$\int h d\mu_1 = \int hf d\mu$$

Then if $h \in A$, we have $hf \in A$, and

$$0 = \psi(hf) = \int hf d\mu = \int h d\mu_1$$

Therefore $\mu_1 \in A^\perp$. Similarly, we get that $\mu_2 \in A^\perp$. But now

$$\begin{aligned}
\|\mu_1\| + \|\mu_2\| &= \int_X d|\mu_1| + \int_X d|\mu_2| \\
&= \int_X f d|\mu| + \int_X (1-f) d|\mu| \\
&= \int_X d|\mu| \\
&= \|\mu\| \\
&= \|\psi\| \\
&= 1
\end{aligned}$$

(We get $\|\psi\| = 1$ since if $\|\psi\| = r < 1$, then $\frac{1}{r}\psi \in K$ and $\psi = r(\frac{1}{r}\psi) + (1-r)0$ is not an extreme point.)

Observe now that $\frac{\mu_1}{\|\mu_1\|}, \frac{\mu_2}{\|\mu_2\|} \in K$ and $\mu = fd\mu + (1-f)d\mu = \|\mu_1\|\frac{\mu_1}{\|\mu_1\|} + \|\mu_2\|\frac{\mu_2}{\|\mu_2\|}$; so μ is not extreme, and ψ is not extreme, a contradiction. $\square$ Claim 183

So $\mathrm{supp}(\mu) = \{ x_0 \}$ for some $x_0 \in X$; so $\mu = \pm\delta_{x_0}$, where δ_x is the point mass:

$$\delta_x(A) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

But then

$$0 = \psi(g) = \int g d\mu = \pm \int g d\delta_{x_0} = \pm g(x_0) \neq 0$$

a contradiction. $\square$ Theorem 182

5 Operator theory

Definition 184. Suppose X and Y are Banach spaces; suppose $T \in \mathcal{B}(X,Y)$. Then there is a map $T^* \in \mathcal{B}(Y^*, X^*)$ called the *adjoint* (or *transpose*) of T given by $(T^*\varphi)(x) = \varphi(Tx)$ for $\varphi \in Y^*$ and $x \in X$.

Theorem 185. *Suppose X, Y, Z are Banach spaces; suppose $T \in \mathcal{B}(X,Y)$ and $S \in \mathcal{B}(Y,Z)$. Then*

1. $\|T^*\| = \|T\|$.
2. $T \mapsto T^*$ *is linear.*
3. $I_X^* = I_{X^*}$.
4. $(ST)^* = T^*S^*$.
5. T^* *is weak-*-weak-*-continuous.*
6. $T^{**} \restriction X = T$.

Proof. For convenience, given $\varphi \in X^*$ and $x \in X$, we write $\langle x, \varphi \rangle = \varphi(x)$.

1. Note that

$$\begin{aligned}
\|T^*\| &= \sup_{\substack{\varphi \in Y^* \\ \|\varphi\| \leq 1}} \\
&= \sup_{\substack{\varphi \in Y^* \\ \|\varphi\| \leq 1}} \sup_{\substack{x \in X \\ \|x\| \leq 1}} |T^*\psi(x)| \\
&= \sup_{\substack{x \in X \\ \|x\| \leq 1}} \sup_{\substack{\varphi \in Y^* \\ \|\varphi\| \leq 1}} |T^*\psi(x)| \\
&= \sup_{\substack{x \in X \\ \|x\| \leq 1}} \|Tx\| \\
&= \|T\|
\end{aligned}$$

by Hahn-Banach.

2. Suppose $a, b \in \mathbb{F}$ and $S, T \in \mathcal{B}(X, Y)$. Suppose $\varphi \in Y^*$ and $x \in X$. Then

$$\begin{aligned}((aS + bT)^*\varphi)(x) &= \varphi((aS + bT)x)\\ &= a\varphi(Sx) + b\varphi(Tx)\\ &= ((aS^* + bT^*)\varphi)(x)\end{aligned}$$

So $T \mapsto T^*$ is linear.

3. Suppose $\varphi \in X^*$. Then, for $x \in X$, we have

$$(I_X^*\varphi)(x) = \varphi(Ix) = \varphi(x) = (I_{X^*}\varphi)(x)$$

So $I_X^* = I_{X^*}$.

4. Suppose $w \in Z^*$ and $x \in X$. Then

$$((ST)^*w)(x) = w(S(Tx)) = (S^*w)(Tx) = (T^*S^*w)(x)$$

So $(ST)^* = T^*S^*$.

5. Note that $T^*\colon Y^* \to X^*$ is norm-continuous. Suppose now that $(\psi_\lambda : \lambda \in \Lambda)$ is a net in Y^* converging to ψ in the weak-* topology. The for all $x \in X$ we have

$$(T^*\psi_\lambda)(x) = \psi_\lambda(Tx) \to \psi(Tx) = (T^*\psi)(x)$$

So $(T\psi_\lambda : \lambda \in \Lambda) \xrightarrow{w^*} T^*\psi$, and T^* is weak-*-weak-*-continuous

6. Note that $T \in \mathcal{B}(X, Y)$ implies that $T^* \in B(Y^*, X^*)$ and $T^{**} \in B(X^{**}, Y^{**})$. Suppose $x \in X$ and $\psi \in Y^*$. Then

$$(T^{**}\widehat{x})(\psi) = \widehat{x}(T^*\psi) = (T^*\psi)(x) = \psi(Tx) = \widehat{Tx}(\psi)$$

which we colloquially interpret to mean $T^{**} \restriction X = T$.

□ Theorem 185

Let X be an n-dimensional Banach space with $n \in \mathbb{N}$; let $e_1, \ldots, e_n$ be a basis. Let Y be an m-dimensional Banach space with $m \in \mathbb{N}$; let $f_1, \ldots, f_m$ be a basis. Then X^* has dual basis $\varepsilon_1, \ldots, \varepsilon_n$ where

$$\varepsilon_j(e_i) = \begin{cases} 1 & i = j\\ 0 & i \neq j\end{cases}$$

and similarly Y^* has dual basis $\delta_1, \ldots, \delta_m$. Let $T \in \mathcal{B}(X, Y)$. Then T has matrix $[t_{ij}]$ and

$$Te_j = \sum_{i=1}^{m} t_{ij} f_i$$

Then $T^* \in \mathcal{B}(Y^*, X^*)$ has matrix $[s_{ij}]$, and

$$T^*\delta_j = \sum_{i=1}^{n} s_{ij}\varepsilon_i$$

But then

$$s_{ij} = (T^*\delta_j)(e_i) = \delta_j(Te_i) = t_{ji}$$

So the matrix of T is the transpose of that of T^*.

Proposition 186. *Suppose $T\colon X \to Y$ is linear. Then T is bounded if and only if T is weak-weak-continuous.*

Proof.

($\Longrightarrow$) Suppose T is bounded; suppose $(x_\alpha : \alpha \in \Lambda) \xrightarrow{w} x$. Then

$$\psi(Tx_\alpha) = (T^*\psi)(x_\alpha) \to (T^*\psi)(x) = \psi(Tx)$$

So $(Tx_\alpha : \alpha \in \Lambda) \xrightarrow{w} Tx$ is weak-weak-continuous.

($\Longleftarrow$) Suppose T is weak-weak-continuous. Then $\psi \circ T$ is continuous for all $\psi \in Y^*$. So, if $\psi \in Y^*$, then

$$\sup_{\|x\|\leq 1} |\psi(Tx)| = \sup_{\|x\|\leq 1} |(\psi \circ T)(x)| = \|\psi \circ T\| < \infty$$

But $(\psi \circ T)(x) = (T^*\psi)(x)$; so $\|T^*\psi\| = \|\psi \circ T\| < \infty$. (Notice that T^* is defined even if T is not bounded.)

Consider $\{ Tx : x \in X, \|x\| \leq 1 \} \subseteq Y \subseteq Y^{**}$. We then have

$$\begin{aligned} \sup_{\substack{x\in X \\ \|x\|\leq 1}} |\widehat{Tx}(\psi)| &= \sup_{\substack{x\in X \\ \|x\|\leq 1}} |\psi(Tx)| \\ &= \|\psi \circ T\| \\ &< \infty \end{aligned}$$

Then by Banach-Steinhaus we have

$$\|T\| = \sup_{\substack{x\in X \\ \|x\|\leq 1}} \|Tx\| = \sup_{\substack{x\in X \\ \|x\|\leq 1}} \|\widehat{Tx}\| < \infty$$

So T is bounded.

□ Proposition 186

Proposition 187. *$T\colon Y^* \to X^*$ is weak-*-weak-*-continuous if and only if there is $S \in \mathcal{B}(X, Y)$ such that $T = S^*$.*

TODO 1. *Conditions on T?*

Proof.

($\Longrightarrow$)

TODO 2. *This.*

($\Longleftarrow$) Part 5 of the previous theorem.

□ TODO 1

Example 188. Consider the inclusion map $i_X\colon X \to X^{**}$ given by $i_X(x) = \widehat{x}$. Then $i_X^*\colon X^{***} \to X^*$. If $\Phi \in X^{***}$ and $x \in X$, we have $i_X^*(\Phi)(x) = \Phi(i_X(x)) = \Phi(\widehat{x})$. So $i_X^*(\Phi) = \Phi \restriction X$.

We also have $i_{X^*}\colon X^* \to X^{***}$. Define $p = i_{X^*} \circ i_X^*\colon X^{***} \to X^{***}$; then

$$p(\Phi) = i_{x^*}(\Phi \restriction X) = \widehat{\Phi \restriction X} \in \widehat{X^*}$$

Also $i_X^* \circ i_{X^*}\colon X^* \to X^*$. For $\varphi \in X^*$ and $x \in X$ we have

$$(i_X^* i_{X^*}(\varphi))(x) = (i_{X^*}\varphi)(i_X(x)) = \widehat{\varphi}(x)$$

So $i_X^* i_{X^*} = I_{X^*}$. But then

$$p^2 = i_{X^*}(i_X^* i_{X^*}) i_X^* = i_{X^*} i_X^* = p$$

So p is a projection of norm 1.

$$\|p\| \leq \|i_{X^*}\| \|i_{X^*}\| = 1 \cdot 1 = 1$$

and $\mathrm{Ran}(p) = \mathrm{Ran}(i_{X^*}) = \widehat{X^*}$. So p projects X^{***} onto $\widehat{X^*}$.

5.1 Hilbert space adjoint

Proposition 189. *Suppose $\mathcal{H}$ is a Hilbert space and $[\cdot,\cdot]$ is a sesquilinear form which is bounded (i.e. $|[x,y]| \leq C\|x\|\|y\|$ for all $x,y \in \mathcal{H}$). Then there is a unique $B \in \mathcal{B}(\mathcal{H})$ such that $[x,y] = \langle x, By\rangle$.*

Proof. Fix $y \in \mathcal{H}$. Define $\Phi_y(x) = [x,y]$; then Φ_y is a linear functional, and

$$\|\Phi_y\| = \sup_{\|x\|\leq 1} |[x,y]| \leq C\|y\|$$

So $\Phi_y \in \mathcal{H}^*$. So there is a unique $z_y \in \mathcal{H}$ such that $[x,y] = \langle x, z_y\rangle$ and $\|z_y\| = \|\Phi_y\| \leq C\|y\|$. Define $By = z_y$; then $\|B\| \leq C$, and B is bounded.

To see linearity, suppose $y_1, y_2 \in \mathcal{H}$, $a, b \in \mathbb{F}$. Then for any $x \in \mathcal{H}$ we have

$$\begin{aligned}
\langle x, B(ay_1 + by_2)\rangle &= [x, ay_1 + by_2] \\
&= \overline{a}[x,y_1] + \overline{b}[x,y_2] \\
&= \overline{a}\langle x, By_1\rangle + \overline{b}\langle x, By_2\rangle \\
&= \langle x, aBy_1 + bBy_2\rangle
\end{aligned}$$

So

$$0 = \langle x, B(Ay_1 + by_2) - (aBy_1 + bBy_2)\rangle$$

for all $x \in \mathcal{H}$. So $B(ay_1 + by_2) = aBy_1 + bBy_2$, and B is linear. □ Proposition 189

Definition 190. If $\mathcal{H}$ is a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$, then the *Hilbert space adjoint of* T is T^* the unique element of $\mathcal{B}(\mathcal{H})$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x, y \in \mathcal{H}$.

Remark 191. To see that this exists, define $[x,y] = \langle Tx, y\rangle$; then by the above proposition there is a unique $B \in \mathcal{B}(\mathcal{H})$ such that $\langle Tx, y\rangle = \langle x, By\rangle$; then $B = T^*$.

Proposition 192.

1. *$\|T^*\| = \|T\|$.*

2. *$(aS + bT)^* = \overline{a}S^* + \overline{b}T^*$.*

3. *$(ST)^* = T^*S^*$.*

4. *$T^{**} = T$.*

Proof. Omitted. □ Proposition 192

Example 193. Consider $L^2(0,1)$ with $M_x(f) = xf$. What is $\|M_x\|$? Well,

$$\begin{aligned}
\|xf\|_2^2 &= \int_0^1 x^2|f(x)|^2 dx \\
&\leq \int_0^1 |f(x)|^2 dx \\
&= \|f\|^2
\end{aligned}$$

So $\|M_x\| \leq 1$. Define

$$\chi_n(x) = \begin{cases} 0 & 0 \leq x \leq 1 - \frac{1}{n} \\ \sqrt{n} & 1 - \frac{1}{n} \leq x \leq 1 \end{cases}$$

Then

$$\|x_n\|^2 = \int_0^1 |\chi_n|^2 = \int_{1-\frac{1}{n}}^1 n dx = 1$$

and

$$\begin{aligned}
\|M_x x_n\|^2 &= \int_{1-\frac{1}{n}}^{1} (x\sqrt{n})^2 dx \\
&= (n\frac{x^3}{3})|_{1-\frac{1}{n}}^{1} \\
&= \frac{n}{3}\left(1-\left(1-\frac{1}{n}\right)^3\right) \\
&= \frac{n}{3}\left(\frac{3}{n}-\frac{3}{n^2}+\frac{1}{n^3}\right) \\
&= 1-\frac{3}{n}+\frac{1}{3n^2}
\end{aligned}$$

M_x is injective since if $xf = xg$ almost everywhere then $f = g$ almost everywhere. But $1 \notin \mathrm{Ran}(M_x)$ because $x^{-1} \notin L^2(0,1)$. So it's not invertible, but has no kernel. Does it have eigenvalues?

Suppose $xf = \lambda f$ for some $\lambda \in \mathbb{C}$. Then $(x-\lambda)f = 0$ almost everywhere. But $x-\lambda \neq 0$ almost everywhere. So $f = 0$. So it has no eigenvalues.

Definition 194. We say $T \in \mathcal{B}(X,Y)$ is *bounded below* if there is $c > 0$ such that $\|Tx\| \geq c\|x\|$.

Example 195.

1. M_x is not bounded below: let
$$y_n(x)\begin{cases}\sqrt{n} & 0 \leq x \leq \frac{1}{n}\\ 0 & x > \frac{1}{n}\end{cases}$$
Then $\|y_n\| = 1$ but
$$\|xy_n\|^2 = \int_0^{\frac{1}{n}} (x\sqrt{n})^2 dx = n\frac{x^3}{3}|_0^{\frac{1}{n}} = \frac{1}{3n^2}$$

2. Let $D\colon \ell_1 \to \ell_1$ be $De_n = \frac{1}{n}e_n$. Then $D((x_1,x_2,x_3,\dots)) = (\frac{x_1}{1},\frac{x_2}{2},\frac{x_3}{3},\dots)$. Then D is injective but is not bounded below; it is also not surjective, as $(1,\frac{1}{4},\frac{1}{9},\dots) \in \ell^1$ but $(1,\frac{1}{2},\frac{1}{3},\dots) \notin \ell^1$.

3. Consider $\mathcal{H} = \ell_2$ with orthonormal basis $e_1, e_2, \dots$. Consider the shift $Se_n = e_{n+1}$. Then $S((x_1,x_2,x_3,\dots)) = (0,x_1,x_2,x_3,\dots)$. We have $\|Sx\| = \|x\|$, so this is an isometry, and in particular is injective. It is also therefore bounded below by 1. It is not surjective: $\mathrm{Ran}(S) = (\mathbb{C}e_1)^\perp$.

   We can consider its Hilbert space adjoint S^*:
$$\begin{aligned}
\langle x, S^*y\rangle &= \langle Sx, y\rangle \\
&= \langle (0,x_1,x_2,\dots),(y_1,y_2,\dots)\rangle \\
&= \sum_{n=1}^{\infty} x_n\overline{y_{n+1}} \\
&= \langle (x_1,x_2,\dots),(y_2,y_3,y_4,\dots)\rangle
\end{aligned}$$
   So $S^*((y_1,y_2,y_3,\dots)) = (y_2,y_3,y_4,\dots)$. i.e. $S^*e_1 = 0$ and $S^*e_{n+1} = e_n$ for $n \geq 1$. Also $\ker(S^*) = \mathbb{C}e_1$, but S^* is surjective.

   Now, neither S nor S^* is invertible. But
$$\begin{aligned}
(S^*S)((x_1,x_2,x_3,\dots)) &= S^*(0,x_1,x_2,x_3,\dots) \\
&= (x_1,x_2,x_3,\dots)
\end{aligned}$$
   So $S^*S = I$. But
$$\begin{aligned}
(SS^*)((x_1,x_2,x_3,\dots)) &= S((x_2,x_3,x_4,\dots)) \\
&= (0,x_2,x_3,x_4,\dots)
\end{aligned}$$
   So $SS^* \neq I$.

Lemma 196. *Suppose $T \in \mathcal{B}(X,Y)$. Then $(\ker(T))^\perp = \overline{\operatorname{Ran}(T^*)}$ (where the closure is taken in the weak-* topology) and $\ker(T^*) = (\operatorname{Ran}(T))^\perp$.*

Proof. We first show $\ker(T^*) = (\operatorname{Ran}(T))^\perp$. Note that for $\psi \in Y^*$, we have

$$\begin{aligned}\psi \in (\operatorname{Ran}(T))^\perp &\iff \psi(Tx) = 0 \text{ for all } x \in X\\ &\iff (T^*\psi)(x) = 0 \text{ for all } x \in X\\ &\iff T^*\psi = 0\\ &\iff \psi \in \ker(T^*)\end{aligned}$$

Now note that $(\operatorname{Ran}(T^*))^\perp = \ker(T^{**})$, and

$$\begin{aligned}(\operatorname{Ran}(T^*))_\perp &= \{\, x \in X : x \perp \operatorname{Ran}(T^*) \,\}\\ &= (\ker(T^{**})) \cap X\\ &= \ker(T)\end{aligned}$$

So $(\ker(T))^\perp = ((\operatorname{Ran}(T^*))_\perp)^\perp$ which is the weak-*-closure of $\operatorname{Ran}(T^*)$. □ Lemma 196

Remark 197. If $T \in \mathcal{B}(X,Y)$, then the *norm* closure of $\operatorname{Ran}(T^*)$ may not be weak-*-closed. Consider $D\colon \ell_1 \to \ell_1$ by $De_n = \frac{1}{n}e_n$. Then $D^*\colon \ell_\infty \to \ell_\infty$ is given by $D^*((x_1,x_2,x_3,\dots)) = (\frac{x_1}{1}, \frac{x_2}{2}, \dots)$. So

$$c_0 = \operatorname{span}\left\{ e_1, \frac{e_2}{2}, \frac{e_3}{3}, \dots \right\} \subseteq \overline{\operatorname{Ran}(D^*)} \subseteq c_0$$

(where here we use the norm closure). So c_0 is the norm-closure of $\operatorname{Ran}(D^*)$. But by Goldstine we have that the weak-*-closure of c_0 is ℓ_∞. So the norm-closure of $\operatorname{Ran}(D^*)$ is not weak-*-closed.

Proposition 198. *Suppose $T \in \mathcal{B}(X,Y)$. Then the following are equivalent:*

1. *T is invertible*
2. *T is bijective*
3. *T is bounded below and has dense range*
4. *T and T^* are bounded below*
5. *T^* is invertible*

Proof.

(1) $\implies$ **(2)** Trivial.

(2) $\implies$ **(1)** Banach isomorphism theorem.

(1) $\implies$ **(3)** Suppose $T^{-1}T = I_X$. Then

$$\|x\| = \|T^{-1}(Tx)\| \le \|T^{-1}\|\|Tx\|$$

So

$$\|Tx\| \ge \frac{1}{\|T^{-1}\|}\|x\|$$

and T is bounded below. Also, T is surjective; so T has dense range.

(3) $\implies$ **(2)** If $x \neq 0$ then $\|Tx\| \ge c\|x\| > 0$; so T is injective. Let $y \in Y = \overline{\operatorname{Ran}(T)}$. Find $(x_n : n \in \mathbb{N})$ in X such that $(Tx_n : n \in \mathbb{N}) \to y$. Let $y_n = Tx_n$. Then $(y_n : n \in \mathbb{N})$ converges, and is thus Cauchy. But

$$\|x_n - x_m\| \le \frac{1}{c}\|y_n - y_m\|$$

Take $\varepsilon > 0$. Then there is $N \in \mathbb{N}$ such that for all $n, m \ge N$ we have $\|y_n - y_m\| < c\varepsilon$; then for all $n, m \ge N$ we have $\|x_n - x_m\| < \varepsilon$. So $(x_n : n \in \mathbb{N})$ is Cauchy, and thus converges to x. But then

$$y = \lim_{n\to\infty} Tx_n = Tx$$

So T is surjective.

(1) $\implies$ (5) T is invertible so $T^{-1}T = I_X$ and $TT^{-1} = I_Y$. Taking adjoints, we find that $T^*(T^{-1})^* = I_X^* = I_{X^*}$ and $(T^{-1})^*T^* = I_Y^* = I_{Y^*}$. So $(T^{-1})^* = (T^*)^{-1}$.

(5) $\implies$ (4) Suppose T^* is invertible. Then, by previous directions, we have that T^* is bounded below and that T^{**} is inverible and bounded below. But $T = T^{**} \restriction X$; so T is bounded below.

(4) $\implies$ (3) Suppose T and T^* are bounded below. Then T is bounded below, and $\ker(T^*) = \{ 0 \}$. But by the lemma we have $(\operatorname{Ran}(T))^\perp = \ker(T^*)$. So $\overline{\operatorname{Ran}(T)} = Y$.

☐ Proposition 198

Definition 199. If $T \in \mathcal{B}(X)$, we define the *spectrum* of T is

$$\sigma(T) = \{ \lambda \in \mathbb{C} : \lambda I - T \text{ is not invertible} \}$$

The *resolvent* of T is $\rho(T) = \mathbb{C} \setminus \sigma(T)$. The *resolvent function* $R(T, \lambda) = (\lambda I - T)^{-1}$ for $\lambda \in \rho(T)$. The *point spectrum* is $\sigma_p(T)$ the set of eigenvalues of T; i.e. $\sigma_p(T) = \{ \lambda : \ker(\lambda I - T) \neq \emptyset \}$. The *approximate point spectrum* is $\pi(T) = \{ \lambda : \lambda I - T \text{ is not bounded below} \}$. The *compression spectrum* is $\gamma(T) = \{ \lambda : \overline{\operatorname{Ran}(\lambda I - T)} \neq X \}$.

Remark 200. By proposition we have that $\sigma(T) = \pi(T) \cup \gamma(T)$.

We let $\mathcal{B}(X)^{-1}$ denote the set of invertible operators in X.

Proposition 201. *$B(X)^{-1}$ is open and contains $b_1(I_X)$.*

Proof. If $A \in \mathcal{B}(X)$ with $\|A\| \leq 1$, we wish to show that $I - A \in b_1(I)$ is invertible. Recall that in $\mathbb{C}$ if $|x| < 1$ we have

$$\frac{1}{1-x} = 1 + x + x^2 + \dots$$

Let

$$B = \sum_{n=0}^{\infty} A^n \in \mathcal{B}(X)$$

This converges because

$$\sum_{n=0}^{\infty} \|A^n\| \leq \sum_{n=0}^{\infty} \|A\|^n = \frac{1}{1 - \|A\|} < \infty$$

Then

$$(I - A)B = \lim_{k \to \infty} ((I - A)(I + A + \dots + A^k)) = \lim_{k \to \infty} (I - A^{k+1}) = I$$

By continuity, since $I - A$ commutes with the partial sums, it commutes with B, and $B(I - A) = (I - A)B = I$. So $I - A$ is invertible.

If $T \in \mathcal{B}(X)^{-1}$ and $\|A\| < \frac{1}{\|T^{-1}\|}$ then $T - A = T(I - T^{-1}A)$. Then $\|T^{-1}A\| \leq \|T^{-1}\|\|A\| < 1$. So $(T - A)^{-1} = (I - T^{-1}A)^{-1}T^{-1}$. So

$$b_{\frac{1}{\|T^{-1}\|}} \subseteq B(X)^{-1}$$

So $B(X)^{-1}$ is open. ☐ Proposition 201

Proposition 202. *If $T \in \mathcal{B}(X)$ then $\rho(T)$ is open and $\sigma(T) \subseteq \overline{b_{\|T\|}(0)}$.*

Proof. $\mathcal{B}(X)^{-1}$ is open and $f\colon \mathbb{C} \to \mathcal{B}(X)$ given by $f(\lambda) = \lambda I - T$ is norm-continuous. Thus $\rho(T) = f^{-1}(\mathcal{B}(X)^{-1})$ is open. If $|\lambda| > \|T\|$, then $\lambda I - T = \lambda(I - \lambda^{-1}T)$. So

$$\|\lambda^{-1}T\| = \frac{\|T\|}{|\lambda|} < 1$$

So

$$(\lambda I - T)^{-1} = \lambda^{-1}(I - \lambda^{-1}T)^{-1} = \lambda^{-1}\sum_{n=0}^{\infty}(\lambda^{-1}T)^n = \sum_{n=0}^{\infty} T^n \lambda^{-n-1}$$

☐ Proposition 202

Proposition 203. *The map $\mathcal{B}(X)^{-1} \to \mathcal{B}(X)^{-1}$ given by $T \mapsto T^{-1}$ is continuous.*

Proof. Suppose $T_0 \in \mathcal{B}(X)^{-1}$; suppose $\|A\| < \frac{1}{\|T_0^{-1}\|}$. Then

$$\begin{aligned}(T_0 + A)^{-1} &= (T_0(I + T_0^{-1}A))^{-1} \\ &= (I + T_0^{-1}A)^{-1}T_0^{-1} \\ &= \sum_{n=0}^{\infty}(-T_0^{-1}A)^n T_0^{-1} \\ &= T_0^{-1} + \sum_{n=1}^{\infty}(-T_0 A)^n T_0^{-1}\end{aligned}$$

So

$$\begin{aligned}\|(T_0 + A)^{-1} - T_0^{-1}\| &= \left\|\sum_{n=1}^{\infty}(-T_0^{-1}A)^n T_0^{-1}\right\| \\ &\le \sum_{n=1}^{\infty}(\|T_0^{-1}\|\|A\|)^n \|T_0^{-1}\| \\ &= \frac{\|T_0^{-1}\|^2\|A\|}{1 - \|T_0^{-1}\|\|A\|} \\ &\to 0 \text{ as } \|A\| \to 0\end{aligned}$$

So the map is continuous at T_0. ☐ Proposition 203

Example 204.

1. Let $X = L^p(0,1)$ for $1 \le p < \infty$. Let $h \in L^\infty(0,1)$ where

$$\|h\|_\infty = \operatorname{ess\,sup}|h| = \sup\{\, r : m(\{\, x : |h(x)| \ge r \,\}) > 0 \,\}$$

Let $M_h f = fh$ for $f \in L^p(0,1)$. Then

$$\begin{aligned}\|M_h f\|_p^p &= \int |hf|^p dm \\ &\le \int \|h\|_\infty^p |f|^p dm \\ &= \|h\|_\infty^p \|f\|_p^p\end{aligned}$$

So $\|M_h\| \le \|h\|_\infty$. Let $f = \chi_A$. Then

$$\|f\|_p = \left(\int \chi_A^p\right)^{\frac{1}{p}} = m(A)^{\frac{1}{p}}$$

and

$$\|fh\|_p = \left(\int (|h|\chi_A)^p\right)^{\frac{1}{p}} \ge r\left(\int \chi_A\right)^{\frac{1}{p}} = r\|f\|_p$$

What is $\sigma(M_h)$? Well, if $h, k \in L^\infty(0,1)$, then

$$M_h M_k f = M_h k f = hkf = M_{hk} f$$

We look at the case of $h = x$. So if $\lambda \notin [0,1]$ then $\frac{1}{x-\lambda} \in L^\infty(0,1)$, so $(M_x - \lambda I)M_{\frac{1}{x-\lambda}} = 1$, and $\lambda \notin \sigma(M_x)$. So $\sigma(M_x) \subseteq [0,1]$.

On the other hand, for $\varepsilon > 0$, let $f_\varepsilon = \chi_{(\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon)}$. Then

$$\|M_{x-\frac{1}{2}} f_\varepsilon\|_p = \left\|\left(x - \frac{1}{2}\chi_{(\frac{1}{2}-\varepsilon,\frac{1}{2}+\varepsilon)}\right)\right\| < \|\varepsilon f_\varepsilon\|_p = \varepsilon\|f_\varepsilon\|_p$$

But this is not bounded below. So $\frac{1}{2} \in \sigma(M_x)$. Similarly, we have that $y \in \sigma(M_x)$ for any $y \in [0,1]$.

Consider now arbitrary $h \in C[0,1]$. We let $h([0,1]) = X = \text{Ran}(h)$. If $\lambda \notin \text{Ran}(h)$ then

$$\frac{1}{h-\lambda} \in C[0,1]$$

and

$$(M_h - \lambda I)M_{\frac{1}{h-\lambda}} = I$$

so $\lambda \notin \sigma(M_h)$. If $h(x_0) = \lambda$, then for all $\varepsilon > 0$ there is $\delta > 0$ such that $h^{-1}(b_\varepsilon(\lambda)) \supseteq b_\delta(x_0)$; then if $f_\varepsilon = \chi_{(x_0-\delta, x_0+\delta)}$, we have $\|M_{h-\lambda}f_\varepsilon\|_p \leq \varepsilon\|f_\varepsilon\|_p$. So $\sigma(M_h) = h([0,1])$.

Consider now arbitrary $h \in L^\infty(0,1)$. Define

$$\text{ess Ran}(h) = \{\, z \in \mathbb{C} : m(h^{-1}(b_\varepsilon(z))) > 0 \text{ for all } \varepsilon > 0 \,\}$$

If $\lambda \notin \text{ess Ran}(h)$ then there is $\varepsilon > 0$ such that $m(h^{-1}(b_\varepsilon(z))) = 0$. Then

$$\left|\frac{1}{h-z}\right| \leq \frac{1}{\varepsilon}$$

almost everywhere, so

$$(M_h - zI)M_{\frac{1}{h-z}} = I$$

and $z \notin \sigma(M_h)$. Conversely, if $z \in \text{ess Ran}(h)$, we let $f_\varepsilon = \chi_{h^{-1}(b_\varepsilon(z))} \neq 0$. Then $\|M_{h-z}f_\varepsilon\| \leq \varepsilon\|f_\varepsilon\|$ is not bounded below, and is thus not invertible. So $\sigma(M_h) = \text{ess Ran}(h)$.

We consider now the Banach space adjoint to M_h. If $f \in L^p$ and $g \in L^q$ (where $\frac{1}{p} + \frac{1}{q} = 1$), then

$$\begin{aligned} \langle M_h f, g\rangle &= \int hfg\,dm \\ &= \int f(hg)\,dm \\ &= \langle f, M_h g\rangle \end{aligned}$$

So the Banach space adjoint $M_h^* = M_h$ on $L^q(0,1)$. To see the Hilbert space adjoint, note that if $f, g \in L^2(0,1)$, then

$$\begin{aligned} \langle M_h f, g\rangle &= \int (hf)\overline{g}\,dm \\ &= \int f(\overline{\overline{h}g})\,dm \\ &= \langle f, \overline{h}g\rangle \end{aligned}$$

So $M_h M_{\overline{h}} = M_{|h|^2} = M_{\overline{h}} M_h$. So M_h commutes with M_h^*, and it is *normal*.

2. Consider the unilateral shift on ℓ^2: $Se_n = e_{n+1}$ for $n \geq 0$. i.e. $S((x_0, x_1, \dots)) = (0, x_0, x_1, \dots)$. We have the backwards shift $S^*((x_0, x_1, x_2, \dots)) = (x_1, x_2, \dots)$. Then $\|S\| = 1 = \|S^*\|$; so $\sigma(S) \subseteq \mathbb{D}$.

   S is not invertible because $\text{Ran}(S) \perp \mathbb{C}e_0$. So $S^*e_0 = 0$, and 0 is an eigenvalue of S^*. On the other hand, clearly S has no eigenvalues, since if $Sx = \lambda x$, then $\|x\| = \|Sx\| = \|\lambda x\| = |\lambda|\|x\|$, and $|\lambda| = 1$; but then $\lambda x_0 = 0$, and $\lambda x_1 = x_0 = 0$, and so on, so $x = 0$, a contradiction.

   Can we have $S^*x = \lambda x$? We need $x_{n+1} = \lambda x_n$ for $n \geq 0$. i.e.

$$x_n = \lambda x_{n-1} = \lambda^2 x_{n-2} = \dots = \lambda^n x_0$$

   So $x = x_0(1, \lambda, \lambda^2, \dots)$. So $S^*(1, \lambda, \lambda^2, \lambda^3, \dots) = (\lambda, \lambda^2, \lambda^3, \lambda^4, \dots) = \lambda(1, \lambda, \lambda^2, \dots)$. If $|\lambda| < 1$, then $x_\lambda = (1, \lambda, \lambda^2, \dots) \in \ell_2$. So $\sigma_p(S^*) = \mathbb{D} = \{\, \lambda : |\lambda| < 1 \,\}$. So $\sigma(S^*) \supseteq \overline{\mathbb{D}}$. So $\sigma(S^*) = \overline{\mathbb{D}}$.

Returning to S, note that if $|\lambda| \le 1$, then $(S - \lambda I)^* = S^* - \overline{\lambda I}$ is not invertible. So $S - \lambda I$ is not invertible. If $|\lambda < 1|$, then

$$\langle (S - \lambda I)x, x_{\overline{\lambda}} \rangle = \langle x, (S^* - \overline{\lambda} I)x_{\overline{\lambda}} \rangle = \langle x, 0 \rangle = 0$$

So $\mathrm{Ran}(S - \lambda I) \perp \mathbb{C}x_{\overline{\lambda}}$; so $\sigma(S) = \overline{\mathbb{D}}$. If $|\lambda| = 1$, let

$$x_n = \frac{1}{\sqrt{n}}(1, \overline{\lambda}, \overline{\lambda}^2, \dots, \overline{\lambda}^{n-1}, 0, 0, \dots)$$

$$\|x_n\|^2 = \frac{1}{n} \sum_{i=0}^{n-1} |\overline{\lambda}^i|^2 = \frac{n}{n} = 1$$

But

$$Sx_n = \frac{1}{\sqrt{n}}(0, 1, \overline{\lambda}, \overline{\lambda}^2, \dots, \overline{\lambda}^{n-2}, \overline{\lambda}^{n-1}, 0)$$

and

$$\lambda x_n = \frac{1}{\sqrt{n}}(\lambda, 1, \overline{\lambda}, \overline{\lambda}^2, \dots, \overline{\lambda}^{n-2}, 0, \dots)$$

So

$$(S - \lambda I)x_n = \frac{1}{\sqrt{n}}(-\lambda, 0, 0, \dots, 0, \overline{\lambda}^{n-1}, 0, \dots)$$

So

$$\|(S - \lambda I)x_n\| = \sqrt{\frac{2}{n}} \to 0$$

is not bounded below.

Definition 205. Suppose $\Omega \subseteq \mathbb{C}$ is open and X is a Banach space. Suppose $f \colon \Omega \to X$. We say f is *strongly analytic* if for all $z_0 \in \Omega$ there exist $x_0, x_1, x_2, \dots, \in X$ such that

$$f(z_0 + w) = \sum_{n=0}^{\infty} x_n w^n$$

converges uniformly for all $|w| \le r$ for all $r > 0$. We say f is *weakly anaytic* if for all $\varphi \in X^*$ we have that $\varphi \circ f \colon \Omega \to \mathbb{C}$ is analytic.

Exercise 206 (Bonus problem). Prove that if f is weakly analytic then it is strongly analytic. Hint:

1. Show
$$\left\{ \varphi\left(\frac{f(z_0 + w) - f(z_0)}{w} \right) : |w| \le r \right\}$$
is bounded.

2. Show f is continuous.

3. For $n \ge 0$, set
$$x_n = \frac{1}{2\pi i} \int_0^{2\pi} f(z_0 + r\exp(i\theta)) \exp(-in\theta) d\theta$$
(as a Riemann integral).

4. Show
$$f(z_0 + w) = \sum_{n=0}^{\infty} x_n w^n$$
for $|w| \le r$.

Theorem 207. *Suppose $T \in \mathcal{B}(X)$; suppose $\lambda, \mu \in \rho(T)$. Then*

1.
$$\frac{R(T,\lambda)-R(T,\mu)}{\lambda-\mu}=-R(T,\lambda)R(T,\mu)$$

2. $\lambda \mapsto R(T,\lambda)$ *is strongly analytic on* $\rho(T)$*.*

3.
$$\lim_{|\lambda|\to\infty} R(T,\lambda)=0$$

Proof.

1. Note that
$$(R(T,\lambda)-R(T,\mu))(\lambda I-T)(\mu I-T)=(\mu I-T)-(\lambda I-T)=(\mu-\lambda)I$$
Multiplying by $R(T,\lambda)R(T,\mu)$, we see
$$\frac{R(T,\lambda)-R(T,\mu)}{\lambda-\mu}=-R(T,\lambda)R(T,\mu)$$

2. Note that
$$\frac{d}{d\lambda}(R(T,\lambda))|_{\lambda=\mu}=\lim_{\lambda\to\mu}\frac{R(T,\lambda)-R(T,\mu)}{\lambda-\mu}=-R(T,\lambda)^2$$
So
$$\frac{d}{d\lambda}\left(\frac{1}{\lambda-z}\right)=\frac{-1}{(\lambda-z)^2}$$
If $\lambda_0\in\rho(T)$, then
$$(\lambda_0+w)I=(\lambda_0 I-T)(\lambda_0 I-T)^{-1}((\lambda_0 I-T)+wI)=(\lambda_0 I-T)(I+w(\lambda_0 I-T)^{-1})$$
If $|w|<\frac{1}{\|(\lambda_0 I-T)^{-1}\|}$, then
$$R(T,\lambda_0+w)=(\lambda_0 I-T)^{-1}\sum_{n=0} t\infty(-(\lambda_0 I-T)^{-1})^n w^n$$
which converges uniformly for $|w|\le r<\frac{1}{\|(\lambda_0 I-T)^{-1}\|}$.

3. Suppose $|\lambda|>\|T\|$. Then
$$R(T,\lambda)=\sum_{n=0}^{\infty}T^n\lambda^{-n-1}$$
So
$$\begin{aligned}\|R(T,\lambda)\| &\le \sum_{n=0}^{\infty}\|T^n\||\lambda|^{-n-1}\\ &\le \sum\|T\|^n|\lambda|^{-n-1}\\ &= \frac{\frac{1}{|\lambda|}}{1-\frac{\|T\|}{|\lambda|}}\\ &= \frac{1}{|\lambda|-\|T\|}\\ &\to 0 \text{ as } |\lambda|\to\infty\end{aligned}$$

□ Theorem 207

Theorem 208. *Suppose* $T\in\mathcal{B}(X)$*. Then* $\sigma(T)\neq\emptyset$*.*

Proof. If $\sigma(T)$ were empty, then $R(T,\lambda)$ is an entire function. But $\|R(T,\lambda)\| \to 0$ as $|\lambda| \to \infty$. So it is a bounded, entire function. Take $\varphi \in \mathcal{B}(X)^*$. Then $\varphi \circ R(T,\lambda)$ is a bounded, scalar-valued, entire functino. Thus it is constant by Liouville's theorem. If $R(T,\lambda)$ were not constant, then by Hahn-Banahc we have φ such that $\varphi \circ R(T,\lambda)$ is not constant, a contradiction. So $R(T,\lambda)$ is constant. But $R(T,\lambda) \to 0$ as $|\lambda| \to \infty$. So $R(T,\lambda) = 0$. This is absurd. So $\sigma(T) \neq \emptyset$. □ Theorem 208

Proposition 209. *If $\lambda \in \rho(T)$ for $T \in \mathcal{B}(X)$ and $\operatorname{dist}(\lambda, \sigma(T)) = r$, then $\|(\lambda I - T)^{-1}\| \geq \frac{1}{r}$.*

Proof. Pick $\lambda_0 \in \sigma(T)$ such that $|\lambda - \lambda_0| = r$. Now, $(\lambda_0 I - T)(\lambda I - T)^{-1}$ is not invertible. But

$$((\lambda_0 - \lambda)I + (\lambda I - T))(\lambda I - T)^{-1} = (\lambda_0 - \lambda)(\lambda I - T)^{-1} + I$$

and $b_1(I) \subseteq \mathcal{B}(X)^{-1}$. So $\|(\lambda_0 - \lambda)(\lambda I - T)^{-1}\| \geq 1$. So $\|(\lambda I - T)^{-1}\| \geq \frac{1}{|\lambda_0 - \lambda|} = \frac{1}{r}$. □ Proposition 209

Corollary 210. *$\partial\sigma(T) \subseteq \pi(T)$; i.e. λ_0 in the boundary of $\sigma(T)$ is an approximate eigenvalue.*

Proof. We show $\lambda_0 I - T$ is not bounded below. Fix $\varepsilon > 0$. Pick $\lambda \in \rho(T)$ such that $|\lambda - \lambda_0| < \varepsilon$. Then $\|(\lambda I - T)^{-1}\| > \frac{1}{\varepsilon}$. Find x with $\|x\| = 1$ such that $\|(\lambda I - T)^{-1}x\| > \frac{1}{\varepsilon}$. Let $y = (\lambda I - T)^{-1}x$. Then

$$\begin{aligned}
\|(\lambda_0 I - T)y\| &= \|(\lambda_0 - \lambda)y + (\lambda I - T)y\| \\
&\leq \|(\lambda_0 - \lambda)y\| + \|x\| \\
&< \varepsilon\|y\| + \varepsilon\|y\| \\
&= 2\varepsilon\|y\|
\end{aligned}$$

So $\lambda_0 I - T$ is not bounded below. □ Corollary 210

5.2 Spectral mapping theorem for rational functions

If $p \in \mathbb{C}[z]$ is a polynomial, say $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ and $T \in \mathcal{B}(X)$, we define $p(T) = a_0 I + a_1 T + a_2 T^2 + \cdots + a_n T^n$. The map $\rho_T \colon \mathbb{C}[z] \to \mathcal{B}(X)$ given by $\rho_T(p) = p(T)$ is a homomorphism. If $q \in \mathbb{C}[z]$ has no roots in $\sigma(T)$, say $q(z) = b(z - \beta_1)(z - \beta_2)\ldots(z - \beta_m)$, then $q(T) = b(T - \beta_1 I)(T - \beta_2 I)\ldots(T - \beta_m I)$ is invertible. We can then define $\left(\frac{p}{q}\right)(t) = p(T)q(T)^{-1}$. If we set $\operatorname{Rat}(\sigma(T))$ to be the set of rationa $\frac{p}{q}$ such that q has no roots in $\sigma(T)$, then $\rho_T \colon \operatorname{Rat}(\sigma(T)) \to \mathcal{B}(X)$ given by $\rho_T\left(\frac{p}{q}\right) = p(T)q(T)^{-1}$ is well-defined and a homomorphism.

Theorem 211 (Spectral mapping theorem—rational case)**.** *Suppose $T \in \mathcal{B}(X)$ and $\frac{p}{q} \in \operatorname{Rat}(\sigma(T))$, then $\sigma(f(T)) = f(\sigma(T))$.*

Proof. Write $f = \frac{p}{q}$ with $\gcd(p,q) = 1$; factor $q(z) = b(z - \beta_1)\ldots(z - \beta_m)$. If $\lambda \in \mathbb{C}$, then

$$f(z) - \lambda = \frac{p}{q} - \lambda = \frac{p_\lambda}{q} = \frac{a(z - \alpha_1)\ldots(z - \alpha_n)}{b(z - \beta_1)\ldots(z - \beta_n)}$$

Then $f(T) - \lambda I = p_\lambda(T)q(T)^{-1}$ is invertible if and only if $p_\lambda(T)$ is invertible. But $p_\lambda(T) = a(T - \alpha_1 I)\ldots(T - \alpha_n I)$ is invertible if and only if $\alpha_1, \ldots, \alpha_n \in \rho(T)$. i.e.

$$\begin{aligned}
\lambda \in \sigma(f(T)) &\iff f(T) - \lambda I \text{ is not invertible} \\
&\iff p_\lambda(T) \text{ is not invertible} \\
&\iff \exists i(\alpha_i \in \sigma(T)) \\
&\iff \exists \alpha(\alpha \in \sigma(T) \land p_\lambda(\alpha) = 0) \\
&\iff \exists \alpha(\alpha \in \sigma(T) \land f(\alpha) = \lambda) \\
&\iff \lambda \in f(\sigma(T))
\end{aligned}$$

□ Theorem 211

If $\lambda \notin f(\sigma(T))$, then $\frac{p(z)}{q(z)} - \lambda = \frac{p_\lambda(z)}{q(z)}$ is invertible in $\mathrm{Rat}(\sigma(T))$ as

$$\frac{1}{f(z) - \lambda} = \frac{q(z)}{p_\lambda(z)}$$

So $(f(T) - \lambda I)^{-1} = q(T)p_\lambda(T)^{-1}$; so one direction is easy.

Definition 212. If $T \in \mathcal{B}(X)$, we define the *spectral radius* of T to be $\mathrm{spr}(T) = \sup\{\, |\lambda| : \lambda \in \sigma(T)\,\}$.

We know $\mathrm{spr}(T) \le \|T\|$. Now, if $\lambda > \|T\|$, then

$$R(\lambda, T) = (\lambda I - T)^{-1} = \sum_{n=0}^{\infty} T^n \lambda^{-n-1}$$

But $R(\lambda, T)$ is analytic on $\{\, \lambda : |\lambda| > \mathrm{spr}(T)\,\}$.

Theorem 213 (Spectral radius formula)**.** *We have*

$$\mathrm{spr}(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} = \inf_{n\ge 1} \|T^n\|^{\frac{1}{n}}$$

Proof. The spectral mapping theorem shows that $\sigma(T^n) = \sigma(T)^n$. Thus $\mathrm{spr}(T) = \mathrm{spr}(T^n)^{\frac{1}{n}} \le \|T^n\|^{\frac{1}{n}}$ for all $n \ge 1$. So

$$\mathrm{spr}(T) \le \inf_{n\ge 1} \|T^n\|^{\frac{1}{n}}$$

But $R(\lambda, T)$ is analytic on $\{\, \lambda : |\lambda| > \mathrm{spr}(T)\,\}$. So if $\varphi \in \mathcal{B}(X)^*$, then $\varphi(R(\lambda, T))$ is an analytic scalar function on the same annulus. For $|\lambda| > \|T\|$, we have

$$\begin{aligned}
\varphi(R(\lambda, T)) &= \varphi\left(\sum_{n=0}^{\infty} T^n \lambda^{-n-1}\right) \text{ (which converges absolutely)} \\
&= \sum_{n=0}^{\infty} \varphi(T^n) \lambda^{-n-1}
\end{aligned}$$

where the latter is sum is the Laurent expansion of $\varphi(R(\lambda, T))$ on $\{\, \lambda : |\lambda| > \|T\|\,\}$. This is analytic on a bigger annulus, namely $\{\, \lambda : |\lambda| > \mathrm{spr}(T)\,\}$. So, by complex analysis, this converges in $\{\, \lambda : |\lambda| > \mathrm{spr}(T)\,\}$. In particular, if $|\lambda| = t > \mathrm{spr}(T)$, then $|\varphi(T^n)\lambda^{-n-1}| = |\varphi(T^n)|t^{-n-1} \to 0$. So

$$\sup_{n\ge 0} \frac{|\varphi(T^n)|}{t^{n+1}} < \infty$$

But this holds for all $\varphi \in \mathcal{B}(X)^*$. So, by Banach-Steinhaus, we have

$$\sup_{n\ge 0} \left\| \frac{T^n}{t^{n+1}} \right\| = C < \infty$$

So $\|T^n\|^{\frac{1}{n}} \le (Ct^{n+1})^{\frac{1}{n}} = C^{\frac{1}{n}}$. So

$$\limsup_{n\to\infty} \|T^n\|^{\frac{1}{n}} \le t$$

Su

$$\begin{aligned}
\limsup \|T^n\|^{\frac{1}{n}} &\le \mathrm{spr}(T) \\
&\le \inf \|T^n\|^{\frac{1}{n}} \\
&\le \liminf \|T^n\|^{\frac{1}{n}} \\
&\le \limsup \|T^n\|^{\frac{1}{n}}
\end{aligned}$$

So $\lim \|T^n\|^{\frac{1}{n}} = \inf \|T^n\|^{\frac{1}{n}} = \mathrm{spr}(T)$. □ Theorem 213

5.3 Compact operators

Definition 214. We say $T \in \mathcal{B}(X, Y)$ is *compact* if $\overline{Tb_1(X)}$ is compact in Y. We write $\mathcal{K}(X, Y)$ for the set of compact operators in $\mathcal{B}(X, Y)$; likewise, we write $\mathcal{K}(X)$ for the set of compact operators in $\mathcal{B}(X)$.

Example 215.

1. If F has finite rank, then it is compact because $\overline{Fb_1(X)} \subseteq \overline{b_{\|F\|}(Y)} \cap \mathrm{Ran}(F)$ is compact by the Heine-Borel theorem.

2. Let $X = \ell_p$ for $1 \leq p < \infty$. Let $(d_n : n \in \mathbb{N}) \in \ell_\infty$. Let $D((x_1, x_2, x_3, \dots)) = (d_1 x_1, d_2 x_2, d_3 x_3, \dots)$ a "diagonal" operator. Then
$$\|D\| = \sup_{n \geq 1} |d_n|$$
Suppose
$$\limsup d_n > 0$$
Say we can find $|d_{n_i}| \geq r$ with $n_1 < n_2 < \dots$. Then $De_{n_i} = d_{n_i} e_{n_i} \in Db_1(\ell_p)$, so D is not compact.

   Suppose on the other hand that
$$\lim_{n \to \infty} d_n = 0$$
Claim 216. *D is compact.*

   Proof. Let $D_N((x_1, x_2, \dots) = (d_1 x_1, \dots, d_N x_N, 0, 0, \dots)$. Then D_N has rank N and
$$\|D - D_N\| = \sup_{n > N} |d_n| \to 0$$
So
$$D = \lim_{N \to \infty} D_N$$
The following proposition will show that the compact operators form a closed set, which then proves the claim. □ Claim 216

Proposition 217. *$\mathcal{K}(X, Y)$ is a $\mathcal{B}(Y)$-$\mathcal{B}(X)$ bimodule; i.e. for $K, L \in \mathcal{K}(X, Y)$, $S \in \mathcal{B}(Y)$, and $T \in \mathcal{B}(X)$, we have*
$$aK + bL \in \mathcal{K}(X, Y)$$
$$SKT \in \mathcal{K}(X, Y)$$
Furthermore, $\mathcal{K}(X, Y)$ is norm-closed. In particular, $\mathcal{K}(X)$ is a closed ideal of $\mathcal{B}(X)$.

Proof. Let $\mathcal{C}_1 = \overline{Kb_1(X)}$; let $\mathcal{C}_2 = \overline{Lb_1(X)}$. Then $\mathcal{C}_1, \mathcal{C}_2$ are compact. Consider
$$f : \mathcal{C}_1 \times \mathcal{C}_2 \to Y$$
$$(c_1, c_2) \mapsto ac_1 + bc_2$$
Then f is continuous, so its image is compact. So $aKx + bLx \in f(\mathcal{C}_1 \times \mathcal{C}_2)$ for all $\|x\| \leq 1$. So
$$\overline{(aK + bL)(b_1(X))} \subseteq f(\mathcal{C}_1 \times \mathcal{C}_2)$$
Now, if $S \in \mathcal{B}(Y)$, $T \in \mathcal{B}(X)$, and $K \in \mathcal{K}(X, Y)$, then
$$\overline{SKTb_1(X)} \subseteq \overline{SK\|T\|\overline{b_1(X)}} \subseteq \overline{(S\|T\|)\overline{\|Kb_1(X)\|}}$$
But this last is the continuous image of a compact set, and is thus compact.

For norm-closure, suppose $K_n \in \mathcal{K}(X, Y)$ with $K_n \to K$.

Claim 218. *$Kb_1(X)$ is totally bounded; i.e. for all $\varepsilon > 0$ there are $y_1, \ldots, y_n \in Kb_1(X)$ such that*

$$Kb_1(X) \subseteq \bigcup_{i=1}^{n} b_\varepsilon(y_i)$$

Proof. Fix $\varepsilon > 0$. Pick N such that $\|K - K_N\| < \frac{\varepsilon}{3}$. Then $K_N b_1(X)$ is totally bounded, so we may pick $y_1, \ldots, y_n$ with $y_i = K_N x_i$ for $\|x_i\| \leq 1$ such that

$$K_N b_1(X) \subseteq \bigcup_{i=1}^{n} b_{\frac{\varepsilon}{3}}(y_i)$$

Let $y_i' = Kx_i$. Then

$$\|y_i' - y_i\| = \|(K - K_N)x_i\| < \frac{\varepsilon}{3}$$

If $\|x\| \leq 1$, then

$$\|K_N x - y_{i_0}\| < \frac{\varepsilon}{3}$$

for some i_0. Then

$$\|Kx - y_{i_0}'\| \leq \|Kx - K_N x\| + \|K_N x - y_{i_0}\| + \|y_{i_0} - y_{i_0}'\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

So

$$Kb_1(X) \subseteq \bigcup_{i=1}^{n} b_\varepsilon(y_i')$$

So $\overline{Kb_1(X)}$ is compact. □ Claim 218

□ Proposition 217

Example 219. Let $D = \operatorname{diag}(1, \frac{1}{2}, \frac{1}{3}, \ldots) \in \mathcal{B}(c_0)$. Then $Db_1(c_0)$ is not closed, since

$$D((1, 1, 1, \ldots, 0, 0, \ldots)) = \left(1, \frac{1}{2}, \ldots, \frac{1}{n}, 0, \ldots\right) \to \left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right) \notin \operatorname{Ran}(D)$$

Example 220 (Hilbert-Schmidt kernels). Let $k(x, y) \in L^2((0, 1)^2)$. Define $K \in \mathcal{B}(L^2(0, 1))$ by

$$(Kf)(x) = \int_0^1 k(x, y) f(y) dy$$

Note $k(\cdot, y), k(x, \cdot) \in L^2(0, 1)$ for almost every x, y. To check boundedness, suppose $f \in L^2(0, 1)$. Then

$$\begin{aligned}
\|Kf\|_2^2 &= \int_0^1 |Kf(x)|^2 dx \\
&= \int_0^1 \left| \int_0^1 k(x, y) f(y) dy \right|^2 dx \\
&\leq \int_0^1 \left(\int_0^1 |k(x, y)||f(y)| dy \right)^2 dx \\
&\leq \int_0^1 (\|k(x, \cdot)\|_2 \|f\|_2)^2 dx \\
&= \|f\|_2^2 \int_0^1 \int_0^1 |k(x, y)|^2 dy dx \\
&= \|f\|_2^2 \|k\|_2^2
\end{aligned}$$

So $\|K\| \leq \|k\|_2$.

Let $\{ e_i(x) : i \geq 1 \}$ be an orthonormal basis for $L^2(0,1)$. Let $\{ f_j(y) : j \geq 1 \}$ be anohter orthonormal basis for $L^2(0,1)$. Then $\{ e_i(x) f_j(y) : i, j \geq 1 \}$ is an orthonormal basis for $L^2((0,1)^2)$ because

$$\left\{ \sum_{m=1}^{M} g_m(x) h_m(y) : g_m \in L^2, h_m \in L^2 \right\}$$

is dense in $L^2((0,1)^2)$. If

$$g_m(x) = \sum a_i e_i(x)$$
$$h_m(y) = \sum b_j f_j(y)$$

then

$$g_m h_m = \sum \sum a_i b_j e_i(x) f_j(y)$$

Take $f_j(y) = \overline{e_j(y)}$. Expand

$$k(x,y) = \sum \sum a_{ij} e_i(x) \overline{e_j(y)}$$

where

$$\|k\|_2^2 = \sum \sum |a_{ij}|^2$$

For $N < \infty$, let

$$k_N(x,y) = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} e_i(x) \overline{e_j(y)}$$

Then $k_N \in L^2((0,1)^2)$ with $\|k - k_N\|_2 \to 0$. If

$$K_N h(x) = \int_0^1 k_N(x,y) h(y) dy$$

then $\|K - K_N\| = \|k - k_N\|_2 \to 0$. So

$$\begin{aligned} K_N h(x) &= \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} \int_0^1 e_i(x) \overline{e_j(y)} h(y) dy \\ &= \sum_{i=1}^{N} e_i(x) \sum_{j=1}^{N} a_{ij} \langle h, e_j \rangle \end{aligned}$$

So $\operatorname{Ran}(K_N) \subseteq \operatorname{span}\{ e_1, \ldots, e_N \}$. So K is a norm limit of finite rank operators, and is thus compact. The "matrix of K_N" is given by, if

$$h = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ \vdots \end{pmatrix}$$

where $h_i = \langle h, e_i \rangle$, then

$$K_N h = \begin{pmatrix} a_{11} & \ldots & a_{1N} & \\ \vdots & \ddots & \vdots & 0 \\ a_{N1} & \ldots & a_{NN} & \\ & 0 & & 0 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_N \\ h_{N+1} \\ \vdots \end{pmatrix}$$

Example 221 (Volterra operator)*.* Let $V \in \mathcal{B}(L^2(0,1))$ be

$$Vh(x) = \int_0^x h(y)dy$$

We may take

$$k(x,y) = \begin{cases} 1 & y \leq x \\ 0 & y > x \end{cases}$$

So V is compact, by the above argument. Then

$$\begin{aligned} V^2h(x) &= \int_0^x (Vh)(y)dy \\ &= \int_0^x \left(\int_0^y h(z)dz \right) dy \\ &= \int_0^x h(z) \int_z^x 1dydz \\ &= \int_0^x h(z)(x-z)dz \\ V^3h(x) &= \int_0^x (V^2h)(y)dy \\ &= \int_0^x \left(\int_0^y h(z)(y-z)dz \right) dy \\ &= \int_0^x h(z) \int_z^y (y-z)dydz \\ &= \int_0^x h(z) \frac{x-z)^2}{2} dz \end{aligned}$$

Claim 222*.*

$$V^n h(x) = \int_0^x h(y) \frac{(x-y)^{n-1}}{(n-1)!} dy$$

Then

$$\|V^n\| = \left\| \frac{(x-y)^{n-1}}{(n-1)!} \chi_{\{\, y \leq x \,\}} \right\|_2 \leq \frac{1}{(n-1)!}$$

Then

$$\operatorname{spr}(V) = \lim_{n\to\infty} \|V^n\|^{\frac{1}{n}} \leq \lim_{n\to\infty} \left(\frac{1}{(n-1)!} \right)^{\frac{1}{n}} = 0$$

So $\sigma(V) \subseteq \{\, 0 \,\}$.

Claim 223*. V is injective.*

Proof. Suppose $Vh = \lambda h$ for $\lambda \neq 0$. Then

$$\lambda h(x) = \int_0^x h(y)dy$$

But $h \in L^2$; so

$$\int_0^x h(y)dy \in C[0,1]$$

So RHS $\in C[0,1]$, so LHS $\in C[0,1]$. So $h \in C[0,1]$, and RHS is C^1. So $h \in C^1$, so RHS is C^2. So $h \in C^\infty$. So $\lambda h'(x) = h(x)$ by the fundamental theorem of calculus. So $h(x) = c\exp(x/\lambda)$ and $h(0) = 0$; so $h = 0$.

In the case of $\lambda = 0$, we have that if $Vh = 0$, then

$$\int_0^x h(y)dy = 0$$

for all $x \in [0,1]$. So $h = 0$ by measure theory. □ Claim 223

Proposition 224. *Suppose $\mathcal{H}$ is a Hilbert space and $K \in \mathcal{K}(\mathcal{H})$. Then K is a limit of finite rank operators.*

Proof. Note that $\overline{Kb_1(\mathcal{H})}$ is compact. Suppose $\varepsilon > 0$. Find $y_1 = Kx_i$ with $\|x_i\| \le 1$ for $1 \le i \le n$ such that $\{ y_1, \dots, y_n \}$ is an ε-net for $\overline{Kb_1(\mathcal{H})}$. i.e. if $\|x\| \le 1$ then there is i such that $\|Kx - y_i\| < \varepsilon$. Let P be the orthogonal projection onto $\operatorname{span}\{ y_1, \dots, y_n \}$. Then PK has rank $\le n$. Then

$$\|(K - PK)(x)\| = \|P^\perp Kx\| = \|P^\perp(Kx - y_i)\| < \varepsilon$$

for all $\|x\| \le 1$. So $\|K - PK\| \le \varepsilon$. □ Proposition 224

Theorem 225 (Schauder)**.** *If $K \in \mathcal{K}(X, Y)$ then $K^* \in \mathcal{K}(Y^*, X^*)$.*

Proof. Let $\mathcal{C} = \overline{Kb_1(X)}$; then $\mathcal{C}$ is a compact subset of Y. Define $\rho\colon Y^* \to C(\mathcal{C})$ be $\rho(\varphi) = \varphi \restriction \mathcal{C}$. Then $\overline{\rho(b_1(Y^*))}$ is closed and bounded (by $\|K\|$) in $C(\mathcal{C})$. It is also equicontinuous since if $y_1, y_2 \in \mathcal{C}$ and $\varphi \in \overline{b_1(Y^*)}$ then $|\varphi(y_1) - \varphi(y_2)| \le \|\varphi\|\|y_1 - y_2\|$. So, by Arzela-Ascoli theorem, we have $\overline{\rho(b_1(Y^*))}$ is compact.

Claim 226. *$\overline{K^* b_1(Y^*)}$ is compact.*

Proof. Suppose $\varphi_1, \varphi_2, \dots \in \overline{b_1(Y^*)}$. Then for $x \in \overline{b_1(X)}$, we have $(K^*\varphi_i)(x) = \varphi_i(Kx)$. But $Kx \in \mathcal{C}$; so $(K^*\varphi_i)(x) = \rho(\varphi_i)(Kx)$. Letting $\psi_i = \rho(\varphi_i)$, we have $\psi_i \in \overline{\rho(b_1(Y^*))} \subseteq C(\mathcal{C})$. So there is a subsequence ψ_{n_i} converging to ψ uniformly in $\mathcal{C}$. So $(K^*\varphi_i)(x) = \varphi_i(Kx) \to \psi(Kx)$. i.e. $K^*\varphi_i \to \Psi \in X^*$. Thus $\overline{K^* b_1(Y)}$ is compact. So K^* is compact. □ Claim 226

□ Theorem 225

5.3.1 Complemented subspaces

Definition 227. Suppose X is a Banach space; suppose $Y \subseteq X$ is a closed subspace. We say Y is *complemented* if there is $Z \subseteq X$ a closed subspace such that $Y \cap Z = \{ 0 \}$ and $Y + Z = X$.

Remark 228. If Y is complemented, we can define

$$T\colon Y \oplus_1 Z \to X$$

by $T(y, z) = y + z$ (where $\oplus_1$ denotes that the norm is the 1-norm on the direct sum). Then by hypotheses we have T is bijective. Also $\|y + z\| \le \|y\| + \|z\| = \|(y, z)\|$, so T is continuous. By the Banach isomorphism theorem, we get that T is invertible. So $X \cong Y \oplus Z$.

We can also define $P\colon Y \oplus_1 Z \to Y \oplus_1 Z$ by $P(y, z) = (y, 0)$. We can then let $Q = TPT^{-1}\colon X \to Y$; then Q is a continuous projection. Conversely, if $Q = Q^2$ with $\operatorname{Ran}(Q) = Y$, let $Z = \operatorname{Ran}(I - Q)$. Then $(I - Q)^2 = I - Q$. So $x = Qx + (I - Q)x$. So $x \in Y \cap Z$. So $x = Qx = (1 - Q)Qx = 0$.

Lemma 229. *There is an uncountable collection $\{ A_r : r \in \mathbb{R} \}$ of subsets of $\mathbb{N}$ such that $|A_r \cap A_s| < \aleph_0$ if $r \ne s$.*

Proof. Identify $\mathbb{N}$ with $\mathbb{Q}$ (as they are both countable). For $r \in \mathbb{R}$, pick a sequence $q_{r,i} \to r$. Then let $A_r = \{ n_{r,i} : i \ge 1 \}$ where $n_{r,i}$ is the natural number corresponding to $q_{r,i}$. □ Lemma 229

Theorem 230. *c_0 is not complemented in ℓ_∞.*

Proof. If $\ell_\infty \cong c_0 \oplus Y$ then $Y \cong \ell_\infty / c_0$. Take A_r as in the lemma. Let $y_r = [\chi_{A_r}] \in \ell_\infty / c_0$. Then

$$\left\| \sum_{i=1}^n a_i y_{r_i} \right\| = \left\| \sum a_i \chi_{A_{r_i}} + c_0 \right\|$$

But if $B_i \subseteq A_{r_i}$ are pairwise disjoint with $|A_{r_i} \setminus B_i| < \infty$, then this is

$$\left\| \sum a_i \chi_{B_i} + c_0 \right\| = \max_{1 \le i \le n} |a_i|$$

Claim 231. *No continuous, linear $T\colon \ell_\infty / c_0 \to \ell_\infty$ is injective.*

Proof. If $Ty_r \neq 0$, then there is n_r such that $(Ty_r)(n_r) = \alpha_r \neq 0$. Then there is $n \in \mathbb{N}$ such that $S = \{ r : n_r = n, |\alpha_r| \geq \varepsilon \}$ is uncountable. Thus uncountably many $|\alpha_r| \geq \varepsilon > 0$. But then

$$T\left(\sum_{\substack{i=1 \\ r_i \in S}}^{N} \overline{\alpha_i} y_{r_i}\right) = \sum |\alpha_i|^2$$
$$> N\varepsilon^2$$

Letting $N \to \infty$, we get a contradiction. □ Claim 231

□ Theorem 230

Proposition 232. *If K is a compact, infinite metric space then c_0 is complemented in $C(K)$.*

Proof. Pick a sequence $x_n \in K$ distinct with $x_n \to x_\infty$. Let

$$Sf(x) = f(x) - f(x_\infty)$$

the projection of $C(K)$ onto $I(x_0) = \{ f \colon f(x_0) = 0 \}$. Pick disjoint balls $b_{r_n}(x_n)$ with $n \geq 1$; let

$$g_n(x) = \max\left\{ \frac{r_n - \operatorname{dist}(x, x_n)}{r_n}, 0 \right\}$$

Let $T \colon I(x_0) \to I(x_0)$ be

$$Tf = \sum_{n \geq 1} f(x_n) g_n$$

The $f(x_n) \to 0$, so $Tf \in C(K)$. Also $P = TS$ is a projection onto a copy of c_0.

$$\left\| \sum a_n g_n \right\| = \max_{n \geq 1} |a_n|$$

$a_n \to$. So $\operatorname{Ran} P \cong c_0$. □ Proposition 232

Theorem 233. *c_0 is not complemented in any dual space. Suppose $X^* \cong c_0 \oplus Y$. Then $X^{***} \cong \ell^\infty \oplus Y^{**}$. We can consider map $\ell^\infty \to c_0$ by mapping to $\ell^\infty \oplus Y^{**} \cong X^{***}$, taking the projection down to X^*, and observing that it will still be in c_0 when we write $X^* \cong c_0 \oplus Y$.*

Corollary 234. *If K is a compact, infinite metric space, then $C(K)$ is not a dual space.*

Corollary 235. *If X is a compact Hausdorff space and $C(X)$ is a dual space, then the only convergent sequences in X are eventually constant.*

6 Compact operators and Fredholm theory

Lemma 236. *If X is a Banach space and V is a closed subspace such that $\dim(V) < \infty$ or $\dim(X/V) < \infty$, then V is complemented.*

Proof. **Case 1.** Suppose $\dim(V) = n < \infty$. Then there is a basis $v_1, \ldots, v_n$ for V, and V^* has dual basis $\varphi_1, \ldots, \varphi_n \in V^*$ such that $\varphi_i(v_j) = \delta_{ij}$. Extend φ_i to $\widetilde{\varphi}_i \in X^*$ by Hahn-Banach. Define

$$P = \sum_{i=1}^{n} v_i \varphi_i \in \mathcal{B}(X)$$

so

$$Px = \sum_{i=1}^{n} v_i \varphi_i(x)$$

So $\operatorname{Ran}(P) = V$, and if $v \in V$, say

$$v = \sum_{i=1}^{n} a_i v_i$$

then

$$Pv = \sum_{i=1}^{n} v_i \varphi_i(v_i) = \sum_{i=1}^{n} a_i v_i = v$$

So $P = P^2$ is a projection onto V. So it is complemented.

Case 2. Suppose $\dim(X/V) = n < \infty$. Pick a basis $\dot{x}_1, \dots, \dot{x}_n$ for X/V. Let $q\colon X \to X/V$ be the quotient map. Pick $x_i \in X$ such that $q(x_i) = \dot{x}_i$. Let $W = \operatorname{span}\{ x_1, \dots, x_n \}$.

Claim 237. *$V + W = X$.*

Proof. Suppose $x \in X$ with

$$q(x) = \sum_{i=1}^{n} a_i \dot{x}_i$$

Let

$$w = \sum_{i=1}^{n} a_i x_i$$

and $v = x - w$. Then

$$q(v) = q(x) - \sum_{i=1}^{n} a_i \dot{x}_i = 0$$

so $v \in V$. But $x = v + w$. □ Claim 237

Claim 238. *$V \cap W = \{ 0 \}$.*

Proof. For $x \in V \cap W$, we have $q(x) = 0$. Since $x \in W$, we have

$$x = \sum_{i=1}^{n} a_i x_i$$

So

$$0 = q(x) = \sum_{i=1}^{n} a_i \dot{x}_i$$

So each $a_i = 0$. So $x = 0$. □ Claim 238

So V is complemented.

□ Lemma 236

Notation 239. If V and W are complements in X, we write $X = V \oplus W$. (One also sees $X = V \dot{+} W$).

Lemma 240 (Key lemma)**.** *Suppose $K \in \mathcal{K}(X)$. Suppose we have closed subspaces $V_0 \subsetneqq V_1 \subsetneqq V_2 \subsetneqq \dots$ and $\alpha_i \in \mathbb{C}$ such that $(K - \alpha_i I)(V_i) \subseteq V_{i-1}$. Then*

$$\lim_{i \to \infty} \alpha_i = 0$$

Proof. Since $V_i \supsetneqq V_{i-1}$, we may pick $x_i \in V_i$ with $\|x_i\| = 1$ and $\mathrm{dist}(x, V_{i-1}) \geq \frac{1}{2}$. We then have $(K - \alpha_i I)x_i = y_i \in V_{i-1}$; so $Kx_i = \alpha_i x_i + y_i$. Suppose $n_1 < n_2 < n_3 < \dots$ satisfies $|\alpha_{n_k}| \geq \delta > 0$ for all $k \in \mathbb{N}$. If $1 \leq \ell < k$, then

$$\begin{aligned}
\|Kx_{n_k} - Kx_{n_\ell}\| &= \|\alpha_{n_k} x_{n_k} + (y_{n_k} - Kx_{n_\ell})\| \\
&\geq \mathrm{dist}(\alpha_{n_k} x_{n_k}, V_{n_k - 1}) \\
&\geq \frac{|\alpha_{n_k}|}{2} \\
&\geq \frac{\delta}{2}
\end{aligned}$$

So $\overline{Kb_1(X)}$ is not compact. □ Lemma 240

Theorem 241. *If $K \in \mathcal{K}(X)$ then $\ker(I - K)$ is finite-dimensional and $\mathrm{Ran}(I - K)$ is closed and has finite codimension.*

Proof. Let $B = \overline{b_1(X)} \cap \ker(I - K)$. If $x \in B$, then $Kx = (K - I)x + x = x$. So $\overline{Kb_1(X)} \supseteq \overline{KB} = B$ is compact. So $\mathrm{null}(I - K) = \dim(\ker(I - K) < \infty$. So $N = \ker(I - K)$ has a complement V; so $X = N \oplus V$. Then $(I - K)X = (I - K)V$ and $(I - K) \restriction V$ is injective.

Claim 242. *$(I - K) \restriction V$ is bounded below.*

Proof. Otherwise there are $v_1, v_2, \dots \in V$ with $\|v_i\| = 1$ and $\|v_i - Kv_i\| = \|(I - K)v_i\| \to 0$. But $Kv_i \in \overline{Kb_1(X)}$, and the latter is compact. So there is a subsequence $(Kv_{i_k} : k \in \mathbb{N}) \to y$. Then $v_{i_k} = (v_{i_k} - Kv_{i_k}) + Kv_{i_k} \to 0 + y \in V$. So

$$(I - K)y = \lim_{k \to \infty} (I - K)v_{i_k} = 0$$

So $y \in V \cap N = \{0\}$. But

$$\|y\| = \lim_{k \to \infty} \|v_{i_k}\| = 1$$

a contradiction. □ Claim 242

So $\mathrm{Ran}(I - K) = (I - K)V$ is closed.

Claim 243. *$X/(I - K)X$ is finite-dimensional.*

Proof. Otherwise, let $V_0 = \mathrm{Ran}(I - K)$. We have X/V_0 is infinite-dimensional, and so contains linearly independent $\dot{x}_1, \dot{x}_2, \dots$. Pick $x_i \in X$ such that $x_i + V_0 = \dot{x}_i$. Let $V_i = V_0 + \mathrm{span}\{x_1, \dots, x_i\}$. Then

$$V_0 \subsetneqq V_1 \subsetneqq \dots$$

with$(K - I)V_i \subseteq \mathrm{Ran}(I - K) = V_0 \subseteq V_{i-1}$. By the Key lemma, we have

$$\lim_{i \to \infty} 1 = 0$$

a contradiction. □ Claim 243

□ Theorem 241

Definition 244. We say $T \in \mathcal{B}(X, Y)$ is *Fredholm* if

- $\mathrm{null}(T) = \dim(\ker(T)) < \infty$.
- $\mathrm{Ran}(T)$ is closed.
- $\dim(Y/TX) < \infty$.

The *index* of T is $\mathrm{ind}(T) = \mathrm{null}(T) - \dim(Y/TX) \in \mathbb{Z}$.

Remark 245.

1. If $\dim(Y/TX) < \infty$, then TX is closed. (Exercise; use closed graph theorem.)
2. $\dim(Y/TX) = \mathrm{null}(T^*)$. (Useful for A6; need to prove it to use on assignment, though.)

Example 246.

1. If $K \in \mathcal{K}(X)$ and $\lambda \neq 0$, then $\lambda I + K$ is Fredholm.
2. If $T \in \mathcal{K}(X, Y)$ is invertible, then T is Fredholm and $\mathrm{ind}(T) = 0$.
3. The unilateral shift $S \in \mathcal{B}(\ell_2)$ given by $S((x_1, x_2, x_3, \dots)) = (0, x_1, x_2, x_3, \dots)$. This is an isometry, injective, and satisfies $\mathrm{Ran}(S) = (\mathbb{C}e_1)^\perp$. Also $\mathrm{null}(S) = 0$ and $\dim(\ell_2/S\ell_2) = 1$. So $\mathrm{ind}(S) = -1$.
4. The backward shift S^* is surjective and has $\ker(S^*) = \mathbb{C}e_1$, so $\ell_2/\mathrm{Ran}(S^*) = \{0\}$. So $\mathrm{ind}(S^*) = 1$.

Theorem 247. *The set $\mathcal{F}(X)$ of all Fredholm operators on X is open in $\mathcal{B}(X)$, and* ind *is a continuous function (and hence locally constant; so constant on connected components).*

Proof. Suppose $T \in \mathcal{B}(X)$ is Fredholm. Let $N = \ker(T)$. Choose a complement V so $X = N \oplus V$. Let $R = \mathrm{Ran}(T)$; choose a finite-dimensional complement R so $X = R \oplus W$. Then $\mathrm{ind}(T) = \dim(N) - \dim(W)$.

The map $\widetilde{T} \in \mathcal{B}(V, R)$ given by $\widetilde{T}V = Tv$ is injective and surjective, and hence is invertible by Banach isomorphism theorem. Suppose $S \in \mathcal{B}(X)$ and

$$\|S - T\| < \frac{1}{\left\|\widetilde{T}^{-1}\right\|}$$

Let $\widetilde{\widetilde{S}} \colon V \oplus W \to X = R \oplus W$ by $\widetilde{\widetilde{S}}(v + w) = Sv + w$. Let $\widetilde{\widetilde{T}} \colon V \oplus W \to X = R \oplus W$ by $\widetilde{\widetilde{T}}(v + w) = Tv + w$. Then $\widetilde{\widetilde{T}}$ is invertible. But

$$\left\|\widetilde{\widetilde{S}} - \widetilde{\widetilde{T}}\right\| = \|(S - T) \restriction V\| < \frac{1}{\left\|\widetilde{\widetilde{T}}^{-1}\right\|}$$

But $\widetilde{\widetilde{T}}$ is invertible; so $\widetilde{\widetilde{S}}$ is invertible. So $X = \widetilde{\widetilde{S}}(V \oplus W) = SV \oplus W$. (The sum is direct since in general if $S \colon V \oplus W \to X$ then $X = SV + SW$ and $SV \cap SW = S(V \cap W) = 0$.) So $\ker(S) \cap V = \{0\}$. So $\mathrm{null}(S) \leq \dim(W) = \mathrm{null}(T)$.

Aside 248. Suppose $V \oplus W = X$ with $N \cap V = \{0\}$. We claim that $\dim(N) \leq \dim(W)$. Suppose $\dim(W) = n$ with $x_1, \dots, x_{n+1} \in N$ linearly independent. Then $q \colon V \oplus W \to W$ given by $q(v + w) = w$ has $q(v_1), \dots, q(v_{n+1})$ are linearly dependent. So there are $a_1, \dots, a_{n+1}$ not all 0 such that

$$q\left(\sum_{i=1}^{n+1} a_i x_i\right) = 0$$

But

$$\sum_{i=1}^{n+1} a_i x_i \in V \cap N \setminus \{0\}$$

a contradiction.

So $SV \subseteq \mathrm{Ran}(S) = SV + SN$. But SV is closed and SN is finite dimensional; so $SV + SN$ is closed.

Aside 249. To see this, suppose $Sv_n + Sk_n \to y$ where $v_n \in V$ and $k_n \in N$. Then we have a subsequence $k_{n_i} \to k \in N$; so $Sk_{n_i} \to Sk \in SN$. So $Sv_n \to y - Sk \in SV$, as SV is closed. So $y \in SN + SV$.

So $\dim(X/SX) \leq \dim(X/SV) = \dim(W) < \infty$. So it is Fredholm. Let $N_S = \ker(S)$. Then $V \cap N_S = \{0\}$; so $V + N_S$ is a direct sum of finite codimension. Pick a complement Z so $V \oplus N_s \oplus Z = X$; then $(V \oplus Z) \oplus N_S = X$. So $V \oplus Z$ is complement to $\ker(S)$. So $S \restriction (V \oplus Z)$ is bounded below. But $SX = S(V \oplus Z) = SV \oplus SZ$; so

$$\begin{aligned}
\operatorname{ind}(S) &= \operatorname{null}(S) - \dim(X/SX) \\
&= \dim(N_S) - \dim(X/(SV \oplus SZ)) \\
&= \dim(N_S) - (\dim(X/SV) - \dim(SZ)) \\
&= \dim(N_S) - (\dim(W) - \dim(Z)) \\
&= (\dim(N_S) + \dim(Z)) - \dim(W) \\
&= \dim(N_T) - \dim(W) \text{ (since } N_S \oplus Z \text{ is a complement to } V\text{, as is } N_T)
\end{aligned}$$

So $\dim(N_T) - \dim(X/TX) = \operatorname{ind}(T)$. □ Theorem 247

Corollary 250. *If $\lambda \neq 0$ and $K \in \mathcal{K}(X)$, then* $\operatorname{ind}(\lambda I + K) = 0$.

Proof. $I + \lambda^{-1}K$ is Fredholm. So $\lambda I + K$ is Fredholm. So $\lambda I + tk$ is Fredholm for $0 \leq t \leq 1$. So $\operatorname{ind}(\lambda I + K) = \operatorname{ind}(\lambda I) = 0$. □ Corollary 250

Corollary 251 (Of proof). *Suppose T is Fredholm. Then*

$$\limsup_{S \to T} \operatorname{null}(S) \leq \operatorname{null}(T)$$

Remark 252. It can be strict; consider

$$\begin{pmatrix} t & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix}$$

as $t \to 0$.

We have $\mathcal{K}(X) \lhd \mathcal{B}(X)$. So $\mathcal{B}(X)/\mathcal{K}(X)$ is a Banach space and a ring (in fact, an algebra over $\mathbb{C}$). If $\pi \colon \mathcal{B}(X) \to \mathcal{B}(X)/\mathcal{K}(X)$, then

$$\pi(xy) = \|\pi(x)\pi(y)\| \leq \|\pi(x)\|\|\pi(y)\|$$

So this is a Banach algebra.

Theorem 253 (Atkinson)**.** *$T \in \mathcal{B}(X)$ is Fredholm if and only if $\pi(T) \in (\mathcal{B}(X)/\mathcal{K}(X))^{-1}$.*

Proof.

($\Longrightarrow$) Suppose T is Fredholm. So if $N_T = \ker(T)$, then N_T is finite-dimensional; so there is a complement $X = N_T \oplus V$. Likewise, if $R_T = \operatorname{Ran}(T)$, then R_T has finite codimension; so there is a complement $X = R_T \oplus W$, where W is finite-dimensional. Then $\widetilde{T} \in \mathcal{B}(V, R_T)$ is invertible. Let $\widetilde{S} \in \mathcal{B}(R_T, V)$ be the inverse. Define $S \in \mathcal{B}(X)$ by $S(r \oplus w) = \widetilde{S}r$, wehere $r \in R_T$ and $w \in W$. Then $ST(n \oplus v) = S(Tv) = v$ for $n \in N_T$ and $v \in V$. So ST is a projection onto V with kernel N_T. So $I - ST$ is a projection onto N_T; so $\operatorname{rank}(I - ST) = \dim(N_T) < \infty$. So $\pi(S)\pi(T) = \pi(I)$.

On the other side,

$$(TS)(Tv \oplus w) = T(STv) = Tv$$

So TS is a projection onto R_T with $\ker(TS) = W$ is finite-dimensional. So $\operatorname{rank}(I - TS) = \dim(W)$. So $\pi(T)\pi(S) = \pi(I)$. So $\pi(T)$ is invertible.

($\Longleftarrow$) Suppose $S \in \mathcal{B}(X)$ has $T \in \mathcal{B}(X)$ such that $\pi(S) = \pi(T)^{-1}$. Then $\pi(ST) = \pi(I)$; so $ST = I + K$ for some $K \in \mathcal{K}(X)$. Likewise, $\pi(TS) = \pi(I)$, so $TS = I + L$ for some $L \in \mathcal{K}(X)$. Then $\ker(T) \subseteq \ker(ST) = \ker(I + K)$ is finite-dimensional, and $\operatorname{Ran}(T) \supseteq \operatorname{Ran}(TS) = \operatorname{Ran}(I + L)$ has finite codimension. So $\operatorname{Ran}(T)$ is closed and has finite codimension. So T is Fredholm.

□ Theorem 253

Corollary 254. *If T is Fredholm and $K \in \mathcal{K}(X)$, then $T+K$ is Fredholm and* $\operatorname{ind}(T+K) = \operatorname{ind}(T)$.

Proof. $\pi(T+K) = \pi(T)$ is invertible, so $T+K$ is Fredholm. But then $T+tK$ is Fredholm for $0 \le t \le 1$; so by continuity we have $\operatorname{ind}(T+K) = \operatorname{ind}(T)$. ☐ Corollary 254

Theorem 255. $\operatorname{ind}\colon (\mathcal{B}(X)/\mathcal{K}(X))^{-1} \to \mathbb{Z}$ *is a homomorphism.*

Proof. Note that if $\pi(S) = \pi(T) \in (\mathcal{B}(X)/\mathcal{K}(X))^{-1}$, then $S - T \in \mathcal{K}(X)$, so $S = T + K$ for some $K \in \mathcal{K}(X)$. But then $\operatorname{ind}(S) = \operatorname{ind}(T)$. So we can define $\operatorname{ind}(\pi(T)) = \operatorname{ind}(T)$, and this is well-defined.

Suppose $S, T \in \mathcal{F}(X)$. Write $X = N_T \oplus V = TX \oplus W$ (where $N_T = \ker(T)$ and $TX = \operatorname{Ran}(T) = TV$); write $X = N_S \oplus U = SX \oplus Y$ similarly. We need to choose W a bit more carefully to make the proof go smoothly.

Well, $TX + N_S$ is closed, as TX is closed and N_S is finite-dimensional. Choose a complement $W_0 \subseteq N_S$ such that $TX \oplus W_0 = TX + N_S$; then $N_S = (TX \cap N_S) \oplus W_0$. Let W_1 be a complement to $(TX + N_S) \oplus W_1 = X$. Let $W = W_0 \oplus W_1$. Then $TX \oplus W = (TX \oplus W_0) \oplus W_1 = TX \oplus N_S \oplus W_1 = X$.

But then

$$\begin{aligned} \ker(ST) &= N_T + \{\, x : Tx \in N_s \,\} \\ &= N_T + \{\, v \in V : Tv \in N_S \cap TX \,\} \\ &= N_T \oplus (T \restriction V)^{-1}(N_S \cap TX) \end{aligned}$$

since $T \restriction V$ is injective, and $N_T \cap V = \{\, 0 \,\}$. So $(ST) = (T) + \dim(N_S \cap TX)$.

Now, $SX = S(TX \oplus W_0 \oplus W_1) = STX \oplus SW_1$ (where the sum is direct since if $STx = Sw_1$, then $S(Tx - w_1) = 0$, so $Tx - w_1 \in N_S$; so $w_1 \in TX + N_S$, and $w_1 = 0$). So

$$\begin{aligned} \operatorname{ind}(ST) &= (ST) - \dim(X/STX) \\ &= (T) + \dim(N_S \cap TX) - (\dim(X/SX) - \dim(SW_1)) \\ &= (T) + \dim(N_S \cap TX) - (\dim(X/SX) - \dim(W_1)) \\ &= (T) - \dim(W_0 \oplus W_1) + \dim(W_0) + \dim(N_S \cap TX) - \dim(X/SX) \\ &= (T) - \dim(W_0 \oplus W_1) + \dim(N_S) - \dim(X/SX) \\ &= \operatorname{ind}(T) + \operatorname{ind}(S) \end{aligned}$$

since $\dim(W_0) + \dim(N_S \cap TX) = \dim((N_S \cap TX) \oplus W_0) = \dim(N_S)$. ☐ Theorem 255

Theorem 256 (Structure of compact operators)**.** *Suppose $K \in \mathcal{K}(X)$ with $\dim(X) = \infty$. Then*

1. $0 \in \sigma(K)$.
2. $\sigma(K) \setminus \{\, 0 \,\} \subseteq \sigma_p(K)$
3. *$\sigma(K)$ is a finite or countable set with 0 as its only cluster point.*
4. *For all $\lambda \in \sigma(K) \setminus \{\, 0 \,\}$ there is $n_\lambda \in \mathbb{N}$ such that*
   - $N(\lambda) = \ker((\lambda I - K)^{n_\lambda}) = \ker((\lambda I - K)^n)$ *if and only if* $n \ge n_\lambda$
   - $R(\lambda) = \operatorname{Ran}((\lambda I - K)^{n_\lambda}) = \operatorname{Ran}((\lambda I - K)^n)$ *if and only if* $n \ge n_\lambda$
5. *Then $X = N(\lambda) \oplus R(\lambda)$.*
6. *If E_λ is the projection onto $N(\lambda)$ with kernel $R(\lambda)$, then $E_\lambda \in \{\, K \,\}''$.*

   Aside 257. For $\mathcal{A} \subseteq \mathcal{B}(X)$, we set $\mathcal{A}' = \{\, T \in \mathcal{B}(X) : AT = TA \text{ for all } A \in \mathcal{A} \,\}$. We then set $\mathcal{A}'' = (\mathcal{A}')'$.
7. *$\sigma(K \restriction N(\lambda)) = \{\, \lambda \,\}$ and $\sigma(K \restriction R(\lambda)) = \sigma(K) \setminus \lambda$.*
8. *If $\lambda \ne \mu \in \sigma(K) \setminus \{\, 0 \,\}$, then $E_\lambda E_\mu = 0$.*

Proof of Theorem 256.

2. Take $\lambda \in \sigma(K) \setminus \{ 0 \}$. If $\ker(\lambda I - K) = \{ 0 \}$, then since $\lambda I - K$ is Fredholm and $0 = \operatorname{ind}(\lambda I - K) = (\lambda I - K) - \dim(X/(\lambda I - K)X)$, then we would have $\lambda I - K$ is a bijective map $X \to X$; so $\lambda I - K$ is invertible, contradicting our assumption that $\lambda \in \sigma(K)$. So there is $0 \neq x \in \ker(\lambda I - K)$; so we have $Kx = \lambda x$, and $\lambda \in \sigma_p(K)$.

4. Fix $\lambda \in \sigma(K) \setminus \{ 0 \}$. Let $N_i = \ker((\lambda I - K)^i)$; then

$$N_1 \subseteq N_2 \subseteq \dots$$

and

$$\operatorname{Ran}(\lambda I - K) \supseteq \operatorname{Ran}((\lambda I - K)^2) \supseteq \dots$$

Now, if $N_n \subsetneqq N_{n+1}$ for all $n \geq 1$, note that $(\lambda I - K)N_{n+1} \subseteq N_n$. So, by the key lemma, we have

$$\lambda = \lim_{n \to \infty} \lambda = 0$$

a contradiction. So there is a least n_λ such that $N_{n_\lambda - 1} \subsetneqq N_{n_\lambda} = N_{n_\lambda + 1}$.

Now, if $n \geq n_\lambda + 1$ and $x \in N_n$, then $(\lambda I - K)^{n - n_\lambda - 1}x \in N_{n_\lambda + 1} = N_{n_\lambda}$. So $(\lambda I - K)^{n-1}x = 0$. So $N_n = N_{n-1} = \dots = N(\lambda)$.

But $0 = \operatorname{ind}((\lambda I - K)^n) = \dim(N_n) - \dim(R_n)$, where $R_n = \operatorname{Ran}((\lambda I - K)^n)$. Thus $R_n = R(\lambda) = R_{n_\lambda}$ if and only if $n \geq n_\lambda$.

5. Suppose $x \in X$. Then $y = (\lambda I - K)^{n_\lambda} x \in R(\lambda) = \operatorname{Ran}((\lambda I - K)^{2n_\lambda})$. Find $z \in X$ such that $(\lambda I - K)^{2n_\lambda} z = y = (\lambda I - K)^{n_\lambda} x$. Then

$$(\lambda I - K)^{n_\lambda}((\lambda I - K)^{n_\lambda} z - x) = 0$$

with $w = (\lambda I - K)^{n_\lambda} z - x \in N(\lambda)$. But then $x = -w + (\lambda I - K)^{n_\lambda} z \in N(\lambda) + R(\lambda)$.

Suppose now that $x \in N(\lambda) \cap R(\lambda)$. Then there is y such that $x = (\lambda I - K)^{n_\lambda} y$; then since $x \in N(\lambda)$, we have $0 = (\lambda I - K)^{n_\lambda} x = (\lambda I - K)^{2n_\lambda} y$. So $y \in \ker((\lambda I - K)^{2n_\lambda})$. So $x = (\lambda I - K)^{n_\lambda} y = 0$. So $X = N(\lambda) \oplus R(\lambda)$.

6. Let E_λ be the projection onto $N(\lambda)$ with kernel $R(\lambda)$. Suppose $T \in \{ K \}'$. If $x \in N(\lambda)$, then $0 = (\lambda I - K)^{n_\lambda} x$; so $(\lambda I - K)^{n_\lambda} T x = T(\lambda I - K)^{n_\lambda} x = 0$. So $TN(\lambda) \subseteq N(\lambda)$. Now, if $y \in R(\lambda)$, then there is x such that $y = (\lambda I - K)^{n_\lambda} x$; then $Ty = T(\lambda I - K)^{n_\lambda} x = (\lambda I - K)^{n_\lambda} T x \in R(\lambda)$.

Now, if $x = n \oplus y$ for $n \in N(\lambda)$ and $y \in R(\lambda)$, then $E_\lambda T x = E_\lambda(Tn \oplus Ty) = Tn = TE_\lambda x$. So $E_\lambda T = TE_\lambda$. So $E_\lambda \in \{ K \}''$.

7. In particular, the above yields that $N(\lambda)$ and $R(\lambda)$ are invariant for K; so $K \restriction N(\lambda) \in \mathcal{B}(N(\lambda))$. But $N(\lambda)$ is finite dimensional, and $(\lambda I_{N(\lambda)} - (K \restriction N(\lambda)))^{n_\lambda} = (\lambda I - K)^{n_\lambda} \restriction N(\lambda) = 0$ and $(\lambda I - K)^{n_\lambda - 1} \restriction N(\lambda) \neq 0$; so $(\lambda - z)^{n_\lambda}$ is the minimal polynomial of $K \restriction N(\lambda)$. So $\sigma(K \restriction N(\lambda)) = \{ \lambda \}$. Also $(\lambda I - K) \restriction R(\lambda)$ has no kernel (since $N(\lambda) \cap R(\lambda) = \{ 0 \}$). So the index is 0, and $(\lambda I - K) \restriction R(\lambda)$ is invertible. So $\lambda \notin \sigma(K \restriction R(\lambda))$. So

$$K \cong \begin{pmatrix} K \restriction N(\lambda) & 0 \\ 0 & K \restriction R(\lambda) \end{pmatrix}$$

So

$$\mu I - K = \begin{pmatrix} (\mu I - K) \restriction N(\lambda) & 0 \\ 0 & (\mu I - K) \restriction R(\lambda) \end{pmatrix}$$

is invertible if and only if both diagonal entries are invertible. So $\sigma(K) = \sigma(K \restriction N(\lambda)) \cup \sigma(K \restriction R(\lambda))$. But $\sigma(K \restriction N(\lambda)) = \{ \lambda \}$, and $\sigma(K \restriction R(\lambda)) \subseteq \sigma(K) \setminus \{ \lambda \}$. So $\sigma(K \restriction R(\lambda)) = \sigma(K) \subseteq \{ \lambda \}$, as desired.

3. Suppose $(\lambda_n : n \in \mathbb{N})$ are distinct points in $\sigma(K) \setminus \{ 0 \}$. Pick x_n such that $Kx_n = \lambda_n x_n$. Let $V_n = \operatorname{span}\{ x_1, \dots, x_n \}$. Then $(\lambda_n I - K)V_n \subseteq V_{n-1}$. By the key lemma, we have

$$\lim_{n \to \infty} \lambda_n = 0$$

So $\sigma(K)$ is countable with 0 as the only cluster point.

8. Suppose $\lambda, \mu \in \sigma(K) \setminus \{ 0 \}$ are distinct. Then $N(\lambda) \cap N(\mu) = \{ 0 \}$ since $N(\mu) \subseteq R(\lambda)$ by decomposition of K. So $E_\mu E_\lambda = 0$.

1. If $0 \notin \sigma(K) = \{ \lambda_1, \dots, \lambda_n \}$, then

$$X = N_{\lambda_1} \oplus N_{\lambda_2} \oplus \cdots \oplus N_{\lambda_n} \oplus \bigcap_{i=1}^{n} R(\lambda_i)$$

by induction. So

$$\sigma\left(K \restriction \bigcap_{i=1}^{n} R(\lambda_i) \right) \subseteq \sigma(K) \setminus \{ \lambda_1, \dots, \lambda_n \} = \emptyset$$

a contradiction. In fact,

$$\sigma\left(K \restriction \bigcap_{i=1}^{n} R(\lambda_i) \right) = \{ 0 \}$$

though it doesn't have to be an eigenvalue.

□ Theorem 256

6.1 Normal operators on Hilbert space

Recall that for $T \in \mathcal{B}(\mathcal{H})$, we have a unique $T^* \in \mathcal{B}(\mathcal{H})$ such that $\langle T^*x, y \rangle = \langle x, Ty \rangle$ for all $x, y \in \mathcal{H}$.

Definition 258.

- $T \in \mathcal{B}(\mathcal{H})$ is *self-adjoint* if $T = T^*$.
- $T \in \mathcal{B}(\mathcal{H})$ is *positive* (written $T \geq 0$) if $T = T^*$ and $\langle Tx, x \rangle \geq 0$ for all $x \in \mathcal{H}$.
- $U \in \mathcal{B}(\mathcal{H})$ is *unitary* if U is a surjective isometry. (Equivalently, by assignment 6, if $U^* = U^{-1}$.)
- $N \in \mathcal{B}(\mathcal{H})$ is *normal* if $NN^* = N^*N$.

Remark 259.

1. On $L^2(0,1)$, if $f \in L^\infty(0,1)$, then $M_f h = fh$ is bounded. Also

$$\begin{aligned} \langle M_f^* g, h \rangle &= \langle g, M_f h \rangle \\ &= \langle g, fh \rangle \\ &= \int g \overline{fh} dx \\ &= \int (\overline{f} g) \overline{h} dx \\ &= \langle M_{\overline{f}} g, h \rangle \end{aligned}$$

   So $M_f^* = M_{\overline{f}}$ and $M^* M_f = M_{\overline{f}} M_f = M_{|f|^2} = M_f M_{\overline{f}} = M_f M_f^*$. So M_f is normal.

2. Diagonal operators are normal. Let $\{ e_n : n \in \mathbb{N} \}$ be an orthonormal basis. Let $De_n = d_n e_n$ where $(d_n : n \in \mathbb{N}) \in \ell_\infty$. Then $D^* e_n = \overline{d_n} e_n$, and D is normal.

3. If $T = T^*$, then $\langle Tx, x \rangle = \langle x, Tx \rangle = \overline{\langle Tx, x \rangle}$; so $\langle Tx, x \rangle \in \mathbb{R}$. If $\mathbb{F} = \mathbb{C}$, then converse is true:

$$\begin{aligned} \langle Tx, y \rangle &= \frac{1}{4}(\langle Tx + y, x + y \rangle - \langle T(x-y), x-y \rangle + i\langle T(x+iy), x+iy \rangle - i\langle T(x-iy), x-iy \rangle) \\ \langle x, Ty \rangle &= \overline{\langle Ty, x \rangle} \\ &= \overline{\frac{1}{4}(\langle Ty + x, y + x \rangle - \langle T(y-x), y-x \rangle + i\langle T(y+ix), y+ix \rangle - i\langle T(y-ix), y-ix \rangle)} \\ &= \langle Tx, y \rangle \end{aligned}$$

since $\langle Tz, z\rangle \in \mathbb{R}$ for all $z \in \mathcal{H}$.

Note that the converse fails over $\mathbb{R}$: let

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

We then have $\langle Tx, x\rangle = 0$ for all $x \in \mathbb{R}^2$ but

$$T^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -T$$

4. If $A \in \mathcal{B}(\mathcal{H})$ then $A^*A \geq 0$, since $(A^*A)^* = A^*A^{**} = A^*A$ and $\langle A^*Ax, x\rangle = \langle Ax, Ax\rangle = \|Ax\|^2 \geq 0$.

Proposition 260. *Suppose N is normal.*

1. $\|Nx\| = \|N^*x\|$ *for all* $x \in \mathcal{H}$.
2. $\|N\| = \operatorname{spr}(N)$.
3. $\ker(N - \lambda I) = \ker((N - \lambda I)^n) = \ker((N-\lambda I)^*)$ *for all* $n \geq 1$ *and all* $\lambda \in \mathbb{C}$.
4. $\ker(N - \lambda I)^\perp = \overline{\operatorname{Ran}(N - \lambda I)}$.
5. *If* $\lambda \neq \mu$ *then* $\ker(N - \lambda I) \perp \ker(N - \mu I)$.
6. *If* $p \in \mathbb{C}[z]$ *then*

$$\|p(N)\| = \sup_{\lambda \in \sigma(N)} |p(\lambda)|$$

Proof.

1. Note that

$$\begin{aligned} \|N^*x\|^2 &= \langle N^*x, N^*x\rangle \\ &= \langle NN^*x, x\rangle \\ &= \langle N^*Nx, x\rangle \\ &= \langle Nx, Nx\rangle \\ &= \|Nx\|^2 \end{aligned}$$

2. By (1), we have $\|N^2x\| = \|N^*(Nx)\| \geq \langle N^*N, x, x\rangle = \|Nx\|^2$. So

$$\|N^2\| = \sup_{\|x\|\leq 1} \|N^2x\| \geq \sup_{\|x\|\leq 1} \|Nx\|^2 = \|N\|^2$$

But $\|N^2\| \leq \|N\|^2$. So $\|N^2\| = \|N\|^2$. So

$$\left\|N^{2^k}\right\|^{\frac{1}{2^k}} = \left(\|N\|^{2^k}\right)^{\frac{1}{2^k}} = \|N\|$$

So

$$\operatorname{spr}(N) = \lim_{k\to\infty} \left\|N^{2^k}\right\|^{\frac{1}{2^k}} = \|N\|$$

3. Well

$$\begin{aligned} x \in \ker(N - \lambda I) &\iff \|(N-\lambda)x\| = 0 = \|(N-\lambda)^*x\| \\ &\iff x \in \ker(N - \lambda I)^* \end{aligned}$$

Also if $x \in \ker((N - \lambda I)^{2^k})$, then

$$0 = \|(N - \lambda I)^{2^k} x\| \geq \|(N - \lambda I)x\|^{2^k}$$

So $\|(N - \lambda I)x\|^{2^k} = 0$, and $x \in \ker(N - \lambda I)$. So $\ker((N - \lambda I)^{2k}) = \ker(N - \lambda I)$.

4. Note that $\ker(N-\lambda I)^\perp = \overline{\operatorname{Ran}(N-\lambda I)}$. Also
$$\overline{\operatorname{Ran}(N-\lambda I)} = (\ker(N-\lambda I)^*)^\perp = \ker(N-\lambda I)^\perp$$
(So $\overline{\operatorname{Ran}(N-\lambda I)^*} = \overline{\operatorname{Ran}(N-\lambda I)}$.)

5. Suppose $\lambda \neq \mu$. Suppose $x \in \ker(N-\lambda I)$ and $y \in \ker(N-\mu I) = \ker(N^* - \overline{\mu} I)$. Then $Nx = \lambda x$, and $N^* y = \overline{\mu} y$, so $Ny = \mu y$. So
$$\lambda\langle x, y\rangle = \langle Nx, y\rangle = \langle x, N^* y\rangle = \langle x, \overline{\mu} y\rangle = \mu\langle x, y\rangle$$
But $\lambda \neq \mu$. So $\langle x, y\rangle = 0$.

6. Well, $p(N)$ is normal. By (2), we have $\|p(N)\| = \operatorname{spr}(p(N))$. But $\sigma(p(N)) = p(\sigma(N))$ by the spectral mapping theorem. So
$$\|p(N)\| = \sup_{\lambda \in \sigma(N)} |p(\lambda)|$$

□ Proposition 260

Corollary 261. *If N is normal and Fredholm then* $\operatorname{ind}(N) = 0$.

Proof. Well, $\ker(N)^\perp = \operatorname{Ran}(N)$ and
$$\begin{aligned}
\operatorname{ind}(N) &= \dim(\ker(N)) - \dim(\mathcal{H}/\operatorname{Ran}(N)) \\
&= \dim(\ker(N)) - \dim((\operatorname{Ran}(N)^\perp)) \\
&= \dim(\ker(N)) - \dim(\ker(N)) \\
&= 0
\end{aligned}$$

□ Corollary 261

Theorem 262 (Spectral theorem for compact normal operators)**.** *Suppose N is a compact normal operator on $\mathcal{H}$. Then $\mathcal{H}$ has an orthonormal basis which diagonalizes N.*

Proof. From the structure of arbitrary compact operators, we have
$$\sigma(N) = \{\lambda_1, \lambda_2, \dots\} \cup \{0\}$$
with
$$\lim_{n\to\infty} \lambda_n = 0$$
Then
$$\bigvee_{n=1}^{\infty} \ker(N-\lambda_i I)^n = \ker(N-\lambda_i I) = M_i = \ker(N^* - \overline{\lambda_i} I)$$
by part (3) of Proposition 260, where $\bigvee$ denotes the closed span. Note that the M_i are finite-dimensional, and by part (5) of Proposition 260, we have $M_m \perp M_n$ if $m \neq n$. Let
$$\mathcal{M} = \bigoplus_{n=1}^{\infty} M_n$$
Then $\mathcal{M}$ is a closed subspace with $N\mathcal{M} \subseteq \mathcal{M}$ and $N^*\mathcal{M} \subseteq \mathcal{M}$. Write $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$, and write
$$N = \begin{pmatrix} N_{11} & 0 \\ 0 & N_{22} \end{pmatrix}$$
where $N_{11} \colon \mathcal{M} \to \mathcal{M}$ and $N_{22} \colon \mathcal{M}^\perp \to \mathcal{M}^\perp$. Then N_{11} and N_{22} are normal, and
$$N^* = \begin{pmatrix} N_{11}^* & 0 \\ 0 & N_{22}^* \end{pmatrix}$$

and

$$0 = N^*N - NN^* = \begin{pmatrix} N_{11}^*N_{11} - N_{11}N_{11}^* & 0 \\ 0 & N_{22}^*N_{22} - N_{22}N_{22}^* \end{pmatrix}$$

So N_{22} is normal, compact, and has no non-zero eigenvalues. So $\sigma(N_{22}) = 0$. So $\|N_{22}\| = \operatorname{spr}(N_{22}) = 0$. So $N_{22} = 0$. So $\mathcal{M}^\perp = \ker(N)$. Choose an orthonormal basis for each M_i; these are then eigenvectors with eigenvalue λ_i. Say $e_{i,1}, \dots, e_{i,n_i}$ are an orthonormal basis for M_i. Choose an orthonormal basis $\{ e_{0,i} : i < \alpha \}$ for $\ker(N) = \mathcal{M}^\perp$; note that α is possibly infinite (indeed, possibly uncountable).

Notation 263. If $x, y \in \mathcal{H}$, then $(xy^*)(z) = z(y^*z) = \langle z, y\rangle x$. Write

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix}$$

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \end{pmatrix}$$

Then

$$xy^* = \begin{pmatrix} x_1\overline{y_1} & x_1\overline{y_2} & \cdots \\ x_2\overline{y_1} & x_2\overline{y_2} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

If N is compact and normal and $\{ e_n : n \in \mathbb{N} \}$ is an orthonormal basis of eigenvectors which span $\mathcal{M} = (\ker(N))^\perp$; say $Ne_n = \lambda_n e_n$. Then

$$N = \sum_{n=1}^{\infty} \lambda e_n e_n^* = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3, \dots) \oplus 0$$

on $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$.

□ Theorem 262

6.2 Invariant subspaces

Definition 264. If $S \subseteq \mathcal{B}(X)$ and $\mathcal{M}$ is a closed subspace of X, we say $\mathcal{M}$ is *invariant* for S if $s\mathcal{M} \subseteq \mathcal{M}$ for all $s \in S$. We write $\operatorname{Lat}(S)$ for the set of all S-invariant subspaces of X.

Remark 265. We have $\{ 0 \}, X \in \operatorname{Lat}(S)$. If all $\mathcal{M}_\alpha \in \operatorname{Lat}(S)$, then

$$\bigcap_\alpha \mathcal{M}_\alpha \in \operatorname{Lat}(S)$$

If we further have that

$$\bigvee_\alpha \mathcal{M}_\alpha \in \operatorname{Lat}(S)$$

then it is called a *complete lattice*.

Definition 266. If $\mathcal{L}$ is a collection of subspaces, we define

$$\operatorname{Alg}(\mathcal{L}) = \{ A \in \mathcal{B}(X) : AM \subseteq M \text{ for all } M \in \mathcal{L} \}$$

Remark 267. $\operatorname{Alg}(\mathcal{L})$ is an algebra containing I: if $AM \subseteq M$ and $BM \subseteq M$, then $(\alpha A + \beta B)M \subseteq M$ and $ABM \subseteq AM \subseteq M$. Furthermore, if $A_\alpha \in \operatorname{Alg}(\mathcal{L})$ with $A_\alpha \xrightarrow{\text{WOT}} A$, then $\varphi(A_\alpha x) \to \varphi(Ax)$ for all $x \in X$ and all $\varphi \in X^*$. If $x \in M$ and $\varphi \in M^\perp$, then

$$\varphi(Ax) = \lim \varphi(A_\alpha x) = 0$$

Ao $Ax \in M$. So $\operatorname{Alg}(\mathcal{L})$ is a WOT-closed unital algebra.

Remark 268. If $\mathcal{A}$ is an algebra, we have $\mathrm{Alg}(\mathrm{Lat}(\mathcal{A})) \supseteq \mathcal{A}$; we say $\mathcal{A}$ is *reflexive* if $\mathcal{A} = \mathrm{Alg}(\mathrm{Lat}(\mathcal{A}))$. **Note: this differs from our prior usage.**

Simlarly, if $\mathcal{L}$ is a lattice, then $\mathrm{Lat}(\mathrm{Alg}(\mathcal{L})) \supseteq \mathcal{L}$.

Example 269. Recall the Volterra operator

$$Vf(x) = \int_0^x f(t)dt$$

on $\mathcal{B}(L^2(0,1))$. Then

$$N_t = \{\, f : \mathrm{supp}(f) \subseteq [t,1] \,\} \in \mathrm{Lat}(V)$$

Theorem 270. $\mathrm{Lat}(V) = \{\, N_t : 0 \leq t \leq 1 \,\}$.

TODO 3. *Last two lectures.*