Lecture Notes

# PMATH 450/650: Lebesgue Integration and Fourier Analysis

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Term: Spring 2018

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Last revised: July 26, 2018

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## Preface and Acknowledgments

This PDF document includes lecture notes for PMATH 450/650 - Lebesgue Integration and Fourier Analysis taught by Stephen New in Spring 2018.

For any questions e-mail me at `c2kent(at)uwaterloo(dot)ca`.

These notes are meant to be used as a supplementary reference for the lecture notes posted by the instructor.

Throughout these notes when we say "textbook" we mean `Real Analysis`, by A. Bruckner, J. Bruckner and B. Thomson.

Calvin KENT

**Start of Lecture 1**

Course webpage: http://www.math.uwaterloo.ca/~snew/

**Definition 1.1:** We define the **outer Jordan Content** ($c^*$) of set $A$ as

$$c^*(A) = \inf\left\{ \sum_{R_{ij}\cap A\neq\varnothing} |R_{ij}| \;\middle|\; \begin{array}{c} R \text{ is a rectangle with} \\ A \subseteq R \text{ and } P \text{ is a} \\ \text{double partition of } R \end{array} \right\}.$$

**Definition 1.2:** We define the **inner Jordan Content** ($c_*$) of set $A$ as

$$c_*(A) = \sup\left\{ \sum_{R_{ij}\cap A\neq\varnothing} |R_{ij}| \;\middle|\; \begin{array}{c} R \text{ is a rectangle with} \\ A \subseteq R \text{ and } P \text{ is a} \\ \text{double partition of } R \end{array} \right\} \text{ or equivalently}$$
$$= \sup\left\{ |R| - \sum_{R_{ij}\cap (R\setminus A)\neq\varnothing} |R_{ij}| \;\middle|\; \begin{array}{c} R \text{ is a rectangle with} \\ A \subseteq R \text{ and } P \text{ is a} \\ \text{double partition of } R \end{array} \right\}.$$

**Remark 1.3:** $A$ has a well-defined Jordan content when $c^*(A) = c_*(A)$. We denote Jordan content was $c(A) = c^*(A) = c_*(A)$.

**Definition 1.4:** We define the **outer Jordan content** of a bounded set $A \subseteq \mathbb{R}$ as

$$c^*(A) = \inf\left\{ \sum_{i=1}^{n} |R_i| \;\middle|\; \begin{array}{c} n \in \mathbb{Z}^+ \text{ and each } R_i \text{ is a bounded} \\ \text{open interval with } A \subseteq \displaystyle\bigcup_{i=0}^{n} R_i \end{array} \right\}.$$

**Theorem 1.5** (Properties of outer Jordan content)**:**

1. (Translation) If $A \subseteq \mathbb{R}$ and $a \in \mathbb{R}$, then $c^*(a + A) = c^*(A)$ where $a + A = \{a + x \mid x \in A\}$.
2. (Scaling) If $0 \neq r \in \mathbb{R}$ then $c^*(rA) = rc^*(A)$.
3. (Inclusion) If$A \subseteq B \subseteq \mathbb{R}$, then $c^*(A) \leq c^*(B)$.
4. If $A \subseteq \mathbb{R}$ is finite, then $c^*(A) = 0$.
5. If $I = (a, b), (a, b], [a, b)$ or $[a, b]$ where $a, b \in \mathbb{R}$ with $a \leq b$ then $c^*(I) = |I| = b - a$.
6. (Subadditivity) If $A, B \subseteq \mathbb{R}$ then $c^*(A \cup B) \leq c^*(A) + c^*(B)$.
7. We have $c^*(\overline{A}) = c^*(A)$.

**Exercise 1.6:** Prove these theorems.

**Remark 1.7:** We would also like to have to property that if $A, B \subseteq \mathbb{R}$ with $A \cap B = \varnothing$ then $c^*(A \cup B) = c^*(A) + c^*(B)$ but this property does not hold.

**Example 1.8:** If $A = [0, 1] \cap \mathbb{Q}$ and $B = [0, 1] \setminus \mathbb{Q}$ then $c^*(A) = 1, c^*(B) = 1$ but $c^*(A \cup B) = c^*([0, 1]) = 1 \neq 2$.

**End of Lecture 1**

**Start of Lecture 2**

**Definition 2.1:** For the interval $I = (a, b), (a, b], [a, b)or[a, b]$ where $a, b \in \mathbb{R}$ with $a \leq b$ we define $|I| = b-a$ and for the unbounded intervals $I = (-\infty, a), (-\infty, a], (a, \infty)[a, \infty), or(-\infty, \infty)$ where $a \in \mathbb{R}$ we define $|I| = \infty$.

**Definition 2.2:** We define the **outer Lebesgue measure** of a bounded set $A \subseteq \mathbb{R}$ as

$$\lambda^*(A) = \inf\left\{ \sum_{i=1}^{n} |R_i| \;\middle|\; \begin{array}{c} \text{each } R_i \text{ is a bounded} \\ \text{open interval with } A \subseteq \bigcup_{i=0}^{n} R_i \end{array} \right\}.$$

**Notation 2.3:** Through out this course, unless otherwise specified, when we say "measure" we mean Lebesgue measure.

**Theorem 2.4** (Properties of outer Lebesgue measure)**:**

1. If $A$ is finite or countable then $\lambda^*(A) = 0$.
2. (Inclusion) If $A \subseteq B \subseteq \mathbb{R}$ then $\lambda^*(A) \leq \lambda^*(B)$.
3. (Translation) If $A \subseteq \mathbb{R}$ and $a \in \mathbb{R}$ then $\underbrace{\lambda^*(a + A) = \lambda^*(A)}_{\text{bijective correspondence}}$ where $a + A = \{a + x \mid x \in A\}$.
4. (Scaling) If $0 \neq r \in \mathbb{R}$ then $\lambda^*(rA) = r\lambda^*(A)$.
5. (Intervals) If $I$ is an interval, then $\lambda^*(I) = |I|$.
6. (Subadditivity) If $A, B \subseteq \mathbb{R}$ then $\lambda^*(A \cup B) \leq \lambda^*(A) + \lambda^*(B)$. More generally, if $A_1, A_2, \ldots \in \mathbb{R}$, then
$$\lambda^*\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \lambda^*(A_k)$$
.

Proof of 1.

Let $A = \{a_1, a_2, \ldots\}$. Let $\varepsilon > 0$. For each $k \in \mathbb{Z}^+$ let $I_k = \left(a_k - \dfrac{\varepsilon}{2^k}, a_k + \dfrac{\varepsilon}{2^k}\right)$. Then $a_k \in I_k$. So $A \subseteq \bigcup_{k=1}^{\infty} I_k$. Thus $\lambda^*(A) \leq \sum_{k=1}^{\infty} |I_k| = \varepsilon + \dfrac{\varepsilon}{2} + \dfrac{\varepsilon}{4} + \ldots = 2\varepsilon$. Since $\lambda^*(A) \leq 2\varepsilon$, for all $\varepsilon > 0$, we have $\lambda^*(A) = 0$.

**Exercise 2.5:** Proofs of *2*, *3*, and *4* are left as exercises.

Proof of 5.

Let $I = (a, b), (a, b], [a, b)or[a, b]$ where $a, b \in \mathbb{R}$ with $a \leq b$. Let $\varepsilon > 0$. Let $I_1 = \big(a - \varepsilon/2, a + \varepsilon/2\big)$ so that $I \subseteq I_1$ and $|I| = (b - a) + \varepsilon = |I| + \varepsilon$ and let $I_2 = I_3 = \ldots = \varnothing$. Then $I \subseteq \bigcup_{k=1}^{\infty} I_k$ and $\sum_{k=1}^{\infty} \lambda^*(I_k) = \lambda^*(I_k) = |I_k| = |I| + \varepsilon$. It follows that $\lambda^*(I) \leq |I| + \varepsilon$. Since this holds for all $\varepsilon > 0$, we have $\lambda^*(I) \leq |I| = b - a$.

**Remark 2.6:** We could also have used transfinite induction on $I_i$'s to arrive this conclusion.

Now let $I_k$ be any bounded open interval such that $(a,b)$ or $[a,b)$ or $(a,b] = I \subseteq \bigcup_{k=1}^{\infty} I_k$. Let $\varepsilon > 0$ arbitrary with $\varepsilon < b-a$. Let $k = \left[a-\varepsilon/2, a+\varepsilon/2\right]$. Since $K$ is compact, we can extract a finite subcover of $\{I_1, I_2, \ldots\}$. Let this subcover be $\{I_1, I_2, \ldots, I_m\}$ (after possibly reordering). Choose $I_{k_1} = (a_1, b_1)$ with $a_1 < a+\varepsilon/2 < b_1$, and if $b_1 \leq b-\varepsilon/2$ then choose $I_{k_2} = (a_2, b_2)$ with $a_2 < b_1$, $b_2 > b_1$. If $b_2 \leq b-\varepsilon/2$ choose $I_{k_3} = (a_3, b_3)$ with $a_3 < b_2$, $b_3 > b_2$. Eventually we obtain intervals,

$$(a_1,b_1),(a_2,b_2),\ldots,(a_\ell,b_\ell) \text{ with } a_1 < a+\frac{\varepsilon}{2}, a_2 < b_1, a_3 < b_2, \ldots, a_\ell < b_{\ell-1}, b_\ell > b-\frac{\varepsilon}{2}.$$

We have

$$\begin{aligned}\sum_{i=1}^{\ell} \lambda^*(a_i,b_i) &= \sum_{i=1}^{\ell}(b_i - a_i) = (b_1-a_1)+(b_2-a_2)+\ldots+(b_\ell-a_\ell)\\ &\geq a_2 - (a+\frac{\varepsilon}{2}) + (a_3-a_2)+(a_4-a_3)+\ldots+(a_\ell-a_{\ell-1})+(b-\frac{\varepsilon}{2})-a_\ell\\ &= (b-a)-\varepsilon.\end{aligned}$$

Since $\varepsilon$ is arbitrary, $\sum_{i=1}^{\ell}(b-a \geq (b-a))$. Thus,

$$\sum_{k=1}^{\infty}\lambda^*(I_k) \geq \sum_{i=1}^{\ell}(b_i-a_i) \geq b-a.$$

Thus, $\lambda^*(I) \geq b-a$. Since we have both $\lambda^*(I) \leq |I| = b-a$ and $\lambda^*(I) \geq b-a$ then $\lambda^*(I) = |I|$.

**End of Lecture 2**

**Start of Lecture 3**

**Remark 3.1:** Recall that we proved *part 5* last lecture in the case of a bounded interval $I$. When $I$ is an unbounded interval, for any $\mathbb{R} > 0$, we can choose a bounded interval $J \subseteq I$ with $|J| = R$ then by the inclusion property, we have $\lambda^*(I) \geq \lambda^*(J) = R$. Since $R$ was arbitrary, we have $\lambda^*(I) = \infty$.

**Recall 3.2:** Recall the $6^{\text{th}}$ (subadditivity) property.

6. (Subadditivity) If $A, B \subseteq \mathbb{R}$ then $\lambda^*(A \cup B) \leq \lambda^*(A) + \lambda^*(B)$. More generally, if $A_1, A_2, \ldots \in \mathbb{R}$, then

$$\lambda^*\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \lambda^*(A_k).$$

Proof of 6.

Let $A_1, A_2, \ldots \subseteq \mathbb{R}$. Let $\varepsilon > 0$. For each $k \in \mathbb{Z}^+$ choose bounded open intervals $I_{k_1}, I_{k_2}, \ldots$ so that

$$A_k \subseteq \bigcup_{i=1}^{\infty} I_{k_i} \text{ and } \sum_{i=1}^{\infty} |I_{k_i}| < \lambda^T A_k + \frac{\varepsilon}{2^k}.$$

Then $\displaystyle\bigcup_{k=1}^{\infty} A_k \subseteq \bigcup_{k,i} I_{k_i}$.

$$\begin{aligned}
\text{So } \lambda^*\left(\bigcup_{k=1}^{\infty} A_k\right) &\leq \sum_{k,i} |I_{k_i}| \\
&= \sum_{k}^{\infty}\sum_{i}^{\infty} |I_{k_i}| \\
&\leq \sum_{k}^{\infty}\left(\lambda^*(A_k) + \frac{\varepsilon}{2^k}\right) \\
&= \sum_{k}^{\infty} \lambda^*(A_k) + \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we get $\displaystyle\lambda^*\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \lambda^*(A_k)$ as required.

**Definition 3.3:** We define the **lower Jordan content** of a bounded set $A \subseteq \mathbb{R}$ as

$$c_*(A) = |I| - c^*(I \setminus A),$$

where $I$ is a bounded interval which contains $A \subseteq I$ (if we wish we can require that $I$ is the smallest closed interval which contains $A$).

**Definition 3.4:** We say that $A$ has a **(well-defined) Jordan content** when $c^*(A) = c_*(A)$ and in this case, we define the Jordan content of $A$ to be $c(A) = c^*(A) = c_*(A)$.

**Remark 3.5:** We could define the lower Lebesgue measure of a bounded set $A \subseteq \mathbb{R}$ to be $\lambda^*(A) = |I| - \lambda^*(I \setminus A)$ where $I$ is any bounded interval containing $A$ (or where $I = [\inf(A), \sup(A)]$) and we could (but don't) define $A$ to be Lebesgue measurable when $\lambda^*(A) = \lambda_*(A)$.

**Definition 3.6:** For $A \subseteq \mathbb{R}$ (A is not necessarily bounded), $A$ is **Lebesgue measurable** when for all sets $X \in \mathbb{R}$,

$$\lambda^*(X) = \lambda^*(X \cap A) + \lambda^*(X \setminus A).$$

In this case, we define the **Lebesgue measure** of $A$ to be $\lambda(A) = \lambda^*(A)$ and denote the set of all measurable subsets of $\mathbb{R}$ as $\mathcal{M}$.

**Theorem 3.7** (Properties of Lebesgue measure)**:**

1. For $A \subseteq \mathbb{R}$ and $a \in \mathbb{R}$, $A$ is measurable iff $a + A$ is measurable. This means translation is measurable.
2. if $0 \neq r \in \mathbb{R}$ then $A$ is measurable iff $rA$ is measurable.
3. $\varnothing$ and $\mathbb{R}$ are measurable.
4. For $A \subseteq \mathbb{R}$, if $\lambda^*(A) = 0$ then $A$ is measurable.
5. For $A \subseteq \mathbb{R}$, if $A$ is measurable then so is $A^c = \mathbb{R} \setminus A$.
6. If $A, B \subseteq \mathbb{R}$ are both measurable then so are $A \cup B$, $A \cap B$ and $A \setminus B$.
7. Every interval $I$ is measurable.
8. If $A_1, A_2, \ldots \subseteq \mathbb{R}$ are all measurable, then so are (countable )$\displaystyle\bigcup_{k=1}^{\infty} A_k$ and (countable) $\displaystyle\bigcap_{k=1}^{\infty} A_k$.
9. (Additivity) If $A_1, A_2, \ldots \subseteq \mathbb{R}$ are measurable and disjoint then $\displaystyle\lambda\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty}(A_k)$

**Remark 3.8:** Before beginning the proof, we remark that for any sets $A, X \subseteq$ we have $(X \cap A) \cup (X \setminus A) = X$. So by subadditivity $\lambda^*(X) \leq \lambda^*(X \cap A) + \lambda^*(X \setminus A)$. thus, $A \subseteq \mathbb{R}$ is measurable iff for all $X \subseteq \mathbb{R}$,

$$\lambda^*(X) \geq \lambda^*(X \cap A) + \lambda^*(X \setminus A).$$

**Exercise 3.9:** Prove *1* and *2*.

Proof of 3.

$\varnothing$ is measurable because for all $X \subseteq \mathbb{R}$ we have

$$\lambda^*(X \cap \varnothing) + \lambda^*(X \setminus \varnothing) = \lambda^*(\varnothing) + \lambda^*(X) = 0 + \lambda^*(X) = \lambda^*(X)$$

and $\mathbb{R}$ is measurable because $\forall\ X \in \mathbb{R}$ we have

$$\lambda^*(X \cap \mathbb{R}) + \lambda^*(X \setminus \mathbb{R}) = \lambda^*(X) + \lambda^*(\varnothing) = \lambda^*(X) + 0 = \lambda^*(X).$$

**Remark 3.10:** Also note after we show $\varnothing$ is measurable, by Property 4, $\varnothing^c = \mathbb{R}$ is also measurable.

Proof of 4.

Let $A \subseteq \mathbb{R}$ with $\lambda^*(A) = 0$. Let $X \subseteq \mathbb{R}$. Then

$$\begin{aligned} \lambda^*(X \cap A) + \lambda^*(X \setminus A) &\leq \lambda^*(A) + \lambda^*(X) \qquad \text{(by inclusion property, since } X \cup A \subseteq A, X \setminus A \subseteq A) \\ &= 0 + \lambda^*(X) = \lambda^*(X). \end{aligned}$$

Proof of 5.

Let $A \subseteq \mathbb{R}$ be measurable. Let $X \subseteq \mathbb{R}$. Then $\lambda^*(X \cap A^c) + \lambda^*(X \setminus A^c) = \lambda^*(X \setminus A) + \lambda^*(X \cap A) = \lambda^*(X)$.

Proof of 6.

Suppose $A, B \subseteq \mathbb{R}$ are measurable. Let $X \subseteq \mathbb{R}$. We have

$$\begin{aligned}
\lambda^*(X) &= \lambda^*(X \cap A)\lambda^*(X \setminus A) && \text{(since } A \text{ is measurable)} \\
&= \lambda^*(X \cap A) + \lambda^*((X \setminus A) \cap B) + \lambda^*((X \setminus A) \setminus B) && \text{(since } B \text{ is measurable)} \\
&\geq \lambda^*(X \cap (A \cup B)) + \lambda^*((X \setminus A) \setminus B) && \text{(by subadditivity)} \\
&= \lambda^*(X \cap (A \cup B)) + \lambda^*(X \setminus (A \cup B)) && \text{(since } (X \cap A) \cup ((X \setminus A) \cap B) = X \cap (A \cup B))
\end{aligned}$$

Hence $A \cup B$ is measurable. Then $A^c \cup B^c$ is measurable and thus $(A^c \cup B^c)^c = A \cap B$ is measurable. Then $A \cap B^c = A \setminus B$ is also measurable.

**End of Lecture 3**

**Start of Lecture 4**

Proof of 7.

This content is covered in section 2.10 in the textbook.

If $I = \varnothing$ or $I = \{a\}$ (i.e $I$ is a degenerate interval) then $I$ is measurable because $\lambda^*(I) = 0$.

Suppose $I = (a, b)$ where $a, b \in \mathbb{R}$ with $a < b$. Let $X \subseteq \mathbb{R}$. Let $\varepsilon > 0$. Choose bounded open intervals $I_1, I_2, \ldots \subseteq \mathbb{R}$ so that $X \subseteq \bigcup_{k=1}^{\infty} I_k$ and $\sum_{k=1}^{\infty} |I_k| \leq \lambda^*(X) + \varepsilon$. For each $n \in Z^+$ let

$$J_n = I_n \cap (a, b), K_n = I_n \cap (-\infty, a), \text{ and } L_n = I_n \cap (b, \infty).$$

Then $X \cap (a, b) \subseteq \bigcup_{n=1}^{\infty} J_n$.

So $\lambda^*(X \cap (a, b)) \leq \sum_{n=1}^{\infty} |J_n|$ and $X \setminus (a, b) \subseteq \bigcup_{n=1}^{\infty} K_n \cup \bigcup_{n=1}^{\infty} L_n \cup \underbrace{(a - \varepsilon, a + \varepsilon)}_{\text{an interval containing } a} \cup \underbrace{(b - \varepsilon, b + \varepsilon)}_{\text{an interval containing } b}$.

So $\lambda^*(X \setminus (a, b)) \leq \sum_{n=1}^{\infty} |K_n| + \sum_{n=1}^{\infty} |L_n| + 4\varepsilon$. Thus,

$$\begin{aligned}
\lambda^*(X \cap (a, b)) + \lambda^*(X \setminus (a, b)) &\leq \sum_{n=1}^{\infty} \big(|J_n| + |K_n| + |L_n|\big) + 4\varepsilon \\
&= \sum_{n=1}^{\infty} |I_n| + 4\varepsilon \\
&\leq \lambda^*(X) + 5\varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, then

$$\lambda^*(X \cap (a, b)) + \lambda^*(X \setminus (a, b)) \leq \lambda^*(X).$$

Thus, $(a, b)$ is measurable.

This proves *Part 6* for bounded open intervals. Note that every interval can be obtained from bounded open intervals using countable unions or intersections and/or compliments. Hence every interval $I$ is measurable.

**Remark 4.1:** To help proving *8* and *9* note that for $X \subseteq \mathbb{R}$, if $A, B \subseteq \mathbb{R}$ are measurable and disjoint then

$$\begin{aligned}
\lambda^*(X \cap (A \cup B)) &= \lambda^*((X \cap (A \cup B)) \cap A) + \lambda^*((X \cap (A \cup B)) \setminus A) \\
&= \lambda^*(X \cap A) + \lambda^*(X \cap B).
\end{aligned}$$

By induction, if $A_1, A_2, \ldots, A_n$ are measurable and disjoint then $\bigcup_{k=1}^{n} A_k$ is measurable and for all sets $X \subseteq \mathbb{R}$ we have

$$\lambda^*(X \cap \bigcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} \lambda^*(X \cap A_k).$$

Proof of 8 and 9.

Let $A_1, A_2, \ldots \subseteq \mathbb{R}$ be measurable and disjoint. For each $n \in \mathbb{Z}^+$, we have

$$\begin{aligned}
\sum_{k=1}^{n} \lambda^*(X \cap A_k) &= \lambda^*\left(X \cap \bigcup_{k=1}^{n} A_k\right) \\
&\leq \lambda^*\left(X \cap \bigcup_{k=1}^{\infty} A_k\right) && \text{(by inclusion)} \\
&= \lambda^*\left(\bigcup_{k=1}^{\infty} X \cap A_k\right) \\
&\leq \sum_{k=1}^{\infty} \lambda^*(X \cap A_k) && \text{(by subadditivity)}
\end{aligned}$$

Taking the limit as $n \to \infty$ we obtain

$$\sum_{k=1}^{\infty} \lambda^*(X \cap A_k) \leq \sum_{k=1}^{\infty} \lambda^*\left(X \cap \bigcup_{k=1}^{\infty} A_k\right) z \leq \sum_{k=1}^{\infty} \lambda^*(X \cap A_k).$$

So,

$$\sum_{k=1}^{\infty} \lambda^*\left(X \cap \bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \lambda^*(X \cap A_k).$$

In particular, taking $X = \mathbb{R}$ gives $\lambda^*\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \lambda^*(A_k)$ which is needed for part *8*.

We still need to show that $\bigcup_{k=1}^{\infty} A_k$ is measurable. Let $X \subseteq \mathbb{R}$. For all $n \in \mathbb{Z}^+$,

$$\begin{aligned}
\lambda^*(X) &= \lambda^*\left(X \cap \bigcup_{k=1}^{n} A_k\right) + \lambda^*\left(X \setminus \bigcup_{k=1}^{n} A_k\right) \\
&= \sum_{k=1}^{n} \lambda^*(X \cap A_k) + \lambda^*\left(X \setminus \bigcup_{k=1}^{n} A_k\right) \\
&\leq \sum_{k=1}^{n} \lambda^*(X \cap A_k) + \lambda^*\left(X \setminus \bigcup_{k=1}^{n} A_k\right) && \text{(by inclusion)}
\end{aligned}$$

Taking the limit as $n \to \infty$ gives us

$$\lambda^*(X) = \lambda^*\left(X \cap \bigcup_{k=1}^{\infty} A_k\right) + \lambda^*\left(X \setminus \bigcup_{k=1}^{n} A_k\right).$$

Thus, $\bigcup_{k=1}^{n} A_k$ is measurable when $A_1, A_2, \ldots \subseteq \mathbb{R}$ are measurable and disjoint.

When $A_1, A_2, \ldots \subseteq \mathbb{R}$ are measurable but not necessarily disjoint we have

$$\bigcup_{k=1}^{\infty} A_k = A_1 \cup (A_2 \setminus A_1) \cup (A_3 \setminus (A_1 \cup A_2) \cup (A_4 \setminus (A_1 \cup A_2 \cup A_3 \cup A_4)) \cup \ldots$$

We can rewrite any countable union of disjoint sets as a union of disjoint sets which is measurable since it is a countable union of disjoint measurable sets.

**Corollary 4.2:**

1. If $A_1, A_2, \ldots \subseteq \mathbb{R}$ are measurable with $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$ (where they form an increasing chain) then
$$\lambda\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \lambda(A_k).$$

2. If $A_1, A_2, \ldots \subseteq \mathbb{R}$ are measurable with $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$ (where they form an descending chain) and $\lambda(A_m) < \infty$ (measure of $A_m$ is finite) then
$$\lambda\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \lambda(A_k).$$

Proof of Corollary 1.

Suppose $A_1, A_2, \ldots \subseteq \mathbb{R}$ are measurable with $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$.
Let
$$B = A_1,\ B_2 = A_2 \setminus A_2,\ \ldots,\ B_k = A_k \setminus A_{k-1}.$$

Then,

$$\begin{aligned}
\lambda\left(\bigcup_{n=1}^{\infty} A_n\right) &= \lambda\left(\bigcup_{k=1}^{\infty} A_k\right) && \left(\text{since } \textstyle\bigcup_{n=1}^{\infty} A_n = \bigcup_{k=1}^{\infty} B_k\right)\\
&= \sum_{n=1}^{\infty} \lambda(B_n) && (\text{since the } B_k \text{ are disjoint})\\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \lambda(B_k)\\
&= \lim_{n\to\infty} \lambda\left(\bigcup_{k=1}^{n} B_k\right) && (\text{since the } B_k \text{ are disjoint})\\
&= \lim_{n\to\infty} \lambda(A_n). && \left(\text{since } A_n = \textstyle\bigcup_{k=1}^{n} B_k\right)
\end{aligned}$$

Proof of Corollary 2.

Suppose $A_1, A_2, \ldots \subseteq \mathbb{R}$ are measurable with $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$ (where they form an descending chain) and $\lambda(A_m) < \infty$. Then

$$\begin{aligned}
\lambda\left(\bigcap_{k=1}^{\infty} A_k\right) &= \lambda\left(\bigcap_{k=m}^{\infty} A_k\right) \\
&= \lambda\left(A_m \setminus \bigcap_{k=m}^{\infty} (A_m \setminus A_k)\right) && \text{(by the equivalence relation of } S = (S^c)^c\text{)}
\end{aligned}$$

Since $A_m$ is the disjoint union of $\displaystyle\bigcup_{k=m}^{\infty} (A_m \setminus A_k)$ and $A_m \setminus \displaystyle\bigcup_{k=m}^{\infty} (A_m \setminus A_k)$ then,

$$\begin{aligned}
\lambda\left(\bigcap_{k=1}^{\infty} A_k\right) &= \lambda(A_m) - \lambda\left(\bigcup_{k=m}^{\infty} (A_m \setminus A_k)\right) \\
&= \lambda(A_m) - \lim_{k\to\infty} \lambda(A_m \setminus A_k)
\end{aligned}$$

Since $A_m$ is the disjoint union of $A_k$ and $A_m \setminus A_k$ then,

$$\begin{aligned}
&= \lambda(A_m) - \lim_{k\to\infty} (\lambda(A_m) - \lambda(A_k)) \\
&= \lambda(A_m) - \lambda(A_m) + \lim_{k\to\infty} \lambda(A_k) \\
&= \lim_{k\to\infty} \lambda(A_k).
\end{aligned}$$

**End of Lecture 4**

**Start of Lecture 5**

**Corollary 5.1:**

1. All open sets and all closed sets in $R$ are measurable.
2. All Borel sets in $R$ are measurable.

Proof of 1.

We make use of the following properties in our proof:

**Recall 5.2:**

1. Any $X \subseteq \mathbb{R}^n$ is equal to the disjoint union of its connected components.
2. If $U \subseteq \mathbb{R}^n$ is open the its connected components are open, also $U$ has at most countable many components.
3. The connected subsets of $\mathbb{R}$ are the intervals.

It follows that every open set $U \subseteq \mathbb{R}$ is equal to a finite or countable disjoint union of open intervals when $U = \bigcup_{k=1}^{n} I_k$ or $U = \bigcup_{k=1}^{\infty} I_k$ where the $I_k$ are the connected components of $U$. Hence, by property *8*, we have

$$\lambda(U) = \sum_k I_k.$$

**Aside:** Supplementary documents on Borel sets and $\sigma-$algebra:

- `http://nptel.ac.in/courses/108106083/lecture7_Borel%20Sets%20and%20Lebesgue%20Measure.pdf`
- `http://stat.math.uregina.ca/~kozdron/Teaching/Regina/451Fall13/Handouts/451lecture05.pdf`

**Notation 5.3:** When $\mathcal{C}$ is a set of subsets of $\mathbb{R}$ with $\varnothing \in \mathcal{C}$ and $\mathbb{R} \in \mathcal{C}$, we write

$$\mathcal{C}_\sigma = \left\{ \bigcup_{k=1}^{\infty} A_k \;\middle|\; \text{each } A_k \in \mathcal{C} \right\} \text{ and } \mathcal{C}_\delta = \left\{ \bigcap_{k=1}^{\infty} A_k \;\middle|\; \text{each } A_k \in \mathcal{C} \right\}.$$

Note that $\mathcal{C}_{\sigma\sigma} = \mathcal{C}_\sigma$ and $\mathcal{C}_{\delta\delta} = \mathcal{C}_\delta$ and also that $\mathcal{G}_\sigma = \mathcal{G}$ and $\mathcal{F}_\delta = \mathcal{F}$.

**Notation 5.4:** We denote the set of all open subsets of $\mathbb{R}$ as $\mathcal{G}$ and the set of all closed subsets of $\mathbb{R}$ as $\mathcal{F}$.

**Definition 5.5:** A set $\mathcal{C}$ of subsets of $\mathbb{R}$ is called a $\sigma-$**algebra** in $\mathbb{R}$ if

1. $\varnothing \in \mathcal{C}$.
2. If $A \in \mathcal{C}$ then $A^c = \mathbb{R} \setminus A \in \mathcal{C}$.
3. If $A_1, A_2, \ldots \in \mathcal{C}$ then $\bigcup_{k=1}^{\infty} A_k \in \mathcal{C}$. Equivalently, if $A_1, A_2, \ldots \in \mathcal{C}$ then $\mathcal{C}_\sigma = \mathcal{C}$.

**Remark 5.6:** Given any set $S$ of subsets $\mathbb{R}$, there is a unique smallest $\sigma-$algebra in $\mathbb{R}$ which contains $S$, namely the intersection of all the $\sigma-$algebras in $\mathbb{R}$ which contain S.

**Definition 5.7:** The **Borel** $\sigma-$**algebra** in $\mathbb{R}$ is the smallest $\sigma-$algebra in $\mathbb{R}$ (denoted as $\mathcal{B}$) which contains $\mathcal{G}$ (hence also $\mathcal{F}$). The elements in $\mathcal{B}$ are called Borel sets.

**Remark 5.8:** Note that $\mathcal{B}$ includes the sets

$$\mathcal{G}, \mathcal{G}_\delta, \mathcal{G}_{\delta\sigma}, \mathcal{G}_{\delta\sigma\delta}, \dots \text{ and } \mathcal{F}, \mathcal{F}_\sigma, \mathcal{F}_{\sigma\delta}, \mathcal{F}_{\sigma\delta\sigma}, \dots$$

where $\mathcal{G}_\delta$ denotes the countable intersection of all open subsets in $\mathbb{R}$ and $\mathcal{G}_{\delta\sigma}$ denotes the countable union of the countable intersection of all open subsets in $\mathbb{R}$ etc. In other words,

$$\bigcap_{k=1}^{\infty} \bigcup_{\ell=1}^{\infty} \bigcap_{m=1}^{\infty} U_{k,\ell,m} \in \mathcal{G}_{\delta\sigma\delta}$$

**Exercise 5.9:** Show that $\mathcal{F} \subseteq \mathcal{G}_\delta$ and $\mathcal{G} \subseteq \mathcal{F}_\delta$.

**Exercise 5.10** (Challenging)**:** Prove of disprove the following statement.

$$\text{If } \mathcal{H} = \mathcal{G} \cup \mathcal{G}_\delta \cup \mathcal{G}_{\delta\sigma} \cup \dots \text{ then } \mathcal{H} \subsetneq \mathcal{H}_\sigma.$$

**Remark 5.11:** The set $\mathcal{M} = \mathcal{L}$ of Lebesgue measurable sets in $\mathbb{R}$ is a $\sigma-$algebra in $\mathbb{R}$ with $G \subseteq \mathcal{L}$. So by definition of $\mathcal{B}$, we have $\mathcal{B} \subseteq \mathcal{L}$.

**End of Lecture 5**

**Start of Lecture 6**

## Cantor Sets

The (standard) cantor set $\mathcal{C} \subseteq [0,1]$ can be constructed as follows:

We remove the open third of $[0,1]$ by letting

$$I_1 = \left(\frac{1}{3}, \frac{2}{3}\right),\ U_1 = I_1,\ \mathcal{C}_1 = U_1^c = [0,1] \setminus U = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$$

and then removing the open thirds of the two components of $C_1$

$$I_2 = \left(\frac{1}{9}, \frac{2}{9}\right),\ I_3 = \left(\frac{7}{9}, \frac{8}{9}\right),\ U_2 = I_1 \cup I_2 \cup I_3,\ \mathcal{C}_2 = U_2^c = [0,1] \setminus U_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{3}{9}\right] \cup \left[\frac{6}{9}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right].$$

We continue this procedure to get

$$U_n = I_1 \cup I_2 \cup \ldots \cup I_{2^n - 1},\ \mathcal{C}_n = U_n^c$$

where $\mathcal{C}_n$ is the disjoint union of $2^n$ closed intervals each of size $\frac{1}{3^n}$ and $U_1 \subseteq U_2 \subseteq \ldots$ and $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \ldots$ We let

$$U = \bigcup_{k=1}^{\infty} U_k \text{ and } \mathcal{C} = U^c = \bigcap_{k=1}^{\infty} \mathcal{C}_k.$$

The $\mathcal{C} \subseteq [0,1]$ is closed (hence measurable) and

$$\lambda(\mathcal{C}_n) = 2^n \frac{1}{3^n} \tag{1}$$

$$\lambda(\mathcal{C}_n) = \lim_{n\to\infty} \lambda(\mathcal{C}_n) = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0. \tag{2}$$

Alternatively, $\mathcal{C}$ is equal to the set of real numbers in $[0,1]$ which can be written in base 3 using only the digits of 0 and 2.

Proof sketch.

$[0,1]$ can be written as $0.*$

$\left[0, \frac{1}{3}\right]$ can be written as $0.0*$

$\left[0, \frac{1}{9}\right]$ can be written as $0.00*$

$\left[\frac{6}{9}, \frac{7}{9}\right]$ can be written as $0.20*$

$\left[\frac{2}{3}, 1\right]$ can be written as $0.2*$

$\left[\frac{2}{9}, \frac{3}{9}\right]$ can be written as $0.02*$

$\left[\frac{8}{9}, 1\right]$ can be written as $0.22*$

etc.

### Properties of Cantor Sets

1. $\mathcal{C}$ is closed.
2. $\mathcal{C}$ is nowhere dense (which means that for every non-degenerate interval $I$, there is a non-degenerate interval $J \subseteq I$ with $J \cap \mathcal{C} = \varnothing$). Or equivalently, that $\overline{\mathcal{C}}^{\circ} = \varnothing$ where $\overline{\mathcal{C}}^{\circ}$ denotes the closure of $\mathcal{C}$'s interior.
3. $\mathcal{C}$ has no isolated points.
4. The cardinality of $\mathcal{C}$ is $|\mathcal{C}| = 2^{\aleph_0}$.
5. $\lambda(\mathcal{C}) = 0$.

We can modify the above construction to obtain a generalized cantor set $\mathcal{C} \subseteq [0,1]$ with any value $\lambda(\mathcal{C}) = \ell$ with $0 \leq \ell < 1$.

Choose any sequence of positive real numbers $a_1, a_2, \ldots$ with $\displaystyle\sum_{k=1}^{\infty} a_k = 1 - \ell$. Let $U_0 = \varnothing$, $\mathcal{C}_0 = U_0^c = [0,1]$. Choose an open interval $I_1 \subseteq \mathcal{C}_0 = [0,1]$ which contains the midpoint $\dfrac{1}{2}$ with $|I_1| = a_1$. We choose midpoint so intervals get smaller in each step (which will be explained next). Then let $U_1 = I_1$ and $\mathcal{C}_1 = U_1^c = [0,1]\backslash U_1$ which is a union of two closed intervals each of size $\leq \dfrac{1}{2}$. Choose two nonempty open intervals $I_2$ and $I_3$ in two components of $\mathcal{C}_1$ containing the midpoints with $|I_2| + |I_3| = a_2$ then let $U_2 = I_1 \cup I_2 \cup I_3$, $\mathcal{C}_2 = U_2^c$ which is a disjoint union of four closed intervals each of size $\leq \dfrac{1}{4}$.
Continue to get sets $U_1 \subseteq U_2 \subseteq$ and $\mathcal{C}_1 \supseteq \mathcal{C}_2 \supseteq$ where each $\mathcal{C}_n$ is disjoint union of $2^n$ disjoint closed intervals, each of size $\leq \dfrac{1}{2^n}$ and $\lambda(U_n) = \displaystyle\sum_{k=1}^{n} a_k$ and $\lambda(C_n) = 1 - \displaystyle\sum_{k=1}^{n} a_k$. Let $U = \displaystyle\bigcup_{k=1}^{\infty} U_k$, and $\mathcal{C} = U^c = \displaystyle\bigcap_{k=1}^{\infty} \mathcal{C}_k$. Then

$$\lambda(\mathcal{C}) = \lim_{n\to\infty} \lambda(\mathcal{C}_n) = \lim_{n\to\infty} \left(1 - \sum_{k=1}^{n} a_k\right) = 1 - \sum_{k=1}^{\infty} a_k = \ell.$$

**Exercise 6.1:** Verify that $\mathcal{C}$ is closed, nowhere dense and $|\mathcal{C}| = 2^{\aleph_0}$.

**Remark 6.2:** There is a bijective correspondence between $\mathcal{C}$ and binary sequences. Such correspondence can be shown from the infinite tree constructed by joining the removed mid points.

### Baire Category Theorem

**Definition 6.3:** $A \subseteq \mathbb{R}$ is **dense** when for every non-degenerate interval $I$, $I \cap A \neq \varnothing$. In other words, $\overline{A} = \mathbb{R}$.

$A \subseteq \mathbb{R}$ is **nowhere dense** when for every non-degenerate interval $I$ there exists a non-degenerate interval $J \subseteq I$ with $J \cap A = \varnothing$ or equivalently $\overline{A}^{\circ} = \varnothing$. Then $B = A^c = \mathbb{R} \setminus A$

$$\begin{aligned} A \text{ is nowhere dense} &\iff \overline{A}^{\circ} = \varnothing \\ &\iff \overline{B^{\circ}} = \mathbb{R} \\ &\iff \text{the interior of } B \text{ is dense} \end{aligned}$$

If $A$ is nowhere dense and $B \subseteq A$, then $B$ is nowhere dense.

**Definition 6.4:** We say $A$ is **first category** (of Baire) when $A$ is a countable union of nowhere dense sets. Some authors also refer first category sets as meager.

We say $A$ is **second category** when $A$ is not first category.

We say $A$ is **residual** when $A^c$ is first category. In other words, $A$ is a countable intersection of sets which have dense interior.

**Theorem 6.5** (Baire Category Theorem)**:** The following are equivalent.

1. If $A$ is first category then $A^\circ = \varnothing$.
2. If $A$ is residual then $\overline{A} = \mathbb{R}$ (that is $A$ is dense).
3. If $A$ is a countable union of closed sets with empty interiors then $A^\circ = \varnothing$.
4. If $A$ is a countable intersection of dense open sets then $A$ is dense.

**End of Lecture 6**

**Start of Lecture 7**

- Course website updated: `http://www.math.uwaterloo.ca/~snew/`
- Assignment #1 is posted.
- Course outline is updated.
- Lecture notes are posted: `http://www.math.uwaterloo.ca/~snew/pmath450-2018-S/Notes/notes.pdf`

The content covered in this lecture is included as snippets from the PDF hosted on above website.

**Definition 7.1:** Let $A \subseteq \mathbb{R}$. We say that $A$ is **first category** (or that $A$ is **meagre**) when $A$ is equal to a countable union of nowhere dense sets. We say that $A$ is **second category** when it is not first category. We say that $A$ **residual** when $A^c$ is first category.

**Example 7.2:** Every countable set is first category since if $A = \{a_1, a_2, a_3, \cdots\}$ then we have $A = \bigcup_{k=1}^{\infty}\{a_k\}$. In particular $\mathbb{Q}$ is first category and $\mathbb{Q}^c$ is residual.

**Remark 7.3:** If $A \subseteq \mathbb{R}$ is first category then so is every subset of $A$.

**Remark 7.4:** If $A_1, A_2, A_3, \cdots \subseteq \mathbb{R}$ are are all first category then so is $\bigcup_{k=1}^{\infty} A_k$.

**Theorem 7.5:** (Baire Category theorem)

1. Every first category set has an empty interior.
2. Every residual set is dense.
3. Every countable union of closed sets with empty interiors has an empty interior.
4. Every countable intersection of dense open sets is dense.

Proof.

Parts (1) and (2) are equivalent by taking complements, and Parts (3) and (4) are special cases of Parts (1) and (2), so it suffices to prove Part (1). Let $A \subseteq \mathbb{R}$ be first category, say $A = \bigcup_{k=1}^{\infty} C_k$ where each $C_k$ is nowhere dense. Suppose, for a contradiction, that $A$ has nonempty interior, and choose a nondegenerate closed interval $I_0$ with $I_0 \subseteq A$. Choose a nondegenerate closed interval $I_1 \subseteq I_0$ such that $I_1 \cap C_1 = \emptyset$ (we can do this because $C_1$ is nowhere dense). Choose a nondegenerate closed interval $I_2 \subseteq I_1$ so that $I_2 \cap C_2 = \emptyset$. Continue this procedure to obtain nested closed intervals $I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots$ with $I_0 \subseteq A$ and $I_k \cap C_k = \emptyset$ for $k \geq 1$. Such nested intervals have nonempty intersection, so we can choose $x \in \bigcap_{k=1}^{\infty} C_k$. Since $x \in I_0 \subseteq A$ we have $x \in A$. But for all $k \geq 1$ we have $x \in I_k$ and $I_k \cap C_k = \emptyset$ so that $x \neq C_k$, and it follows that $x \notin \bigcup_{k=1}^{\infty} C_k$, that is $x \notin A$.

**Example 7.6:** Recall that $\mathbb{Q}$ is first category and $\mathbb{Q}^c$ is residual. The Baire Category Theorem shows that $\mathbb{Q}^c$ cannot be first category because if $\mathbb{Q}$ and $\mathbb{Q}^c$ were both first category then $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^c$ would also be first category, but this is not possible since $\mathbb{R}$ does not have empty interior.

**Example 7.7:** For each $n \in \mathbb{Z}^+$, let $f_n : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose that for all $x \in \mathbb{R}$ there exists $n \in \mathbb{Z}^+$ such that $f_n(x) \in \mathbb{Q}$. Prove that there exists $n \in \mathbb{Z}^+$ such that $f_n$ is constant in some nondegenerate interval.

Proof.

Say $\mathbb{Q} = \{a_1, a_2, \ldots\}$ since $\forall\, x \in \mathbb{R}$, $\exists\, n \in \mathbb{Z}^+$, such that

$$f_n(x) \in \mathbb{Q} \text{ and } \bigcup_{n,k \in \mathbb{Z}^+} f_n^{-1}(a_k) = \mathbb{R}.$$

Also, each set $f_n^{-1}(a_k)$ is closed because $f_n$ is continuous and singleton $\{a_k\}$ is closed. So by Part 3 pf the Baire Category Theorem, one of the sets $f_n^{-1}(a_k)$ must have nonempty interior. So we can choose a non-degenerate interval $I \subseteq f^{-1}(a_k)$ and then we have $f_n(x) = a_k$ for all $x \in I$.

**Example 7.8:** If $A$ is countable (say $A = \{a_1, a_2, \ldots\}$) then $A$ is first category since $A = \displaystyle\bigcup_{k=1}^{\infty} \{a_k\}$ and singletons are nowhere dense. So $A^c$ is residual but note that $A^c$ is not first category (equivalently, $A$) is not residual because if $A^c$ was first category, then $\mathbb{R} = A \cup A^c$ would be first category but $\mathbb{R}$ is not first category because $\mathbb{R}$ has nonempty interior.

**Remark 7.9:** Each of the following sets $\mathcal{C}$ of subsets of $\mathbb{R}$

$$\begin{aligned} \mathcal{C} &= \big\{A \subseteq \mathbb{R} \big| \, A \text{ is finite or countable}\big\} \\ \mathcal{C} &= \big\{A \subseteq \mathbb{R} \big| \, \lambda(A) = 0\big\} \\ \mathcal{C} &= \big\{A \subseteq \mathbb{R} \big| \, A \text{ is first category}\big\} \end{aligned}$$

has the following properties:

1. If $A \subseteq B$ and $B \in \mathcal{C}$ then $A \in \mathcal{C}$,
2. If $A_1, A_2, A_3, \dots \in \mathcal{C}$ then $\bigcup_{k=1}^{\infty} A_k \in \mathcal{C}$, and
3. If $A \in \mathcal{C}$ then $A^0 = \emptyset$.

Because of this, it seems reasonable to consider the sets in $\mathcal{C}$ to be, in some sense, “small". The following theorem, then, states that every set in $\mathbb{R}$ is the union of two small sets.

**Theorem 7.10:** Every subset of $\mathbb{R}$ is equal to the disjoint union of a set of measure zero and a set of first category.

Proof.

Let $\mathbb{Q} = \{a_1, a_2, a_3, \cdots\}$. For $k, \ell \in \mathbb{Z}^+$, let $I_{k,\ell} = \big(a_\ell - \dfrac{1}{2^{k+\ell}}\,,\, a_\ell + \dfrac{1}{2^{k+\ell}}\big)$ and for $k \in \mathbb{Z}^+$, let $U_k = \displaystyle\bigcup_{\ell=1}^{\infty} I_{k,\ell}$. Note that $U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots$ and for each $k \in \mathbb{Z}^+$ we have $\mathbb{Q} \subseteq U_k$ and $\lambda(U_k) \leq \displaystyle\sum_{\ell=1}^{\infty} |I_{k,\ell}| = \dfrac{1}{2^{k-1}}$ and we have $U_1 \supset U_2 \supseteq U_3 \supseteq \cdots$. Let $B = \displaystyle\bigcap_{k=1}^{\infty} U_k$. Note that $B$ is residual (it is a countable intersection of dense open sets) and we have $\lambda(B) = \displaystyle\lim_{k\to\infty} \lambda(U_k) = 0$ since $\lambda(U_k) \leq \dfrac{1}{2^k}$ for all $k \in \mathbb{Z}^+$. Finally note that any sett $A$ is equal to the disjoint union $A = (A \cap B) \cup (A \cap B^c)$, and we have $\lambda(A \cap B) = 0$ and the set $A \cap B^c$ is first category.

**End of Lecture 7**

**Start of Lecture 8**

**Remark 8.1:** $A$ is first category, say $A = \bigcup_{k=1}^{\infty} C_k$ where $C_k$ is nowhere dense, and if $B \subseteq A$, then $B = \bigcup_{k=1}^{\infty}(B \cap C_k)$ with each $B \cap C_k$ being nowhere dense.

**Exercise 8.2:** Show such $B$ is uncountable.

**Theorem 8.3:** There exists a non-measurable set in $\mathbb{R}$.

Proof.

Define an equivalence relation on the set $[0,1]$ by defining $x \sim y$ when $y - x \in \mathbb{Q}$. Let $C$ denote the set of equivalence classes. For each $c \in C$, choose an element $x_c \in c$ and let $A = \{x_c \mid c \in C\} \subseteq [0,1]$. We shall prove that the set $A$ is not measurable. Let $\mathbb{Q} \cap [0,2] = \{a_1, a_2, a_3, \cdots\}$, with the $a_k$ distinct. For each $k \in \mathbb{Z}^+$, let $A_k = a_k + A \subseteq [0,3]$. We claim that the sets $A_k$ are disjoint. Let $k, \ell \in \mathbb{Z}^+$ and suppose that $A_k \cap A_\ell \neq \emptyset$. Choose $y \in A_k \cap A_\ell$, say $y = a_k + x_c = a_\ell + x_d$ where $c, d \in C$. Since $x_c - x_d = a_\ell - a_k \in \mathbb{Q}$ we have $x_c \sim x_d$ and hence $c = d$ (since we only chose one element from each class). Since $c = d$ we have $x_c = x_d$, hence $a_k = a_\ell$, and hence $k = \ell$. Thus the sets $A_k$ are disjoint, as claimed. Next, we claim that $[1,2] \subseteq \bigcup_{k=1}^{\infty} A_k$. Let $y \in [1,2]$. Since $y - 1 \in [0,1]$ we have $y - 1 \in c$ for some $c \in C$. Since $y - 1 \in c$ we have $y - 1 - x_c \in \mathbb{Q}$ hence also $y - x_c \in \mathbb{Q}$. Since $y \in [1,2]$ and $x_c \in [0,1]$ we have $y - x_c \in [0,2]$. Since $y - x_c \in \mathbb{Q} \cap [0,2]$ we have $y - x_c = a_k$ for some $k \in \mathbb{Z}^+$ so that $y \in A_k$. This proves that $[1,2] \subseteq \bigcup_{k=1}^{\infty} A_k$.

Suppose, for a contradiction, that the set $A$ is measurable. By translation, each of the sets $A_k = a_k + A$ is measurable with $\lambda(A_k) = \lambda(A)$. Since the sets $A_k$ are disjoint and measurable, additivity gives

$$\lambda\Big(\bigcup_{k=1}^{\infty} A_k\Big) = \sum_{k=1}^{\infty} \lambda(A_k) = \sum_{k=1}^{\infty} \lambda(A) = \begin{cases} 0, & \text{if } \lambda(A) = 0, \\ \infty, & \text{if } \lambda(A) > 0. \end{cases}$$

But since $[0,1] \subseteq \bigcup_{k=1}^{\infty} A_k \subseteq [0,3]$ we also have $1 \leq \lambda\big(\bigcup_{k=1}^{\infty} A_k\big) \leq 3$, giving the desired contradiction.

**Remark 8.4:** We have $\sigma-$algebras:

$$\begin{aligned} \{\varnothing, \mathbb{R}\} \subseteq \mathcal{B} &= \{A \subseteq \mathbb{R} \mid A \text{ is Borel}\} \\ &\subseteq \mathcal{M} = \{A \subseteq \mathbb{R} \mid A \text{ is measurable}\} \\ &\subseteq \mathcal{P}(\mathbb{R}) = \{A \subseteq \mathbb{R}\} \\ \big|\mathcal{P}(\mathbb{R})\big| &= 2^{2^{\aleph_0}} = 2^{\mathfrak{c}} \end{aligned}$$

where the cardinalities of these sets are

$$\begin{aligned} |\mathcal{B}| &= 2^{\aleph_0} = \mathfrak{c} \text{ — can be shown with transfinite induction} \\ |\mathcal{M}| &= 2^{2^{\aleph_0}} = 2^{\mathfrak{c}} \text{ — since every subset of the (standard) Cantor set has measure zero} \end{aligned}$$

*Proof.* Let $C$ be the standard Cantor set. Since $\lambda(C) = 0$, it follows that every subset of $C$ is measurable. Because $|C| = 2^{\aleph_0}$ we have

$$2^{2^{\aleph_0}} = \big|\{A \mid A \subseteq \mathbb{R}\}\big| \geq |\mathcal{M}| \geq \big|\{A \mid A \subseteq C\}\big| = 2^{2^{\aleph_0}}.$$

**Exercise 8.5:** Show that $|\mathcal{G}| = 2^{\aleph_0}$ and $|\mathcal{G}_\delta| = 2^{\aleph_0}$.

**End of Lecture 8**

**Start of Lecture 9**

`Chapter 1: Lebesgue Measure`, Lectures 1-8. Content is available online: `http://www.math.uwaterloo.ca/~snew/`

`Chapter 2: Lebesgue Integration`, Refer to chapter 4.1,4.2 and 5 in textbook.

## Newton's Interpretation of Integration

If $F$ is differentiable with $F' = f$ of $[a, b]$ we have

$$\int_a^b f = F(b) - F(a)$$

So $f$ is Newton Integrable when $f$ has an antiderivative.

## Cauchy's Interpretation of Integration

When $f$ is continuous on $[a, b]$,

$$\lim_{|p|\to 0} \sum_{k=1}^{n} f(x_k)(x_k - x_{k-1}).$$

exists where $p = (x_0, x_1, \ldots, x_n)$ with $a = x_0 < x_1 < \ldots < x_n = b$. We write $\exists\, I \subseteq \mathbb{R}, \forall\, \varepsilon > 0, \exists\, \delta > 0$ such that $\forall\, n \in \mathbb{Z}^+, \forall\, x_0, x_1, \ldots, x_n$ with $a = x_0 < x_1 < \ldots < x_n = b$ and with $x_k - x_{k-1} < \delta \forall\, k$ we have

$$\left| \sum_{k=1}^{n} f(x_k)(x_k - x_{k-1}) - I \right| < \varepsilon.$$

In this case, $\int_a^b f = I$.

## Riemann's Interpretation of Integration

For $f = [a, b] \to \mathbb{R}$ bounded, we say $f$ is Riemann integrable when

$$\lim_{|p|\to 0} \sum_{k=1}^{n} f(c_k)(x_{k-1})$$

exists and the integral then equals to $\int_a^b = I$ as above. Alternatively, we can also define the Riemann integrability as follows:

**Definition 9.1:** For $S \subseteq A$, the **characteristic function** $\mathcal{X}_S$ on $A$ is the function $\mathcal{X}_S : A \to \{0, 1\}$ is given by

$$\mathcal{X}_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases}$$

**Definition 9.2:** For $a < b$ a **step function** on $[a, b]$ is a function $s : [a, b] \to \mathbb{R}$ which is of the form

$$s = \sum_{k=1}^{n} c_k \mathcal{X}_{I_k}$$

where each $c_k \in \mathbb{R}$ and the $I_k$ are disjoint non-degenerate intervals with $[a, b] = \bigcup_{k=1}^{n} I_k$.
The expression $s = \sum_{k=1}^{n} c_k \mathcal{X}_{I_k}$ is unique if we required that $c_{k-1} \neq c_k$ for $1 \leq k \leq n$.

For the step function $s$ as above, we define the Riemann integral of $s$ to be

$$\int_a^b s = \sum_{k=1}^{n} c_k |I_k|.$$

**Definition 9.3:** For $f : [a, b] \to \mathbb{R}$ bounded, we define the **upper and lower Riemann integrals** of $f$ on $[a, b]$ be

$$U(f) = \inf\left\{ \int_a^b \;\middle|\; s \text{ is a step function on } [a,b] \text{ with } s \geq f \right\}$$
$$L(f) = \sup\left\{ \int_a^b \;\middle|\; s \text{ is a step function on } [a,b] \text{ with } s \leq f \right\}$$

We say that $f$ is Riemann integrable on $[a, b]$ when $U(f) = L(f)$ and in this case we define the Riemann integral of $f$ on $[a, b]$ to be

$$\int_a^b f = U(f) = L(f).$$

**Theorem 9.4** (Properties of Riemann integral)**:** Let $f, g : [a, b] \to \mathbb{R}$ be bounded and let $c \in \mathbb{R}$. Then,

1. If $f$ and $g$ are both Riemann integrable and $f \leq g$ then $\displaystyle\int_a^b f \leq \int_a^b g$.
2. If $c \in (a, b)$ then $f$ is Riemann integrable on $[a, b]$ iff $f$ is Riemann integrable on both $[a, c]$ and $[c, b]$. In this case, $\displaystyle\int_a^b f = \int_a^c f + \int_c^b f$.
3. If $f(x) = g(x)$ for all but finitely many $x \in [a, b]$ then $f$ is Riemann integrable iff $g$ is Riemann integrable and in this case, $\displaystyle\int_a^b f = \int_a^b g$.
4. If $f$ and $g$ are Riemann integrable on $[a, b]$ then so are the functions $cf$ and $f + g$ and in this case,
$$\int_a^b (cf) = c\int_a^b f \text{ and } \int_a^b (f+g) = \int_a^b f + \int_a^b g.$$
5. If $f$ is continuous on $[a, b]$ then $f$ is Riemann integrable on $[a, b]$.
6. If $f$ is monotonic on $[a, b]$ then it's Riemann integrable on $[a, b]$.

**Theorem 9.5** (Fundamental Theorem of Calculus)**:** Suppose $F$ is differentiable with $F' = f$ on $[a, b]$ and suppose that $f$ is Riemann integrable on $[a, b]$ then

$$\int_a^b f = F(b) - F(a).$$

**Theorem 9.6** (Lebesgue)**:** For $f : [a, b] \to \mathbb{R}$ bounded, $f$ is Riemann integrable on $[a, b]$ iff the set of points at which $f$ is not continuous is a set of Lebesgue measure zero. This theorem is covered in William R. Wade's Analysis book.

**Definition 9.7:** For $A \subseteq \mathbb{R}$, a **simple function** on $A$ is a function $s : A \to \mathbb{R}$ of the form

$$s = \sum_{k=1}^{n} c_k \mathcal{X}_{A_k} \text{ where } c_k \in$$

and the $A_k$ are measurable disjoint sets with $\bigcup_{k=1}^{n} A_k = A$.

**Remark 9.8:** Note that we can make the expression so $\sum_{k=1}^{n} c_k \mathcal{X}_{A_k}$ to be unique by requiring that the $c_k$ are distinct. Which gives us Range$(s) = \{c_1, c_2, \ldots, c_n\}$ and $A_k = s^{-1}(c_k)$

**Remark 9.9:** For $s$ above, we define

$$\int_A s = \sum_{k=1}^{n} c_k \lambda(A_k), \text{ where}$$

$$\int_A f = \sup\{\int_A s \ \mid \ s \text{ is a simple function on } [a,b] \text{ with } s \leq f\}$$

**Example 9.10:** For $f : [0,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases},$$

$f$ is not Riemann integrable because upper sums and lower sums are not equal. But, $f$ is Lebesgue integrable.

**Example 9.11:** Define $f : \{0,1\} \to \mathbb{R}$ by $f\left(\frac{a}{b}\right) = \frac{1}{b}$ when $a, b \in \mathbb{Z},\ b > 0,\ 0 \leq a \leq b$ with $\gcd(a,b) = 1$ and $f(x) = 0$ when $x \notin \mathbb{Q}$. The $f$ is Riemann integrable with $\int_a^b f = 0$ because $f$ is discontinuous at the rationals.

**Remark 9.12:** A function $f$ is Riemann integrable iff the Lebesgue measure of its set of discontinuities is zero.

**Example 9.13:** Let $s(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$. Define $f : [0,1] \to [0,1]$ by letting $\mathbb{Q} \cap [0,1] = \{a_1, a_2, \ldots\}$ (enumerating rationals) then setting $f(x) = \sum_{\infty}^{k=1} \frac{s(x - a_k)}{2^k}$. Then, $f$ is strictly increasing with a jump discontinuity at each $a_k \in \mathbb{Q} \cap [0,1]$. It is Riemann integrable with $0 < \int_0^1 f < 1$ .

**End of Lecture 9**

**Start of Lecture 10**

**Definition 10.1:** Let $C$ be a Cantor set, say $U = [0,1] \setminus C$. Say $U = \cup_{k=1}^{\infty} I_k$. We define the associated Cantor function $f : [0,1] \to [0,1]$ as follows.

$$\begin{aligned} \text{We define } f(x) &= \frac{1}{2} \text{ for } x \in I_1, \\ f(x) &= \frac{1}{4} \text{ for } x \in I_2, \\ f(x) &= \frac{3}{4} \text{ for } x \in I_3, \\ f(x) &= \frac{1}{8} \text{ for } x \in I_4, \\ f(x) &= \frac{3}{8} \text{ for } x \in I_5, \\ f(x) &= \frac{5}{8} \text{ for } x \in I_6, \\ f(x) &= \frac{7}{8} \text{ for } x \in I_7, \\ \vdots &= \vdots \text{ for } \vdots \end{aligned}$$

Verify that $f$ can be extended (uniquely) to give continuous function $f : [0,1] \to [0,1]$. This function is called the Cantor function. Note that $f'(x) = 0$ for all $x \in U$. When $C$ is the standard Cantor set, we have $\lambda(C) = 0$. Also note that if $f$ was differentiable everywhere, on $[0,1]$ then it would contradict the fundamental theorem of calculus. We would have $\int_0^1 f'(t)\,\mathrm{d}t = f(1) - f(0) = 1$ but all lower Riemann sums would be zero.

**Example 10.2:** Let $C$ be a Cantor set. Say $U = [0,1] \setminus C$, and $U = \cup_{k=1}^{\infty} I_k$. Let $f : [0,1] \to [0,1]$ be the Cantor function as above. Then define $g : [0,1] \to [0,2]$ by $g(x) = x + f(x)$ is strictly increasing and its inverse $h : [0,2] \to [0,1]$ is continuous. So $g : [0,1] \to [0,2]$ is a homeomorphism. When $C$ is the standard Cantor set, we have $\lambda(C) = 0$, $\lambda(U) = 1$. Note that $g$ sends each interval $I_k$ to an interval $g(I_k)$ of the same size. So, $\lambda(g(U)) = \lambda(U)$. Since $[0,2]$ is the disjoint union $g(U) \cup g(C)$, it follows that $\lambda(g(C)) = 1$. Since $\lambda(g(C)) = 1$, we can choose a non-measurable set $B \subseteq g(C)$ and then for $A = g^{-1}(B)$, so $B = g(A)$. We have $A \subseteq C$. So that $A$ is measurable with $\lambda(A) = 0$ but $g(A)$ is not.

**Example 10.3:** Let $C$ be a Cantor set. Say $U = [0,1] \setminus C$, and $U = \cup_{k=1}^{\infty} I_k$. For each $k$ choose an open interval $J_k \subseteq I_k$ with the same center and with $|J_k| = \frac{1}{2}|I_k|$ and choose continuous function $f_k : [0,1] \to [0,1]$ with $f(x) = 0$ for $x \notin J_k$ and $f_k(x) = 1$ when $x$ is midpoint of $J_k$.

Define $g : [0,1] \to [0,1]$ by $g(x) = \sum_{\infty}^{k=1} f_k(x)$. Then, $g$ is continuous on $U$ but not on $C$. Define $h(x)$ by

$$h(x) = \sum_{k=1}^{\infty} \int_0^x f_k(t)\,\mathrm{d}t.$$

Verify that $h'(x) = g(x)$ for all $x$. When we use a Cantor set $C$ with $\lambda(C) > 0$, we obtain a differentiable function $h$ on $[0,1]$ with $h' = g$ and $g$ is not Riemann integrable.

**Example 10.4:** Let $\mathbb{Q} \cap [0,1] = \{a_1, a_2, \ldots\}$. Let $f(x) = \displaystyle\sum_{k=1}^{\infty} \frac{(x - a_k)^{\frac{1}{3}}}{2^k}$. Note that $f : [0,1] \to [-1,1]$

strictly increasing. Verify that

$$f^{-1}(x) = \sum_{k=1}^{\infty} \frac{1}{3 \cdot 2^k (x - a_k)^{\frac{2}{3}}} \text{ for } x \notin \mathbb{Q}$$

and $f'(x) = \infty$ when $x \in \mathbb{Q}$ and $f'(x) \geq \dfrac{1}{3}$ for all $x$. $f$ gives a homeomorphism from $[0,1]$ to some closed interval $[a,b] \subseteq [-1,1]$ and the inverse function $h : [a,b] \subseteq [0,1]$ is differentiable with $h'(f(x)) = \dfrac{1}{f'(x)}$ for all $x$ so $h'(x) \geq 3$ for all $x$ and $h'(x) = 0$ for $x \in \mathbb{Q}$.

If the Fundamental Theorem of Calculus held, $\displaystyle\int_a^b h'(t)\,\mathrm{d}t = h(b) - h(a) = 1 - 0 = 1$ but since $h'(x) = 0 \; \forall \, x \in \mathbb{Q}$ , the lower Riemann sums are all zero.

**End of Lecture 10**

**Start of Lecture 11**

### Lebesgue Measurable Functions

**Definition 11.1:** Sometimes it is convenient to allow functions to take the values $\pm\infty$. To do so, we use **extended real numbers**

$$[-\infty, +\infty] = \mathbb{R} \cup \{\pm\infty\}.$$

We give $[-\infty, \infty]$ its usual ordering. We use the usual partially-defined $+$ and $\times$ (we leave some expressions undefined, such as $\infty + (-\infty)$, $0 \cdot \pm\infty$). We give $[-\infty, \infty]$ its usual topology. A set $A \subseteq [-\infty, \infty]$ is open when for all $a \in A$ there exists $0 \leq r \in \mathbb{R}$ such that $B(a, r) \subseteq A$, where $B(-\infty, r) = [-\infty, -\frac{1}{r})$, $B(\infty, r) = \left(\frac{1}{r}, \infty\right]$ and $B(a, r) = (a - r, a + r)$ when $a \in \mathbb{R}$.

**Example 11.2:** Every nonempty open set in $[-\infty, \infty]$ is a finite or countable union of nonempty open intervals in $[-\infty, \infty]$ and nonempty open intervals are the sets of the form

$$(a, b), (-\infty, a), (a, \infty), (-\infty, \infty) = \mathbb{R} \text{ and } [-\infty, a), (a, \infty], [-\infty, \infty] \text{ where } a, b \in \mathbb{R} \text{ with } a < b.$$

**Definition 11.3:** For $f : \mathbb{R} \to B \subseteq extendedrealnumbers$ we say that $f$ is Lebesgue measurable when $f^{-1}(U)$ is measurable (in $\mathbb{R}$) for every open set $U$ in $[-\infty, \infty]$ or equivalently, for every open set $U$ in $B$.

**Remark 11.4:** For $f : \mathbb{R} \to B \subseteq extendedrealnumbers$ if $f$ is measurable then $A$ must be measurable since $A = f^{-1}\big([-\infty, \infty]\big)$.

**Theorem 11.5:** Let $f : A \subseteq \mathbb{R} \to [-\infty, \infty]$. Then, $f$ is measurable

$$\begin{aligned}
&\iff f^{-1}\big((a, \infty]\big) \text{ is measurable for all } a \in \mathbb{R}, \\
&\iff f^{-1}\big([a, \infty]\big) \text{ is measurable for all } a \in \mathbb{R}, \\
&\iff f^{-1}\big([-\infty, a)\big) \text{ is measurable for all } a \in \mathbb{R}, \\
&\iff f^{-1}\big([-\infty, a]\big) \text{ is measurable for all } a \in \mathbb{R}.
\end{aligned}$$

Proof.

Proof of 1st equivalence If $f$ is measurable then $f^{-1}(U)$ is measurable for every open set $U$ in $[-\infty, \infty]$. So $f^{-1}((a, \infty])$ for every open set $U$ in $[-\infty, \infty]$. So $f^{-1}((a, \infty])$ is measurable for every $a \in \mathbb{R}$. Suppose that $f^{-1}((a, \infty])$ is measurable for every $a \in \mathbb{R}$. Then $f^{-1}([-\infty, a]) = A \setminus f^{-1}((a, \infty])$ is measurable for all $a \in \mathbb{R}$. So,

$$f^{-1}([-\infty, a)) = \bigcup_{n=1}^{\infty} f^{-1}\left(\left[-\infty, a - \frac{1}{n}\right]\right) \text{ is measurable } \forall\, a \in \mathbb{R}.$$

Hence,

$$f^{-1}((a, b)) = f^{-1}([-\infty, b)) \cap f^{-1}((a, \infty]) \text{ is measurable } \forall\, a, b \in \mathbb{R}.$$

Moreover, every nonempty open set in $[-\infty, \infty]$ is a finite or countable union of open intervals, each of one of the forms $[-\infty, a], (a, \infty], (a, b)$ with $a, b \in \mathbb{R}$.

**Theorem 11.6:** Let $A$ be measurable and $f : A \subseteq \mathbb{R} \to [-\infty, \infty]$. Then,

1. If $f$ is continuous, then it's measurable.
2. If $f$ is monotonic, then it's measurable.

Proof of 1..

If $f$ is continuous then for every open set $U$ in $[-\infty, \infty]$ $f^{-1(U)}$ is open in $A$. So $f^{-1}(U) = A \cap V$ for some open set $V$ in $\mathbb{R}$. Hence $f^{-1}(U)$ is measurable.

Proof of 2..

Suppose that $f$ is increasing. For $x, y \in A$ with $x \leq y$ if $x \in f^{-1}((a, \infty])$ then $f(x) > a$. So $f(y) \geq f(x) > a$. Hence $y \in f^{-1}(a, \infty])$. It follows that for $a \in \mathbb{R}$, $f^{-1}((a, \infty])$ is one of the forms

$$\varnothing \text{ or } A \cap (b, \infty) \text{ or } A \cap [b, \infty) \text{ or } \mathbb{R} \cap A$$

and these are all measurable.

**Remark 11.7:** If $f : A \subseteq \mathbb{R} \to B \subseteq [-\infty, \infty]$ is measurable and $\varphi : B \subseteq [-\infty, \infty] \to C \subseteq [-\infty, \infty]$ is continuous then $\varphi \circ f = A \subseteq \mathbb{R} \to C \subseteq [-\infty, \infty]$ is measurable because $\varphi \circ f^{-1}(U) = f^{-1}\big(\varphi^{-1}(U)\big)$.

**End of Lecture 11**

**Start of Lecture 12**

**Theorem 12.1** (Operations on measurable functions)**:** If $f, g : A \subseteq \mathbb{R} \to [-\infty, \infty]$ are measurable and $c \in \mathbb{R}$ then the following functions

$$cf, \quad f+g, \quad fg, \quad |f|, \quad f^+, \quad f^-$$

are all measurable, provided they are well defined.

**Definition 12.2:** We define the **positive part** of function $f : A \subseteq \mathbb{R} \to [-\infty, \infty]$ to be $f^+ : A \subseteq \mathbb{R} \to [-\infty, \infty]$

$$f^+(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ 0 & \text{if } f(x) \leq 0 \end{cases}$$

and similarly, we define the **negative part** of this function as $f^- : A \subseteq \mathbb{R} \to [-\infty, \infty]$, where

$$f^+(x) = \begin{cases} 0 & \text{if } f(x) \geq 0 \\ \big|f(x)\big| & \text{if } f(x) \leq 0 \end{cases}.$$

Proof.

When $c \neq 0$, the function $\varphi : [-\infty, \infty] \to [-\infty, \infty]$ given by $\varphi(x) = cx$ is continuous. Thus, if $f$ is measurable, then so if $cf = \varphi \circ f$.

When $f, g$ are measurable so if $f + g$ because for $a \in \mathbb{R}$,

$$(f+g)^{-1}(a, \infty] = \{x \in A \mid f(x) + g(x) > a\} = \bigcup_{r \in \mathbb{Q}} \{x \in A \mid f(x) > r \text{ and } g(x) > a - r\} \qquad (\star)$$

Since given $x \in A$ with $f(x) + g(x) > a$, we can choose $r \in \mathbb{Q}$ with

$$\underbrace{f(x) - (f(x) + g(x) - a)}_{a - g(x)} < r < f(x),$$

so that $f(x) > r$ and $g(x) < a - r$. Hence,

$$(\star) \bigcup_{r \in \mathbb{Q}} \Big( f^{-1}(a, \infty] \cap g^{-1}(a - r, \infty] \Big),$$

which is measurable.
The map $\varphi : [-\infty, \infty] \to [-\infty, \infty]$ given by $\varphi(x) = x^2$ is continuous. So, if $f$ is measurable then so is $f^2 = \varphi \circ f$ and it follows that if $f$ and $g$ are both measurable, then so is $\frac{(f+g)^2 - (f-g)^2}{4} = fg$.
The map $\varphi : [-\infty, \infty] \to [-\infty, \infty]$ given by $\varphi(x) = |x|$ is continuous. So if $f$ is measurable then so is $|f| = \varphi \circ f$. Thus, if $f$ is measurable then so is $f^+$ and $f^-$ because

$$f^+ = \frac{1}{2}\big(|f| + f\big) \text{ and } f^- = \frac{1}{2}\big(|f| - f\big)$$

**Theorem 12.3:** Let $f_n : A \subseteq \mathbb{R} \to [-\infty, \infty]$ be measurable. Then each of the functions

$$\sup_{n \geq 1} f_n \text{ and } \inf_{n \geq 1} f_n \text{ and } \limsup_{n \to \infty} f_n \text{ and } \liminf_{n \to \infty} f_n$$

are all well-defined and measurable.

Proof.

For $\sup\limits_{n\geq 1} f_n$: Let $g(x) = \sup\{f_n(x) \mid n \in \mathbb{Z}^+\}$ for all $x \in A$. Then, for $a \in \mathbb{R}$ we have

$$\begin{aligned} g^{-1}(a,\infty] &= \{x \in A \mid g(x) > a\} \\ &= \{x \in A \mid \sup\{f_1(x), f_2(x), \ldots\} > a\} \\ &= \{x \in A \mid f_n(x) > 0 \text{ for some} n \in \mathbb{Z}^+\} \\ &= \bigcup_{n=1}^{\infty} f_n^{n-1}(a,\infty] \end{aligned}$$

which is measurable.
For $\inf\limits_{n\geq 1} f_n$:
Equivalently, ( so $\forall\, x \in A,\ g(x) = \inf\{f_1(x), f_2(x), \ldots\}$ ) we have

$$\begin{aligned} g^{-1}(a,\infty] &= \{x \in A \mid g(x) = \inf\{f_1(x), f_2(x), \ldots\} \geq a\} \\ &= \{x \in A \mid f_n(x) \geq 0 \text{ for some} n \in \mathbb{Z}^+\} \\ &= \bigcap_{n=1}^{\infty} f_n^{n-1}(a,\infty] \end{aligned}$$

which is measurable.
For $\limsup\limits_{n\to\infty} f_n$:
Equivalently, (so $g(x) = \limsup\limits_{n\to\infty} f_n$) we have

$$\begin{aligned} g(x) &= \limsup_{n\to\infty} f_n \\ &= \text{ the limit as } n \to \infty \text{ of the sequence } \sup_{n\geq 1} f_n(x), \sup_{n\geq 2} f_n(x), \sup_{n\geq 3} f_n(x), \ldots \\ &= \lim_{\ell\to\infty} \sup_{n\geq \ell} f_n(x)\ \forall\, x \in A \end{aligned}$$

Then, since the $\sup\limits_{n\geq \ell} f_n(x)$ sequence is decreasing (not necessarily strictly) with $\ell$ for any fixed $x \in A$, we have

$$\lim_{\ell\to\infty} \sup_{n\geq \ell} f_n = \inf_{\ell > 1} \sup_{n\geq \ell} f_n$$

which is measurable. This is because each function $g_\ell = \sup\limits_{n\geq \ell} f_n$ is measurable.

**Definition 12.4:** We say that a property or statement about $x$ holds for almost every (written a.e) $x \in A$ or holds almost everywhere in $A$. When the property of statement holds for all $x \in A \setminus E$ for some set $E \subseteq A$ with $\lambda(E) = 0$.

**Example 12.5:** For $f, g : A \subseteq \mathbb{R} \to [-\infty, \infty]$ we say that $f(x) = g(x)$ for a.e$x \in A$, or that $f = g$a.e $\in A$, when $f(x) = g(x)$ for all $x \in A \setminus E$ for some set $E \subseteq A$ with $\lambda(E) = 0$.

**Theorem 12.6:** Let $f, g : A \subseteq \mathbb{R} \to [-\infty, \infty]$.

1. If $\lambda(A) = 0$ then $f$ is measurable.
2. If $B, C \subseteq \mathbb{R}$ are disjoint and measurable with $A = B \cup C$ then, $f$ is measurable on $A$ if and only if the restrictions of $f$ to $B$ and to $C$ are both measurable on $B$ and $C$.
3. If $f = g$ on a.ein $A$, then $f$ is measurable if and only if $g$ is measurable.

**Exercise 12.7:** Prove these 3 theorems.

## Lebesgue Integration

**Definition 12.8:** A **simple function** on $A$ is a function $s : A \to \mathbb{R}$ of the form

$$s = \sum_{k=1}^{n} c_k \mathcal{X}_{A_k}$$

where $n \in \mathbb{Z}^+$, each $c_k \in \mathbb{R}$ and the sets $A_k$ are disjoint and measurable.

**Remark 12.9:** We can ensure the numbers $c_k$ and the sets $A_k$ are uniquely determined from the function by requiring that $c_1 < c_2 < c_3 < \ldots < c_n$ (and then $A_k = s^{-1}(c_k)$ for each $k$).

**Exercise 12.10:** Integral sum of two simple functions is the sum of two integrals.

**End of Lecture 12**

**Start of Lecture 13**

**Definition 13.1:** We define the **Lebesgue integral of a simple function** $s$ **on** $A$ to be

$$\int_A s = \int_A s \, \mathrm{d}\lambda = \sum_{k=1}^{n} c_k \lambda(A_k).$$

**Theorem 13.2:** Let $r, s : A \subseteq \mathbb{R} \to \mathbb{R}$ be simple functions and let $c \in \mathbb{R}$. Then,

1. If $r \leq s$ then $\int_A r \leq \int_A s$.
2. $\int_A (cS) = c \int_A s$.
3. If $\lambda(A) = 0$ then $\int_A s = 0$.
4. If $A = B \cup C$ where $B$ and $C$ are disjoint measurable sets then $\int_A s = \int_B s + \int_C s$ where $\int_B s$ means $\int_B s_B$ where $s_B : B \to \mathbb{R}$ is given by $s_B(x) = s(x)$ for $x \in B$.
5. If $B \subseteq A$ is measurable then $\int_B s = \int_A s \mathcal{X}_B$.
6. If $r = s$ a.ein $A$ then $\int_A r = \int_A s$.

Proof.

To prove 1 and 3, say

$$r = \sum_{k=1}^{n} a_k \mathcal{X}_{A_k} \text{ and } s = \sum_{\ell=1}^{m} b_\ell \mathcal{X}_{B_\ell}.$$

For each pair of indices $k, \ell$, let $C_{k,\ell} = A_k \cap B_\ell$. Note that the sets $C_{k,\ell}$ are measurable and disjoint. We have

$$\bigcup_{\ell=1}^{m} C_{k,\ell} = \bigcup_{\ell=1}^{m} (A_k \cap B_\ell) = A_k \cap \bigcup_{\ell=1}^{m} B_\ell = A_k \cap A = A_k.$$

$$\text{So, } \sum_{\ell=1}^{m} \lambda\big(C_{k,\ell}\big) = \lambda(A_k) \text{ and } \sum_{\ell=1}^{m} \mathcal{X}_{C_{k,\ell}} = \mathcal{X}_{A_k}.$$

Similarly, we also have $\bigcup_{k=1}^{n} C_{k,\ell} = B_\ell$. To prove 1, note that if $r \leq s$ then $\forall\, x \in C_{k,\ell}$, we have

$$a_k = r(x) \leq s(x) = b_\ell.$$

$$\begin{aligned}
\text{Hence, } \int_A r &= \sum_{k=1}^{n} a_k \lambda(A_k) \\
&= \sum_{k=1}^{n} a_k \sum_{\ell=1}^{m} C_{k,\ell} \\
&= \sum_{k,\ell} a_k \lambda(C_{k,\ell}) \\
&\leq \sum_{k,\ell} b_\ell \lambda(C_{k,\ell}) \\
&= \sum_{\ell=1}^{m} b_\ell \sum_{k=1}^{n} \lambda(C_{k,\ell})
\end{aligned}$$

$$\begin{aligned} &= \sum_{\ell=1}^{m} b_\ell \lambda(B_\ell) \\ &= \int_A s. \end{aligned}$$

To prove 2 note that

$$r + s = \sum_{k=1}^{n} a_k \mathcal{X}_{A_k} + \sum_{\ell=1}^{m} b_\ell \mathcal{X}_{B_\ell} = \sum_{k=1}^{n} a_k \sum_{\ell=1}^{m} \mathcal{X}_{C_{k,\ell}} + \sum_{\ell=1}^{m} b_\ell \sum_{k=1}^{n} \mathcal{X}_{C_{k,\ell}} = \sum_{k,\ell} (a_k + b_\ell) \mathcal{X}_{C_{k,\ell}}.$$

$$\begin{aligned} \text{So, } \int_A (r+s) &= \sum_{k,\ell} (a_k + b_\ell) \mathcal{X}_{C_{k,\ell}} \\ &= \sum_{k,\ell} a_k \mathcal{X}_{C_{k,\ell}} + \sum_{k,\ell} b_\ell \mathcal{X}_{C_{k,\ell}} \\ &= \sum_k a_k \sum_\ell \lambda(C_{k,\ell}) + \sum_\ell b_\ell \sum_k \lambda(C_{k,\ell}) \\ &= \sum_k a_k \lambda(A_k) + \sum_\ell b_\ell \lambda(B_\ell) \\ &= \int_A r + \int_B s. \end{aligned}$$

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**Exercise 13.3:** Prove parts $3 - 6$.

### Non-negative Measurable Functions

**Remark 13.4:** We use $[0, \infty] \subseteq [-\infty, \infty]$. In $[0, \infty]$ we define $0 \cdot \infty = 0$. So the operations $+$ and $\times$ are always well-defined in $[0, \infty]$. We could also define

$$\frac{1}{0} = \infty \text{ and } \frac{1}{\infty} = 0$$

and the map $\varphi : [0, \infty] \to [0, \infty]$ (reciprocal map) given by $\varphi(x) = \frac{1}{x}$ is well-defined and continuous.

**Definition 13.5:** For $f : A \subseteq \mathbb{R} \to [0, \infty]$ non-negative and measurable we define the Lebesgue integral of $f$ on $A$ to be

$$\int_A f = \int_A f \, d\lambda = \sup \left\{ \int_A s \;\middle|\; \begin{array}{c} s \text{ is a simple function} \\ \text{on the set } A \text{ with } s \le f \end{array} \right\}$$

**Theorem 13.6:** For non-negative measurable functions $f, g : A \subseteq \mathbb{R} \to [0, \infty]$ and for $c \in \mathbb{R}$,

1. If $f \le g$ then $\int_A f \le \int_A g$.
2. $\int_A (cf) = c \int_A f$ and $\int_A (f + g) = \int_A f + \int_A g$.
3. If $\lambda(A) = 0$ then $\int_A f = 0$.
4. If $A = B \cup C$ where $B$ and $C$ are disjoint measurable sets then $\int_A f = \int_B f + \int_C f$ where $\int_B f$ means $\int_B f_B$ where $f_B : B \to [0, \infty]$ is given by $f_B(x) = f(x)$ for $x \in B$.
5. If $B \subseteq A$ is measurable then $\int_B f = \int_A f \mathcal{X}_B$.

6. If $f = g$ a.e in $A$ then $\int_A f = \int_A g$.

**Remark 13.7:** The proof of these all follow fairly easily from the analogous properties of simple functions except for the fact that

$$\int_A (f+g) = \int_A f + \int_A g$$

which we will prove later.

**End of Lecture 13**

**Start of Lecture 14**

**Theorem 14.1** (Fatou's Lemma)**:** Let $f_n : A \subseteq \mathbb{R} \to [0,\infty]$ be measurable for $n \in \mathbb{Z}^+$ then

$$\int_A \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int_A f_n.$$

Proof.

We show that for all non-negative simple functions $s$ on $A$ with $s \leq \liminf\limits_{n\to\infty} f_n$ we have

$$\int_A s \leq \liminf_{n\to\infty} \int_A f_n.$$

Let $s : A \subseteq \mathbb{R} \to [0,\infty]$ be any non-negative simple function on $A$ with $s \leq \liminf\limits_{n\to\infty} f_n$.

We write $s = \sum\limits_{k=1}^{m} c_k \mathcal{X}_{A_k}$. For all $x \in A_k$ we have $c_k = s(x) \leq \liminf\limits_{n\to\infty} f_n(x)$. It follows that $\forall\, 0 \leq r < 1$, $\exists\, n \in \mathbb{Z}^+$, $\forall\, \ell \geq n$, $f_\ell(x) \geq rc_k$. For each $k, n \in \mathbb{Z}^+$, let

$$B_{k,n} = \{x \in A_k \mid f_\ell(x) \geq rc_k \forall\, \ell \geq n\} = \bigcap_{\ell \geq n} f_\ell^{-1}\big([rc_k, \infty]\big).$$

Note that each set $B_{k,n}$ is measurable and $B_{k,1} \subseteq B_{k,2} \subseteq \ldots$ and $\bigcup\limits_{n=1}^{\infty} B_{k,n} = A_k$. For $x \in B_{k,n}$ we have $f_\ell \geq rc_k$ for all $\ell \geq n$, so in particular, $f_n(x) \geq rc_k$ for $x \in B_{k,n}$. Therefore for all $x$, $f_n(x) \geq \sum\limits_{k=1}^{m} rc_k \mathcal{X}_{B_{n,k}}$. So

$$\int_A f_n(x) \geq \sum_{k=1}^{m} rc_k \lambda(B_{k,n}).$$

Take the $\liminf$ to get

$$\liminf_{n\to\infty} f_n \geq \liminf_{n\to\infty} \sum_{k=1}^{m} rc_k \lambda(B_{k,n}) = \sum_{k=1}^{m} rc_k \lambda(A_k).$$

Since $0 \leq r \leq 1$ is arbitrary and since

$$\lambda(A_k) = \lambda\left(\bigcup_{n=1}^{\infty} B_{k,n}\right) = \lim_{n\to\infty} \lambda\big(B_{k,n}\big) = r\int_A s,$$

then it follows that $\liminf\limits_{n\to\infty} \int_A f_n \geq \int_A s$. Thus, we obtain $\int_A \liminf\limits_{n\to\infty} f_n \leq \liminf\limits_{n\to\infty} \int_A f_n$ as required.

**Corollary 14.2:** Let $f_n : A \subseteq \mathbb{R} \to [0,\infty]$ be nonnegative and measurable for $n \in \mathbb{Z}^+$. Suppose $\lim\limits_{n\to\infty} f_n(x)$ exists and $f_n(x) \leq \lim\limits_{n\to\infty} f_n(x)$, $\forall\, x \in A$ and $\forall\, n \in \mathbb{Z}^+$. Then

$$\int_A \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_A f_n.$$

Proof.

Since $f_n \leq \lim\limits_{n\to\infty} f_n$ for all $n \in \mathbb{Z}^+$, we have

$$\int_A f_n \leq \int_A \lim_{n\to\infty} f_n,\ \forall\, n \in \mathbb{Z}^+.$$

Thus, $\limsup\limits_{n\to\infty} \int_A f_n = \lim\limits_{n\to\infty} \int_A f_n \le \int_A \lim\limits_{n\to\infty} f_n$. By Fatou's lemma, we have

$$\int_A \liminf_{n\to\infty} f_n = \int_A \lim_{n\to\infty} f_n \le \liminf_{n\to\infty} \int_A f_n = \lim_{n\to\infty} \int_A f_n.$$

**Corollary 14.3:** (Lebesgue's Monotone Convergence theorem) Let $f_n : A \subseteq \mathbb{R} \to [0,\infty]$ be nonnegative measurable functions such that $\{f_n(x)\}$ is increasing for every $x \in A$. Then

$$\int_A \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_A f_n.$$

Proof.

This is a special case of the previous corollary.

**Remark 14.4:** We now return to the proof of the second formula in Part (2) of section 13.6. We suppose that $f, g : A \subseteq \mathbb{R} \to [0,\infty]$ are nonnegative measurable functions, and we need to prove that

$$\int_A (f+g) = \int_A f + \int_A g.$$

Proof.

Given any nonnegative measurable function $f : A \subseteq \mathbb{R} \to [0,\infty]$, we can construct an increasing sequence $\{s_n\}$ of nonnegative simple functions $s_n : A \to [0,\infty)$ with $\lim\limits_{n\to\infty} s_n = f$ as follows. For $n \in \mathbb{Z}^+$, we let

$$s_n(x) = \begin{cases} \dfrac{k-1}{2^n} & \text{, if } \dfrac{k-1}{2^n} \le f(x) < \dfrac{k}{2^n} \text{ with } k \in \{1, 2, \cdots, n2^n\}, \\ n & \text{, if } f(x) \ge n, \end{cases}.$$

that is $s_n = \sum\limits_{k=1}^{n2^n} \dfrac{k-1}{2^n} \mathcal{X}_{A_k}$ where $A_k = f^{-1}[\dfrac{k-1}{2^n}, \dfrac{k}{2^n})$ for $1 \le k < n2^n$ and $A_{n2^n} = f^{-1}[n,\infty]$. Using the construction described above, choose increasing sequences $\{r_n\}$ and $\{s_n\}$ of nonnegative simple functions on $A$ such that $\lim\limits_{n\to\infty} r_n = f$ and $\lim\limits_{n\to\infty} s_n = g$. Then the sequence $\{r_n + s_n\}$ is also increasing with $\lim\limits_{n\to\infty}(r_n + s_n) = f + g$. By the Monotone Convergence Theorem, along with Part (2) of section 13.6, we have

$$\begin{aligned} \int_A (f+g) &= \int_A \lim_{n\to\infty}(r_n + s_n) = \lim_{n\to\infty} \int_A (r_n + s_n) = \lim_{n\to\infty} \left( \int_A r_n + \int_A s_n \right) \\ &= \lim_{n\to\infty} \int_A r_n + \lim_{n\to\infty} \int_A s_n = \int_A \lim_{n\to\infty} r_n + \int_A \lim_{n\to\infty} s_n = \int_A f + \int_A g. \end{aligned}$$

**End of Lecture 14**

**Start of Lecture 15**

From the proof of Remark 14.4 we obtain the following corollary.

**Corollary 15.1:** Let $A \subseteq \mathbb{R}$ be measurable and let $\{f_n\}$ be a sequence of nonnegative measurable functions $f_n : A \to [0, \infty]$. Then

$$\int_A \sum_{n-1}^{\infty} f_n = \sum_{n=1}^{\infty} \int_A f_n \,.$$

Proof.

Apply MCT to partial sums $g_n = \sum_{k=1}^{n} f_n$ and note that $\int_A \sum_{k=1}^{n} f_n = \sum_{k=1}^{n} \int_A f_n$ by induction.

**Corollary 15.2:** Let $A_1, A_2, \ldots \subseteq \mathbb{R}$ be disjoint and measurable with $A = \bigsqcup_{k=1}^{\infty} A_k$ and let $f : A \subseteq \mathbb{R} \to [0, \infty]$ be nonnegative and measurable. Then

$$\int_A f = \sum_{k=1}^{\infty} \int_{A_k} f.$$

Where $A_k \subseteq A$ and $\int_{A_k} f$ means $\int_A f_{A_k}$ where $f_{A_k}$ is the restriction of $f$ to $A_k$ with $f(x) = f_{A_k}(x)$ in $A_k$.

Proof.

This follows from the above corollary using $f_n = f \cdot \mathcal{X}_{A_k}$.

**Definition 15.3:** For a $\sigma$-algebra $\mathcal{C}$, a **measure** on $\mathcal{C}$ is a function $\mu : \mathcal{C} \to [0, \infty]$ such that

1. $\mu(\emptyset) = 0$, and
2. If $A_1, A_2, A_3, \dots \in \mathcal{C}$ are disjoint then $\mu\Big( \cup_{k=1}^{\infty} A_k \Big) = \sum_{k=1}^{\infty} \mu(A_k)$.

When $\mathcal{M}$ is the $\sigma$-algebra of Lebesgue measurable sets in $\mathbb{R}$, and $f : \mathbb{R} \to [0, \infty]$ is any nonnegative measurable function on $\mathbb{R}$, the above corollary shows that we can define a measure $\mu$ on $\mathcal{M}$ by

$$\mu(A) = \int_A f \,.$$

**Theorem 15.4:** Let $f : [a, b] \to \mathbb{R}$ be bounded where $a \leq b$. Then,

1. $f$ is Riemann integrable on $[a, b]$ if and only if $f$ is continuous a.e in $[a, b]$.
2. If $f$ is Riemann integrable on $[a, b]$ then $f$ is Lebesgue integrable on $[a, b]$ and they are the same as follows:
$$\int_a^b f(x)\, \mathrm{d}x = \int_{[a,b]} f \,\mathrm{d}\lambda.$$
3. If we define upper Lebesgue integral and lower Lebesgue integral as
$$U(f) = \inf\left\{ \int s \;\middle|\; \begin{array}{c} s \text{ is a simple function} \\ \text{on } [a,b] \text{ with } s \geq f \end{array} \right\} \text{ and } L(f) = \sup\left\{ \int s \;\middle|\; \begin{array}{c} s \text{ is a simple function} \\ \text{on } [a,b] \text{ with } s \leq f \end{array} \right\},$$
then $f$ is Lebesgue integrable if and only if $U(f) = L(f)$. In this case,
$$\int_A f = U(f) - L(f).$$

Proof.

Chapter 5.5 in textbook includes proof for this theorem.

**Definition 15.5:** For a measurable function $f : A \subseteq \mathbb{R} \to [-\infty, \infty]$, we say that $f$ is (Lebesgue) **integrable** (on $A$) when the functions $f^+$ and $f^-$ are both Lebesgue integrable on $A$ and, in this case, we define the (Lebesgue) **integral** of $f$ on $A$ to be

$$\int_A f = \int_A f \,\mathrm{d}\lambda = \int_A f^+ - \int_A f^- .$$

**Remark 15.6:** For $f : A \subseteq \mathbb{R} \to [-\infty, \infty]$, $f$ is integrable if and only if $|f|$ is integrable.

**Theorem 15.7:** Let $f, g : A \subseteq \mathbb{R} \to [-\infty, \infty]$ be measurable and let $c \in \mathbb{R}$.

1. We have $\left| \int_A f \right| \leq \int_A |f|$ .
2. If $f \leq g$ then $\int_A f \leq \int_A g$.
3. The functions $cf$ and $f + g$ are Lebesgue integrable with $\int_A (cf) = c \int_A f$ and $\int_A (f + g) = \int_A f + \int_A g$.
4. If $A = B \cup C$ where $B$ and $C$ are disjoint and measurable then $\int_A f = \int_B f + \int_C f$.
5. If $B \subseteq A$ is measurable then $\int_B f = \int_A f \cdot \mathcal{X}_B$.
6. If $\lambda(A) = 0$ then $\int_A f = 0$.
7. If $f = g$ a.e. on $A$ then $\int_A f = \int_A g$.
8. $\int_A |f| = 0 \iff f = 0$ a.e in $A$.

Proof of 1.

We want to show $\displaystyle\int_A |f| \geq \int_A f \geq -\int_A |f|$. We have

$$\int_A f = \int_A f^+ - f^- = \int_A f^+ - \int_A f^- \leq \int_A f^+ + \int_A f^- = \int_A |f|.$$

We also have

$$\int_A f = \int_A f^+ - f^- = \int_A f^+ - \int_A f^- \geq -\int_A f^+ - \int_A f^- = -\left( \int_A f^+ + \int_A f^- \right) = -\int_A |f|.$$

**Exercise 15.8:** Prove properties $2 - 8$.

**Theorem 15.9:** (Lebesgue's dominated convergence theorem) Let $A \subseteq \mathbb{R}$ be a measurable set and let $f_n : A \to [-\infty, \infty]$ be measurable functions for $n \in \mathbb{Z}^+$. Suppose $\lim_{n\to\infty} f_n(x)$ exists pointwise $\forall\, x \in A$. Suppose there exists an integrable function $g : A \to [0, \infty]$ such that $\left| f_n(x) \right| \leq g(x)$, $\forall\, n \in \mathbb{Z}^+$, $x \in A$. Then

$$\int_A \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_A f_n \,.$$

Proof.

Let $f = \lim_{n\to\infty} f_n$. By Fatou's Lemma, applied to the function $g + f_n$, we have

$$\int_A g + \int_A \lim_{n\to\infty} f_n = \int_A \liminf_{n\to\infty} (g + f_n) \leq \liminf_{n\to\infty} \int_A (g + f_n) = \int_A g + \liminf_{n\to\infty} f_n.$$

It follows that

$$\liminf_{n\to\infty} \int_A f_n \geq \int_A \lim_{n\to\infty} f_n.$$

By Fatou's Lemma, applied to the function $g - f_n$, we have

$$\int_A g - \int_A \lim_{n\to\infty} f_n = \int_A \liminf_{n\to\infty} (g - f_n) \leq \liminf_{n\to\infty} \int_A (g - f_n) = \int_A g - \limsup_{n\to\infty} \int_A f_n.$$

It follows that

$$\limsup_{n\to\infty} \int_A f_n \leq \int_A \lim_{n\to\infty} f_n.$$

**End of Lecture 15**

**Start of Lecture 16**

**Example 16.1:** Let $f_n : A \subseteq \mathbb{R} \to [0,\infty]$ be nonnegative and measurable for $n \in \mathbb{Z}^+$ with $f_1(x) \geq f_2(x) \geq \ldots$. Do we have $\int_A \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_A f_n$?

No we don't. We give two examples. Consider $A = (0,1)$ and $f_n(x) = \frac{1}{nx}$. So $\int_A f_n = \infty$, $\forall$ $n$ but $\lim_{n\to\infty} = 0$.

Also consider $A = \mathbb{R}$. $f_n(x) = \frac{1}{n}$, $\forall$ $x$. We have $f_n \to 0$ but $\int_A f_n = \infty$, $\forall$ $n$. Note that if $\int_A f_1 < \infty$ then $\int_A \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_A f$ by Lebesgue's Dominated Convergence Theorem. Simply by applying Lebesgue's Monotone Convergence Theorem to $g_n = f_1 - f_n$.

**Example 16.2:** If $f : \mathbb{R} \to \mathbb{R}$ is differentiable then $f'$ is measurable and if so, $f'$ is integrable.

We have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{n\to\infty} \frac{f\Big(x + \frac{1}{n}\Big) - f(x)}{\frac{1}{n}}.$$

Thus $f' = \lim_{n\to\infty} g_n$ where

$$g_n(x) = \frac{f\Big(x + \frac{1}{n} - f(x)\Big)}{\frac{1}{n}}$$

which is measurable.

$$\text{For } f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x \neq 0 \end{cases}$$

$$f'(x) = \begin{cases} 2x \sin\frac{1}{x} - \cos\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x \neq 0 \end{cases}$$

$$\text{But for } f(x) = \begin{cases} x^2 \sin\frac{1}{x^2}, & \text{if } x \neq 0 \\ 0, & \text{if } x \neq 0 \end{cases}$$

$$f'(x) = \begin{cases} 2x \sin\frac{1}{x^2} - \frac{2}{x}\cos\frac{1}{x^2}, & \text{if } x \neq 0 \\ 0, & \text{if } x \neq 0 \end{cases}$$

For further explanation see section 5.5 in textbook.

**Theorem 16.3** (Fundamental theorem of calculus)**:** If $f : [a,b] \to [-\infty,\infty]$ differentiable and $f'$ is bounded, then $f'$ is integrable on $[a,b]$ and $\int_a^b = f(b) - f(a)$.

**Definition 16.4:** $\forall$ $x, y, z \in V$ and $c \in \mathbb{R}$, an **inner product** on a real vector space $V$ satisfies the following properties.

1. $\langle x, x\rangle \geq 0$ with $\langle x, x\rangle = 0 \iff x = 0$.
2. $\langle x, y\rangle = \langle y, x\rangle$.
3. $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$.
4. $\langle cx, y\rangle = c\langle x, y\rangle = \langle x, cy\rangle$.

A vector space with inner product is called an **inner product space**.

**Definition 16.5:** $\forall$ $x, y \in V$ and $c \in \mathbb{R}$, a norm on a vector space $V$ satisfies the following properties.

1. $\|x\| \geq 0$ with $\|x\| = 0 \iff x = 0$.
2. $\|cx\| = |c|\|x\|$.
3. $\|x + y\| \leq \|x\| + \|y\|$.

A vector space with a norm is called a **normed linear space**.

**Definition 16.6:** $\forall\, x, y, z \in X$, a **metric** on a set $X$ satisfies the following properties.

1. $d(x, y) \geq 0$ with $d(x, y) = 0 \iff x = y$.
2. $d(x, y) = d(y, x)$.
3. $d(x, z) \leq d(x, y) + d(y, z)$.

A set with a metric is called **metric space**.

**Definition 16.7:** A **topology** on a set $X$ is a set $\mathcal{T}$ whose elements are subsets of $X$ and satisfies the following properties.

1. $\varnothing \in \mathcal{T}$ and $X \in \mathcal{T}$.
2. If $A, B \in \mathcal{T}$ then $A \cap B \in \mathcal{T}$.
3. If $K$ is a set and $A_k \in \mathcal{T}$ for every $k \in K$ then $\bigcup_{k \in K} A_k \in \mathcal{T}$.

The elements in $\mathcal{T}$ (which are subsets of $X$) are called the **open sets** in $X$. A set with a topology is called a **topological space**.

**End of Lecture 16**

**Start of Lecture 17**

**Definition 17.1:** In given an inner product in a vector space we define the **associated norm on** $V$ by all $\|x\| = \sqrt{\langle x, x\rangle}$. In this case, the inner product and its norm satisfy

1. Cauchy-Schwarz inequality: $\big\|\langle x, y\rangle\big\| \le \|x\|\|y\|$.
2. Polarization identity: $\langle x, y\rangle = \dfrac{1}{4}\Big(\|x+y\|^2 - \|x-y\|^2\Big)$.

**Definition 17.2:** Given a norm on a vector space $V$ we define the **associated metric on** $V$ (or any subset $X \subseteq V$) by $d(x,y) = |x-y|$

**Definition 17.3:** Given a metric on a set $X$, we define the **associated topology on** $X$ by defining a set $A \subseteq X$ to be open when $\forall\, a \in A$, $\exists\, r > 0$ such that $B(a,r) \subseteq A$ where $B(a,r) = \{x \in X \ \mid\ d(x,a) < r\}$.

**Definition 17.4:** For a sequence $\{x_n\}$ in a metric space $X$, we say

$$\begin{aligned}\{x_n\} \textbf{ is convergent} \text{ in } X &\iff \exists\, a \in X \text{ such that } \lim_{n\to\infty} x_n = a,\\ &\iff \exists\, a \in X,\ \forall\, \varepsilon > 0\ \exists\, n \in \mathbb{Z}^+,\ \forall\, k \text{ with } \mathbb{Z}^+ \ni k \ge n, d(x_k, a) < \varepsilon.\end{aligned}$$

**Definition 17.5:** We say the sequence $\{x_k\}$ **is Cauchy** if $\forall\, \varepsilon > 0$, $\exists\, n \in \mathbb{N}^+$ such that $\forall\, k, \ell \in \mathbb{N}, k, \ell \ge n \implies d(x_k, x_\ell) < \varepsilon$.

**Remark 17.6:** It is easy to show that if $\{x_n\}$ converges then $(x_n)$ is Cauchy. When $X = \mathbb{R}^n$, it can be shown that if $(x_n)$ is Cauchy, then $(x_n)$ converges.

**Definition 17.7:** We say that a metric space is **complete** when every Cauchy sequence in $X$ converges in $X$.

**Definition 17.8:** A topological space is separable when t contains a countable dense set.

**Definition 17.9:** For $A \subseteq X$, we say that $A$ is **closed** if $A^c = X \setminus A$ is open.

**Definition 17.10:** The **closure** of $A$, denoted by $\overline{A}$, is the smallest closed set which contains $A$, that is, the intersection of the set of all closed sets in $X$ which contain $A$. We say that $A$ is **dense** in $X$ when $\overline{A} = X$.

**Example 17.11:** We have various subspaces such as

$$\begin{aligned}\mathbb{R}^\omega &= \{\text{sequences } (x_n) \in \mathbb{R} \ \mid\ \exists\, n \in \mathbb{N},\ \forall\, k \ge n, x_k = 0\}\\ &= \{\text{sequences } (x_n) \in \mathbb{R} \ \mid\ (x_n) \text{ is eventually constant}\}\\ &= \{\text{sequences } (x_n) \in \mathbb{R} \ \mid\ (x_n) \text{ converges}\}\end{aligned}$$

We also have

$$\ell_1 = \left\{\text{sequences } (x_n) \in \mathbb{R} \ \middle|\ \sum_{n=1}^{\infty} |x_n| < \infty\right\}$$

$$\ell_2 = \left\{\text{sequences } (x_n) \in \mathbb{R} \ \middle|\ \sum_{n=1}^{\infty} |x_n|^2 < \infty\right\}$$

$$\ell_p = \left\{\text{sequences } (x_n) \in \mathbb{R} \ \middle|\ \sum_{n=1}^{\infty} |x_n|^p < \infty\right\}$$

$$\ell_\infty = \left\{\text{sequences } (x_n) \in \mathbb{R} \ \middle|\ \begin{array}{c}(x_n) \text{ is bounded,}\\ \text{or equivalently,} \sup |x_k| < \infty\end{array}\right\}$$

In $\ell_p$ we have norms

$$\|x\|_1 = \sum_{n=1}^{\infty} |x_n|$$
$$\|x\|_2 = \left(\sum_{n=1}^{\infty} |x_n|^2\right)^{\frac{1}{2}}$$

Note that we can define inner product $\langle x, y\rangle = \sum_{k=1}^{\infty} x_k y_k$.

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}}$$
$$\|x\|_\infty = \sup\{|x_n| \mid n \in \mathbb{N}^+\}$$

**Definition 17.12:** A complete inner product space is called a **Hilbert space** and a complete normed linear space is called **Banach space**.

**Example 17.13:** $\mathcal{C}([a, b]) = \{\text{continuous functions } f : [a, b] \to \mathbb{R}\}.$ has several norms.

$$\|f\|_1 = \int_a^b |f|$$
$$\|f\|_2 = \left(\int_a^b |f|^2\right)^{\frac{1}{2}}, \langle f, g\rangle = \int_a^b fg$$
$$\|f\|_p = \left(\int_a^b |f|^p\right)^{\frac{1}{p}}$$
$$\|f\|_\infty = \max\{\left|f(x)\right| \mid a \le x \le b\}$$

**Example 17.14:** Let $R([a, b]) = \{\text{Riemann integrable functions } f : [a, b] \to \mathbb{R}\}.$ Note that on $R([a, b]),\ \|f\|_1 = \int_a^b |f|$ does not give a norm.

**End of Lecture 17**

**Start of Lecture 18**

**Remark 18.1:**

1. In Assignemnt #2, we will show that for $f : A \subseteq \mathbb{R} \to [0, \infty]$ measurable, we have $\int_A f = 0 \iff f = 0$ a.e in $A$.
2. When $f : A \subseteq \mathbb{R} \to [-\infty, \infty]$,
$$f \text{ is integrable } \iff f^+ \text{ and } f^- \text{ are integrable. In this case } \int_A f = \int_A f^+ - \int_A f^-.$$
$$\iff |f| \text{ is integrable. In this case } \int_A |f| = \int_A f^+ + \int_A f^-.$$
3. When $f : A \subseteq \mathbb{R} \to [-\infty, \infty]$ is integrable, the sets
$$B = \{x \in A \mid f(x) = +\infty\} \text{ and } C = \{x \in A \mid f(x) = -\infty\}$$
both have measure zero.
4. We define $0 \cdot \pm\infty = 0$ and $(\pm\infty) \pm (\pm\infty) = \infty$ for this chapter.

**Definition 18.2:** For $f : A \subseteq \mathbb{R} \to [-\infty, \infty]$ measurable, we define
$$\|f\|_1 = \int_A |f|$$
$$\|f\|_2 = \left(\int_A |f|^2\right)^{\frac{1}{2}}$$
$$\|f\|_p = \left(\int_A |f|^p\right)^{\frac{1}{p}} \text{ for } 1 \leq p < \infty$$
$$\|f\|_\infty = \operatorname{ess\,sup} |f| \text{ (ess sup) is explained below.}$$

Note that $|f|$ is bounded
$$\iff \exists\, a \geq 0 \; \{x \in A \mid \left|f(x)\right| > a\} = \varnothing$$
$$\iff \exists\, a \geq 0 \; |f|^{-1}(a, \infty] = \varnothing$$

with $\sup |f| = \inf\left\{a \geq 0 \;\middle|\; |f|^{-1}(a, \infty]\right\}$.

**Definition 18.3:** We say that $f$ is **essentially bounded** when $\exists\, a \geq 0,\ \lambda\Big(|f|^{-1}(a, \infty]\Big) = 0$ and we define the **essential supremum** of $|f|$ as $\operatorname{ess\,sup} |f| = \inf\left\{a \geq 0 \;\middle|\; \lambda\Big(|f|^{-1}(a, \infty]\Big)\right\}$.

**Definition 18.4:** Let $A \subseteq \mathbb{R}$ be measurable. We define
$$L_1(A) = \left\{f : A \to [-\infty, \infty] \;\middle|\; f \text{ is measurable and } \int_A |f| < \infty\right\} / \sim$$
$$L_2(A) = \left\{f : A \to [-\infty, \infty] \;\middle|\; f \text{ is measurable and } \int_A |f|^2 < \infty\right\} / \sim$$
$$L_p(A) = \left\{f : A \to [-\infty, \infty] \;\middle|\; f \text{ is measurable and } \int_A |f|^p < \infty\right\} / \sim$$
$$L_\infty(A) = \left\{f : A \to [-\infty, \infty] \;\middle|\; f \text{ is measurable and } \operatorname{ess\,sup} |f| < \infty\right\} / \sim$$

where in all cases $f \sim g \iff f = g$ a.e in $A$. Note that in all cases $\{f \mid f = 0 \text{ a.e in } A\}$ is a subspace. So $L_p(A)$ is a vector space.

**Remark 18.5:** We shall show that for $1 \leq p \leq \infty$, $\|f\|_p$ defines a norm on the vector space $L_p(A)$.

**Exercise 18.6:** Show that $\|f\|_\infty$ defines a norm on $L_\infty(A)$.

**Remark 18.7:** For $1 \leq p < \infty$ we have the following

1. For $f \in L_p(A)$, $\|f\|_p = \left(\int |f|^p\right)^{\frac{1}{p}} \in [0, \infty)$ with $\|f\|_p \geq 0$ and

$$\begin{aligned} \|f\|_p = 0 &\iff \int |f|^p = 0 &&\iff |f|^p = 0 \text{ a.e in } A \\ &\iff f = 0 \text{ a.e in } A \\ &\iff f = 0 \text{ in } L_p(A). \end{aligned}$$

2. For $f \in L_p(A)$ and $c \in \mathbb{R}$,

$$\|cf\|_p = \left(\int |cf|^p\right)^{\frac{1}{p}} = \left(\int |c|^p|f|^p\right)^{\frac{1}{p}} = |c|\left(\int |f|^p\right)^{\frac{1}{p}} = c\|f\|_p.$$

   It remains to prove the triangle inequality.

3. For $f, g \in L_p(A)$, $\|f + g\|_p \leq \|f\|_p + \|g\|_p$. This is called **Minkowski's inequality.**

**Definition 18.8:** For $p, q \in [1, \infty]$, we say that $p$ and $q$ are **conjugate** when $\frac{1}{p} + \frac{1}{q} = 1$ with $\frac{1}{\infty} = 0$.

**Theorem 18.9** (Hölder's inequality)**:** Let $p, q \in (1, \infty)$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$, then

1. For $x \in \ell_p, y \in \ell_q$, we have $xy \in \ell_1$ and $\|xy\|_1 \leq \|x\|_p\|y\|_q$ with the equality when $\exists\ s, t \in \mathbb{R}$ both nonzero such that $s|x|^p = t|y|^q$ where $s|x_k|^p = t|y_k|^q, \forall\ k$.
2. For $f \in L_p(A), g \in L_q(A)$ where $A \subseteq \mathbb{R}$ is measurable, we have $fg \in L_1(A)$ and $\|fg\|_1 \leq \|f\|_p\|g\|_q$ with the equality when $\exists\ s, t \in \mathbb{R}$ both nonzero such that $s|f|^p = t|g|^q$ a.e in $A$.

Proof of 1.

We claim that for all $a, b \geq 0$, $ab \leq \dfrac{a^p}{p} + \dfrac{b^q}{q}$. Note that since $\dfrac{1}{p} + \dfrac{1}{q} = 1$ we have $\dfrac{1}{q} = 1 - \dfrac{1}{p} = \dfrac{p-1}{p}$. So $q = \dfrac{p}{p-1}$. Hence $q(p-1) = p$ and similarly $p(q-1) = q$. For $x, y \geq 0$ we have

$$\begin{aligned} y = x^{p-1} &\iff y^q = x^{q(p-1)} = x^p \\ &\iff x = y^{q-1} \end{aligned}$$

So the functions $f(x) = x^{p-1}$ and $g(y) = y^{q-1}$ are inverse functions. Graph We have

$$\begin{aligned} \int_0^a x^{p-1}\,\mathrm{d}x \int_0^b y^{q-1}\,\mathrm{d}y &\geq ab \\ \left[\frac{x^p}{p}\right]_0^a + \left[\frac{y^q}{q}\right]_0^b &\geq ab \\ \frac{a^p}{p} + \frac{b^q}{q} &\geq ab \end{aligned}$$

with equality when

$$\begin{aligned} b = a^{p-1} &\iff b^q = a^p \\ &\iff a = b^{q-1.} \end{aligned}$$

To prove Part 1, apply the above inequality to $a = \dfrac{|x_k|}{\|x\|_p}$, $b = \dfrac{|y_k|}{\|y\|_q}$ to get

$$\frac{|x_k y_k|}{\|x\|_p \|y\|_q} \leq \frac{|x_k|^p}{p\|x\|_p^p} + \frac{|y_k|^q}{q\|y\|_q^q}$$

Sum over $k$ to get

$$\frac{\|xy\|_1}{\|x\|_p \|y\|_q} \geq \frac{1}{p} + \frac{1}{q}.$$

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**End of Lecture 18**

**Start of Lecture 19**

Proof of 2.

Recall that for $a, b \geq 0$, $ab \leq \dfrac{a^p}{p} + \dfrac{b^q}{q}$ with equality when $b = a^{p-1} \iff b^q = a^p \iff a = b^{q-1}$. Apply this inequality to

$$a = \frac{|f(x)|}{\|f\|_p} \text{ and } b = \frac{|g(x)|}{\|g\|_p}$$

and obtain

$$\frac{|f(x)g(x)|}{\|f\|_p\|g\|_q} \leq \frac{|f(x)|^p}{p\|f\|_p^p} + \frac{|g(x)|^q}{q\|g\|_q^q},\ \forall\, x \in A.$$

Integrate over $A$ to get

$$\frac{\|fg\|_1}{\|f\|_p\|g\|_q} \leq \frac{1}{p} + \frac{1}{q} = 1.$$

So that $\|fg\|_1 \leq \|f\|_p\|g\|_q$ and we have equality when

$$\frac{|f(x)|^p}{\|f\|_p^p} = \frac{|g(x)|^q}{\|g\|_q^q}.$$

This is equivalent to $\exists\, s, t > 0$ such that $s|f|^p = t|g|^q$ a.e in $A$. Since if $s|f|^p = t|g|^q$ a.e in $A$ then we can integrate to get $s\|f\|_p^p = t\|g\|_q^q$. So that

$$t = \frac{s\|f\|_p^p}{\|q\|_q^q}.$$

Hence, $s|f|^p = s\dfrac{\|f\|_p^p}{\|g\|_q^q}|g|^q$ a.e in $A$. Thus $\dfrac{|f|^p}{\|f\|_p^p} = \dfrac{|g|^q}{\|g\|_q^q}$ a.e in $A$.

**Theorem 19.1** (Minkowski's Inequality)**:** Let $p \in (1, \infty)$ and let $A \subseteq \mathbb{R}$ be measurable. Then

1. For $x, y \in \ell_p$ we have $\|x + y\|_p \leq |x|_p + \|y\|_p$ with equality
2. For $f, g \in L_p(A)$ we have $\|f + g\|_p \leq \|f\|_p + \|g\|_p$ with equality when $\exists\, s, t \geq 0$ with $(s, t) \neq (0, 0)$ such that $sf = tg$ a.e in $A$.

Proof of Part 1.

Note that for $a, b \in \mathbb{R}$ we have

$$|a + b|^p = |a + b||a + b|^{p-1} \leq \big(|a| + |b|\big)|a + b|^{p-1} = |a||a + b|^{-1} + |b||a + b|^{p-1}$$

with equality when $a$ and $b$ have the same sign.

To prove Part 1, apply this inequality with $a = x_k$ and $b = y_k$ to get

$$|x_k + y_k|^p \leq |x_k||x_k + y_k|^{p-1} + |y_k||x_k + y_k|^{p-1}.$$

Take the sum over $k$ and get

$$\begin{aligned} \|x + y\|_p^p &\leq \Big\||x||x + y|^{p-1}\Big\|_1 + \Big\||y||x + y|^{p-1}\Big\|_1 \\ &\leq \|x\|_p\Big\||x + y|^{p-1}\Big\|_q + \|y\|_p\Big\||x + y|^{p-1}\Big\|_q \end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1,\ \frac{1}{q}=1-\frac{1}{p}=\frac{p-1}{p},\ q=\frac{p}{p-1}$.

$$\begin{aligned}
&= \left(\|x\|_p+\|y\|_p\right)\left(\sum |x+y|^{q(p-1)}\right)^{\frac{1}{q}} \\
&= \left(\|x\|_p+\|y\|_p\right)\left(\sum |x+y|^{p}\right)^{\frac{p-1}{p}} \\
&= \left(\|x\|_p+\|y\|_p\right)\|x+y\|_p^{p-1} \text{ and so,} \\
\|x+y\|_p &\le \|x\|_p+\|y\|_p
\end{aligned}$$

Equality holds when $x_k$ and $y_k$ have the same sign for all $k$ and when $x, y$ both nonzero, $\exists\ r, s, t > 0$ such that $r|x|^p = s|x+y|^q = t|y|^p$ that is when $\exists\ u, v > 0$ such that $ux = vy$.

**End of Lecture 19**

**Start of Lecture 20**

Proof of Part 2.

To prove Part 2, we apply the inequality $|a+b|^p \leq |a||a+b|^{p-1} + |b||a+b|^{p-1}$ with $a = f(x)$ and $b = g(x)$ to get

$$\left|f(x)+g(x)\right|^p \leq \left|f(x)\right|\left|f(x)+g(x)\right|^{p-1} + \left|g(x)\right|\left|f(x)+g(x)\right|^{p-1},\ \forall\, x \in A.$$

Integrate and use Hölder's inequality to obtain

$$\begin{aligned}
\|f+g\|_p^p &\leq \left\||f||f+g|^{p-1}\right\|_1 + \left\||g||f+g|^{p-1}\right\| + 1 \\
&\leq \|f\|_p \left\||f+g|^{p-1}\right\|_q + \|g\|_p \left\||f+g|^{p-1}\right\|_q \\
&= \left(\|f\|_g + \|g\|_p\right)\left\||f+g|^{p-1}\right\|_q \\
&= \left(\|f\|_g + \|g\|_p\right)\left(\int_A |f+g|^{q(p-1)}\right)^{\frac{1}{q}} \\
&= \left(\|f\|_g + \|g\|_p\right)\left(\int_A |f+g|^{something}\right)^{\frac{1}{q}} \\
&= \left(\|f\|_g + \|g\|_p\right)\|f+g\|_p^{p-1}
\end{aligned}$$

Thus, either $\|f+g\|_p = 0$ in which case the equality holds, or we can divide by $\|f+g\|_p^{p-1}$ to get $\|f+g\|_p \leq \|f\|_p + \|g\|_p$. Equality holds when $f(x)$ and $g(x)$ have the same sign for a.e $x \in A$ and either $f = 0$ or $g = 0$ or $f + g = 0$ a.e in $A$ or $\exists\, r, s, t > 0$ such that $rf = s(f+g) = tg$ a.e in $A$. That is,

$$\begin{aligned}
&f = 0 \text{a.e} \text{in } A \\
\text{or } &g = 0 \text{a.e} \text{in } A \\
\text{or } &\exists\, s, t > 0 \text{ such that } sf = tg \text{a.e in } A
\end{aligned}$$

Case 1: $p = 1$.

When $x, y \in \ell_1$ we have

$$\begin{aligned}
\|x+y\|_1 &= \sum_{k=1}^{\infty} |x_k + y_k| \\
&\leq \sum_{k=1}^{\infty} (|x_k| + |y_k|) \\
&= \sum_{k=1}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| \\
&= \|x\|_1 + \|y\|_1
\end{aligned}$$

For $f, g \in L_1(A)$ where $A \subseteq \mathbb{R}$ is measurable, we have

$$\begin{aligned}
\|f+g\|_1 &= \int_A |f+g| \\
&\leq \int_A (|f| + |g|) \\
&= \int_A |f| + \int_A |g| \\
&= \|f\|_1 + \|g\|_1
\end{aligned}$$

Case 2: $p = 2$.

In $\ell_2$ we have an inner product given by $\langle x, y\rangle = \sum_{k=1}^{\infty} x_k y_k$ and $\|x\|_2 = \sqrt{\langle x, x\rangle}$. Note that for $a, b \in \mathbb{R}$ we have

$$\begin{aligned} 0 &\leq (a+b)^2 = a^2 + 2ab + b^2 \\ 0 &\leq (a=b)^2 = a^2 - 2ab + b^2 \end{aligned}$$

Thus $\pm 2ab \leq a^2 + b^2$. Thus $|ab| \leq \frac{1}{2}(a^2 + b^2)$. Thus,

$$\sum_{k=1}^{\infty} |x_k y_k| \leq \sum_{k=1}^{\infty} \frac{1}{2}(x^2 + y_k^2) \leq \frac{1}{2}(\|x\|_2 + \|y\|_2).$$

Similarly, in $L_2(A)$ we have an inner product given by $\langle f, g\rangle = \int_A fg$ since when $f, g \in L_2(A)$, we have

$$\begin{aligned} \left|\int_A fg\right| &\leq \int_A |fg| \\ &\leq \int_A \frac{1}{2}(f^2 - g^2) \\ &= \frac{1}{2}(\|f\|_2 + \|g\|_2) < \infty \end{aligned}$$

and properties of inner product hold.
Case 3: $p = \infty$.

In $\ell_\infty = \{$ bounded sequences in $\mathbb{R}\}$ we have the norm $\|x\|_\infty = \sup(|x_k| \mid k \in \mathbb{Z}^+)$. The triangle inequality holds because

$$\begin{aligned} \|x+y\|_\infty &= \sup_{k\geq 1} |x_k + y_k| \\ &\leq \sup_{k\geq 1}(|x_k| + |y_k|) \\ &\leq \sup_{k\geq 1} |x_k| + \sup_{k\geq 1} |y_k| \\ &= \|x\|_\infty + \|y\|_\infty \end{aligned}$$

In $L_\infty(A)$ we have the norm $\|f\|_\infty = \text{esssup}\, \|f\| = \inf\{a > 0 \mid \lambda\left|f^{-1}\right|(a, \infty] = 0\}$. Let us verify the triangle inequality. For $f, g \in L_\infty(A)$, we have

$$\begin{aligned} \|f+g\|_\infty &= \text{esssup}\, |f+g| \\ &\leq \text{esssup}\, (|f| + |g|) \\ &\leq \text{esssup}\, |f| + \text{esssup}\, |g| = \|f\|_\infty + \|g\|_\infty. \end{aligned}$$

Indeed, for $h, k : A \to [0, \infty]$ with $h(x) \leq k(x)\ \forall\, x \in A$. If $h(x) > a$ then $k(x) > a$.

$$\begin{aligned} &\text{So } k^{-1}(a, \infty] \subseteq h^{-1}(a, \infty] \\ &\text{So if } \lambda k^{-1}(a, \infty] = 0 \text{ then } \lambda h^{-1}(a, \infty] \\ &\text{So } \{a \mid \lambda k^{-1}(a, \infty] = 0\} \subseteq \{a \mid \lambda h^{-1}(a, \infty] = 0\} \\ &\text{So } \inf\{a \mid \lambda k^{-1}(a, \infty] = 0\} \leq \inf\{a \mid \lambda h^{-1}(a, \infty] = 0\} \end{aligned}$$

Hence esssup $(h+k) \leq$ esssup $h$ + esssup $k$. Let $\varepsilon > 0$. Choose $a > 0$ so that $\lambda h^{-1}(a, \infty] = 0$ and $a \leq$ esssup $h + \varepsilon$. Choose $h > 0$ so that $\lambda k^{-1}(b, \infty] = 0$ and $b \leq$ esssup $k + \varepsilon$. If $h(x) + k(x) > a + b$ then either $h(x) > a$ or $k(x) > b$. Thus,

$$(h+k)^{-1}(a+b, \infty] \subseteq h^{-1}(a, \infty] \cup k^{-1}(b, \infty]$$

Since $\lambda h^{-1}(a, \infty] = \lambda k^{-1}(b, \infty] = 0$. We have $\lambda (h+k)^{-1}(a+b, \infty] = 0$. It follows that

$$\text{esssup } (h+k) \leq a + b \leq \text{esssup } h + \text{esssup } k + 2\varepsilon.$$

Since $q > 0$ was arbitrary, we have esssup $(h+k) \leq$ esssup $h$ + esssup $k$ as required.

**End of Lecture 20**

**Start of Lecture 21**

**Recall 21.1:** For $f, g \in L_\infty(A)$, $\|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty$.

$1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. For $f \in L_p(A)$, $g \in L_q(A)$, $\|fg\|_1 \le \|f\|_p \|g\|_q$.

**Theorem 21.2** (Special case for Hölder's inequality)**:** For $p = 1$, $q = \infty$,

1. If $x \in \ell_1, y \in \ell_\infty$, we have $xy \in \ell_1$ and $\|xy\|_1 \le \|x\|_1 \|y\|_\infty$
2. If $f \in L_1(A), g \in L_\infty(A)$ where $A \subseteq \mathbb{R}$ is measurable, we have $fg \in L_1(A)$ and $\|fg\|_1 \le \|f\|_1 \|g\|_\infty$

Proof of 2.

Recall that we define $\|f\|_\infty = \inf\{a \ge 0 \mid \lambda |f|^{-1}(a, \infty] = 0\}$.
We claim $\{x \in A \mid |f(x)| > \|f\|_\infty\}$ has measure zero. Let $A_n = \{x \in A \mid |f(x)| > \|f\|_\infty + \frac{1}{n}\}$. Note that $\lambda(A_n) = 0$, $\forall\, n$ (since by the definition of $\|f\|_\infty$, given $n$ e can choose $a \ge 0$, so $\|f\|_\infty < a < \|f\|_\infty + \frac{1}{n}$ with $\lambda |f|^{-1}(a, \infty] = 0$ and then $\{x \in A \mid |f(x)| > \|f\|_\infty + \frac{1}{n}\} \subseteq |f|^{-1}(a, \infty]$ ). Then, $A_1 \subseteq A_2 \subseteq \ldots$ and $\bigcup A_n = \{x \in A \mid |f(x)| > \|f\|_\infty\}$. Thus, $\{x \in A \mid |f(x)| > \|f\|_\infty\}$ has measure zero.
Let $f \in L_1(A)$ and $g \in L_\infty(A)$ then $fg \in L_1(A)$ with $\|fg\|_1 \le \|f\|_1 \|g\|_\infty$. Let $B = \{x \in A \mid |g(x)| > \|g\|_\infty\}$ and $C = A \setminus B = \{x \in A \mid |g(x)| < \|g\|_\infty\}$. Then,

$$\begin{aligned}
|fg|_1 &= \int_A |fg| \\
&= \int_B |f||g| + \int_C |f||g| \\
&= \int_C |f||g| \text{ since } \lambda(B) = 0 \\
&\le \int_C |f| \|g\|_\infty \text{ since } |g(x)| \le \|g\|_\infty,\ \forall\, x \in C \\
&= \|g\|_\infty \int_C |f| \\
&\le \|g\|_\infty \int_A |f| \text{ since } C \subseteq A \\
&= \|g\|_\infty |f|_1.
\end{aligned}$$

**Theorem 21.3:** Let $A \subseteq \mathbb{R}$ be measurable and let $p \in [1, \infty]$. Then $L_p(A)$ is a complete using the $p-$norm.

Proof.

In the case that $1 \le p < \infty$, we have $\|f\|_p = \left(\int_A |f|^p\right)^{\frac{1}{p}}$. Let $\{f_n\}$ be a Cauchy sequence in $L_p(A)$. Then each $f_n \in L_p(A)$ and $\forall\, \varepsilon$, $\exists\, m \in \mathbb{Z}^+$, $\forall\, k, \ell \in \mathbb{Z}^+$, if $k, \ell \ge m$ then $\|f_k - f_\ell\|_p < \varepsilon$. Choose a subsequence $\{f_{n_k}\}$ so that $\left\|f_{n_{k+1}} - f_{n_k}\right\|_p \le \dfrac{1}{2^k}$, $\forall\, k \in \mathbb{Z}^+$. Note that

$$f_{n_\ell} = f_{n_1} + \sum_{k=1}^{\ell-1} \left(f_{n_{k+1}} - f_{n_k}\right).$$

Let

$$g_\ell = \sum_{k=1}^{\ell-1} \left| f_{n_{k+1}} - f_{n_k} \right| \text{ and } g = \lim_{\ell\to\infty} g_\ell = \sum_{k=1}^{\infty} \left| f_{n_{k+1}} - f_{n_k} \right|$$

Note that $g(x)$ exists in $[0,\infty]$, $\forall\, x \in A$. We have

$$\begin{aligned} \|g_\ell\|_p &\le \sum_{k=1}^{\ell-1} \left\| f_{n_{k+1}} - f_{n_k} \right\|_p && \text{(by Minkowski's inequality)} \\ &\le \sum \ell - 1_{k=1} \frac{1}{2^k} \\ &\le 1,\ \forall\, \ell \in \mathbb{Z}^+ \end{aligned}$$

It follows that

$$\begin{aligned} \|g\|_p^p &= \int_A |g|^p \\ &= \int_A \lim |g_\ell|^p \\ &\le \liminf_{\ell\to\infty} \int_A |g_\ell|^p && \text{(by Fatou's lemma)} \\ &= \liminf_{\ell\to\infty} \|g_\ell\|_p^p \\ &\le 1. \end{aligned}$$

So that $\|g\|_p \le 1$. Since $\int_A |g|^p < \infty$, it follows that $\left|g(x)\right| < \infty$ for a.e $x \in A$. Thus, the sum

$$\sum_{k=1}^{\infty} \left| f_{n_{k+1}} \right| (x) f_{n_k}(x)$$

converges to the finite number $g(x)$ for a.e $x \in A$. Hence

$$\sum_{k=1}^{\infty} \Big( f_{n_{k+1}}(x) - f_{n_k}(x) \Big)$$

converges to a finite real number for a.e $x \in A$.
We define

$$f(x) = \begin{cases} f_{n_1}(x) + \sum_{k=1}^{\infty} \Big( f_{n_{k+1}}(x) - f_{n_k}(x) \Big) = \lim_{\ell\to\infty} f_\ell(x) & \text{provided this limit exists and finite} \\ 0 & \text{otherwise} \end{cases}.$$

We claim that $f_n \to f$ in $L_p(A)$. That is, $\|f_n - f\|_p \to 0$ as $n \to \infty$. Given $\forall\, \varepsilon$, we can choose $m \in \mathbb{Z}^+$ so that $\forall\, k, \ell \in \mathbb{Z}^+$, if $k, \ell \ge m$ then $\|f_k - f_\ell\|_p < \varepsilon$. Let $n \ge m$. So, $\forall\, k \ge m$, $\|f_k - f_n\|_p < \varepsilon$. Then,

$$\|f_n - f\|_p^p = \int_A |f_n - f|^p = \int_A \lim_{k\to\infty} \left| f_n - f_{n_k} \right|^p \le \liminf_{k\to\infty} \int_A \left| f_n - f_{n_k} \right|^p = \liminf_{k\to\infty} \left\| f_n - f_{n_k} \right\|_p^p \le \varepsilon^p$$

and so $\|f_n - f\|_p \le \varepsilon$. Thus $\|f_n - f\|_p \to 0$ as $n \to \infty$. Finally, note if we choose $n$ so that $\|f - f_n\| \le 1$. Then

$$\|f\|_p = \|f - f_n + f_n\|_p \le \|f - f_n\|_p + \|f_n\|_p \le 1 + \|f_n\|_p < \infty$$

So that $f \in L_p(A)$. Since $\|f - f_n\|_p \to 0$ as $n \to \infty$, $f_n \to f$ in $L_p(A)$.
Now we prove for the case $p = \infty$. Let $\{f_n\}$ be a Cauchy sequence in $L_\infty(A)$. So $\forall\ \varepsilon > 0\ \exists\ m \in \mathbb{Z}^+\ \forall\ k, \ell \in \mathbb{Z}^+$ if $k, \ell \geq$ then $\|f_k - f_\ell\|_\infty < \varepsilon$. let $B_n\{x \in A \mid \left|f_n(x)\right| > \|f\|_\infty\}$ and $C_{k,\ell} = \left\{x \in A \;\middle|\; \left|f_k(x) - f_\ell(x)\right| > \|f_k - f_\ell\|_\infty\right\}$ for $k, \ell, n \in \mathbb{Z}^+$. Recall that $B_n$ and $C_{k,\ell}$ all have measure zero. Let

$$E = \bigcup_{n=1}^{\infty} B_n \cup \bigcup_{k,\ell=1}^{\infty} C_{k,\ell}.$$

Then $\lambda(E) = 0$. For $x \in A \setminus E$, $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$ because given $\varepsilon > 0$ we can choose $m \in \mathbb{Z}^+$ so that $k, \ell \geq m \implies \|f_k - f_\ell < \varepsilon\|$. Thus for $k, \ell \geq m$, $\left|f_k(x) - f_\ell(x)\right| \leq \|f_k - f_\ell\|_\infty < \varepsilon$ for all $x \in A \setminus E$. Indeed the sequence of functions $\{f_n\}$ defined on $A \setminus E$ converges to a function $f$ defined on $A \setminus E$. Define $f : A \to \mathbb{R}$ as

$$f(x) = \begin{cases} \lim\limits_{n\to\infty} f_n(x) & \text{for } x \in A \setminus E \\ 0 & \text{for } x \in E \end{cases}$$

Note that $f$ is measurable since each $f_n$ is measurable in $A \setminus E$. So $\lim\limits_{n\to\infty} f_n$ is measurable in $A \setminus E$. So $f$ is measurable on $A \setminus E$. Hence on $A$ (since $\lambda(E) = 0$).
Also note that $f \in L_\infty(A)$. Indeed we can take $\varepsilon = 1$. Choose $m \in \mathbb{Z}^+$ so that $k, \ell \geq m \implies \|f_k - f_\ell\|_\infty \leq 1$. Then for all $n \geq m$, we have $\left|f_n(x) - f_m(x)\right| \leq \|f_n - f_m\|_\infty \leq 1\ \forall\ x \in A \setminus E$. Thus

$$\begin{aligned} \left|f_n(x)\right| &= \left|f_n(x) - f_m(x) + f_m(x)\right| \\ &\leq \left|f_n(x) - f_m(x)\right| + \left|f_m(x)\right| \\ &\leq 1 + \|f_m\|_\infty \\ \text{Hence } \left|f(x)\right| &= \left|\lim_{n\to\infty} f_n(x)\right| \\ &\leq 1 + \|f_m\|_\infty\ \forall\ x \in A \setminus E \\ \text{Hence } \left|f(x)\right| &\leq 1 + \|f_m\|_\infty\ \forall\ x \in A \\ \text{Hence } \|f\|_\infty &\leq 1 + \|f_m\|_\infty < \infty. \end{aligned}$$

Finally note that $f_n \to f$ in $L_\infty(A)$. That is $\|f_n - f\|_\infty \to 0$ as $n \to \infty$. Indeed since $f_n \to f$ uniformly in $A \setminus E$ given $\varepsilon > 0$ we can choose $m \in \mathbb{Z}^+$ so that $\forall\ n \in \mathbb{Z}^+\ \forall\ x \in A \setminus E,\ n \geq m \implies \left|f_n(x) - f(x)\right| < \varepsilon$. So $\forall\ n \in \mathbb{Z}^+$, if $n \geq m$ then $\forall\ x \in A \setminus E$, $\left|f_n(x) - f(x)\right| < \varepsilon \implies \|f_n - f\|_\infty \leq \varepsilon$. So we have $f_n \to f$ in $L_\infty(A)$.

**End of Lecture 21**

**Start of Lecture 22**

This chapter is in Section 13.8 in the textbook.

**Theorem 22.1:** Let $1 \leq p < q \leq \infty$ and let $A \subseteq \mathbb{R}$ be measurable. Then,

1. $\ell_q \subseteq \ell_p$ and,
2. if $\lambda(A) < \infty$ then $L_q(A) \subseteq L_p(A)$.

Note that if $f \in L_q(A)$ then $\|f\|_p \leq \lambda(A)^{\frac{1}{p}-\frac{1}{q}} \cdot \|f\|_q$.

Proof of 2.

Suppose $q < \infty$. Suppose $f \in L_q(A)$. Then

$$\begin{aligned} \|f\|_p^p &= \int_A |f|^p \\ &= \left\||f|^p \cdot 1\right\|_1 \\ &\leq \left\||f|^p\right\|_u \cdot \|1\|_v \text{ where } \frac{1}{u} + \frac{1}{v} = 1 \\ &= \left(\int_A |f|^{p\cdot u}\right)^{\frac{1}{u}} \left(\int_A 1^v\right)^{\frac{1}{v}}. \end{aligned}$$

Choose $u$ so $up = q$. take $u = \frac{q}{p}$ and we want $\frac{1}{v} = 1 - \frac{1}{u} = \frac{u-1}{u}$. So we choose

$$\begin{aligned} v = \frac{u}{u-1} &= \frac{\frac{q}{p}}{\frac{q}{p}-1} = \frac{q}{q-p} \text{ to get} \\ \|f\|_p^p &\leq \left(\int_A |f|^q\right)^{\frac{p}{q}} \lambda(A)^{\frac{q-p}{q}} \\ &= \|f\|_q^p \lambda(A)^{1-\frac{p}{q}} \\ \text{Hence } \|f\|_p &\leq \|f\|_q \cdot \lambda(A)^{\frac{1}{p}-\frac{1}{q}}. \end{aligned}$$

Suppose that $q = \infty$. Suppose $f \in L_\infty(A)$. Let $B = \{x \in A \mid |f(x)| > \|f\|_\infty\}$. So $\lambda(B) = 0$. Let $C = A \setminus B$. Then

$$\begin{aligned} \|f\|_p^p &= \int_A |f|^p \\ &= \int_B |f|^p + \int_C |f|^p \\ &= \int_C |f|^p \\ &\leq \int_C \|f\|_\infty^p \\ &= \|f\|_\infty^p \lambda(C) \\ &= \|f\|_\infty^p \lambda(A) \\ \text{and so } \|f\|_p &\leq \|f\|_\infty \lambda(A)^{\frac{1}{p}}. \end{aligned}$$

**End of Lecture 22**

**Start of Lecture 23**

**Theorem 23.1:** Let $1 \le p < q < r \le \infty$. Let $A \subseteq \mathbb{R}$ be measurable. Then,

1. $\ell_p \cap \ell_r \subseteq \ell_q \subseteq \ell_p + \ell_r$
2. $L_p(A) \cap L_r(A) \subseteq L_q(A) \subseteq L_p(A) + L_r(A)$.

Proof.

Note that Part 1 is trivial since

$$\ell_p \cap \ell_r \subseteq \ell_r \subseteq \ell_q \subseteq \ell_p \subseteq \ell_p + \ell_r.$$

We claim that $L_q(A) \subseteq L_p(A) + L_r(A)$. So every $f \in L_q(A)$ is equal to a sum $f = g + h$ with $g \in L_p(A)$ and $h \in L_r(A)$. To show this is true suppose $f \in L_q(A)$ and let $B = \left\{x \in A \;\middle|\; \left|f(x)\right| \ge 1\right\}$ and $C = A \setminus B = \{x \in A \mid \left|f(x)\right| < 1\}$. Let $g = f \cdot \mathcal{X}_B$ and $h = f \cdot \mathcal{X}_c$. Then $f = g + h$ and when $x \in B$ so $\left|f(x)\right| \ge 1$ and

$$\left|f(x)\right|^p \le \left|f(x)\right|^q \le \left|f(x)\right|^r$$

and when $x \in C$ so $\left|f(x)\right| < 1$ and

$$\left|f(x)\right|^p \ge \left|f(x)\right|^q \ge \left|f(x)\right|^r.$$

Thus

$$\|g\|_p^p = \int_A |g|^p = \int_A |f|^p \le \int_B |f|^q \le \int_A |f|^q = \|f\|_q^q < \infty.$$

Moreoever, $h \in L_r(A)$ since

$$\|h\|_r^r = \int_A |h|^r = \int_C |f|^r \le \int_C |f|^q \le \int_A |f|^q = \|f\|_q^q < \infty \text{ for } r < \infty$$

For $r = \infty$ we have $\|h\|_r = \|h\|_\infty \le 1$. Since $\left|h(x)\right| < 1$ for all $x \in A$. This proves our claim.
For the reverse inclusion we claim $L_p(A) \cap L_r(A) \subseteq L_q(A)$. Let $f \in L_p(A) \cap L_r(A)$. Then,

$$\|f\|_q^q = \int_A |f|^q = \left\||f|^q\right\|_1 \le \left\||f|^a\right\|_u \left\||f|^b\right\|_v \tag{$\star$}$$

by Hölder's inequality for any $a, b, u, v$ with $a + b = q$ and $1 \le u, v$ with $\dfrac{1}{u} + \dfrac{1}{v} = 1$. Thus, $\star$ becomes

$$\left(\int_A |f|^{au}\right)^{\frac{1}{u}} \left(\int_A |f|^{bv}\right)^{\frac{1}{v}}$$

If we choose $a, b, u, v$ so that $au = p$ and $bv = r$. Then we get

$$\|f\|_q^q \le \left(\int_A |f|^p\right)^{\frac{a}{p}} \left(\int_A |f|^r\right)^{\frac{b}{r}} = \|f\|_r^a \|f\|_r^b < \infty.$$

We need $a + b = q,\ au = p,\ bv = r,\ \dfrac{1}{u} + \dfrac{1}{v} = 1$. That is, $\dfrac{a}{p} + \dfrac{b}{r} = 1$. In matrix form, $\begin{pmatrix} 1 & 1 \\ \frac{1}{p} & \frac{1}{r} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} q \\ 1 \end{pmatrix}$.

Therefore,

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\frac{1}{r}-\frac{1}{p}} \begin{pmatrix} \frac{1}{r} & -1 \\ -\frac{1}{p} & 1 \end{pmatrix} \begin{pmatrix} q \\ 1 \end{pmatrix} = \frac{1}{\frac{1}{p}-\frac{1}{r}} \begin{pmatrix} -\frac{1}{r} & 1 \\ \frac{1}{p} & -1 \end{pmatrix} \begin{pmatrix} q \\ 1 \end{pmatrix} = \frac{pr}{r-p} \begin{pmatrix} 1-\frac{q}{r} \\ \frac{q}{p}-1 \end{pmatrix} = \begin{pmatrix} \frac{p(r-q)}{r-p} \\ \frac{r(q-p)}{r-p} \end{pmatrix}.$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{p}{a} \\ \frac{r}{b} \end{pmatrix} = \begin{pmatrix} \frac{r-p}{r-q} \\ \frac{r-p}{q-p} \end{pmatrix}$$

So $a, b > 0,\ u, v > 1$. In the case that $r < \infty$ we found that $\|f\|_q \leq \|f\|_p^{\frac{a}{q}} \|f\|_r^{\frac{b}{q}} = \|f\|_p^{\frac{p(r-q)}{q(r-p)}} \|f\|_r^{\frac{r(q-p)}{q(r-p)}}$. When $r = \infty$, let $B = \{x \in A \mid |f(x)| \leq \|f\|_\infty\},\ C = A \setminus B$. Then

$$\begin{aligned}
\|f\|_q^q &= \int_A |f|^q \\
&= \int_A |f|^p |f|^{q-p} \\
&= \int_B |f|^p |f|^{q-p} + \int_C |f|^p |f|^{q-p} \\
&= \int_B |f|^p |f|^{q-p} \text{ since } \lambda(C) = 0 \\
&\leq \int_B |f|^p \|f\|_\infty^{q-p} \\
&= \left(\int_B |f|^p\right) \|f\|_\infty^{q-p} \\
&\leq \left(\int_A |f|^p\right) \|f\|_\infty^{q-p} \\
&= \|f\|_p^p \|f\|_\infty^{q-p}.
\end{aligned}$$

So that

$$\|f\|_q \leq \|f\|_p^{\frac{p}{q}} \|f\|_\infty^{1-\frac{p}{q}}$$

**Theorem 23.2:**

1. $\ell_p$ is separable when $p < \infty$ but $\ell_\infty$ is not.
2. When $a < b$, $L_p([a, b])$ is separable but $L_\infty([a, b])$ is not.

**Exercise 23.3:** Prove Part 1 of this theorem.

Proof of Part 2.

Claim 1: The set of simple functions on $[a, b]$ is dense in $L_p([a, b])$.

Let $A = [a, b]$. Let $f \in L_p([a, b])$ and let $B = \{x \in A \mid f(x) \geq 0\}$ and $C = \{x \in A \mid f(x) < 0\}$. So that we have $f^+ = f\mathcal{X}_B,\ f^- = f\mathcal{X}_C$. For $n \in \mathbb{Z}^+$, define $S_n^+ : B \to \mathbb{R}$ by

$$S_n^+(x) = \begin{cases} \dfrac{k-1}{2^n} & \text{if } \dfrac{k-1}{2^n} \leq f^+(x) < \dfrac{k}{2^n} \text{ with } k \in \{1, 2, \ldots, n2^n\} \\ n & \text{if } f^+(x) \geq n \end{cases}$$

Then, $\lim_{n\to\infty} S_n^+(x) = f^+(x)\ \forall\, x \in B$. Apply the Lebesgue's Dominated Convergence Theorem to the sequence $\left|f^+ - S_n^+\right|^p$ to get

$$\begin{aligned} \lim_{n\to\infty} \left\|f^+ - S_n^+\right\|_p^p \lim_{n\to\infty} \int_B \left|f^+ - S_n^+\right|^p \\ &= \int_B \lim_{n\to\infty} \left|f^+ - S_n^+\right|^p \\ &= \int_B 0 \\ &= 0 \end{aligned}$$

So $\lim_{n\to\infty} S_n^+ = f^+$ in $L_p(B)$ or in $L_p(A)$ where $S^+(x) = f^+(x) = 0$ in $A \setminus B$. Similarly, construct simple functions $S_n^-$ on $C$ such that

$$\lim_{n\to\infty} \left\|f^- - S_n^-\right\|_p^p = \lim_{n\to\infty} \int_C \left|f^- - S_n^-\right|^p = 0.$$

Let $S_n = S_n^+ - S_n^-$. Then,

$$\begin{aligned} \|f - S_n\|_p^p = \int_A |f - S_n|^p = \int_B |f - S_n|^p + \int_C |f - S_n|^p = \int_B \left|f^+ - S_n^+\right|^p + \int_C \left|f^- - S_n^-\right|^p \\ = \left\|f^+ - S_n^+\right\|^p + \left\|f^- - S_n^-\right\|_p^p. \end{aligned}$$

Which goes to zero as $n \to \infty$. This proves Claim 1. The proof is continued in next lecture.

**End of Lecture 23**

**Start of Lecture 24**

Continuation of proof.

Claim 2: The set of step functions on $[a,b]$ is dense in the set of simple functions on $[a,b]$

To prove this claim we note that to approximate the simple function

$$s = \sum_{k=1}^{m} c_k \mathcal{X}_{A_k},$$

it suffices to approximate each characteristic function $\mathcal{X}_{A_k}$ by a step function since

$$\left\| \sum_{k=1}^{m} c_k \mathcal{X}_{A_k} - \sum_{k=1}^{m} c_k r_k \right\|_p = \left\| \sum_{k=1}^{m} c_k (\mathcal{X}_{A_k} - r_k) \right\|_p \leq \max |c_k| \sum_{k=1}^{m} \left\| \mathcal{X}_{A_k} - r_k \right\|_p$$

where each $r_k$ is a step function. When $A \subseteq [a,b]$ is measurable, we can approximate $\mathcal{X}_A$ as follows. Given $\varepsilon > 0$, we can choose open intervals $I_k$ so $A \subseteq \bigcup_{k=1}^{\infty} I_k$ and $\lambda(A) \leq \sum_{k=1}^{\infty} |I_k| \leq \lambda(A) + \varepsilon$ we can take the intervals $I_k$ to be disjoint. Note that since the intervals $I_k$ can be replaced by the connected components of the open set $\bigcup_{k=1}^{m} I_k$. Let $J_k = I_n \cap [a,b]$. So the $J_k$ are disjoint intervals with

$$A \subseteq \bigcup_{k=1}^{\infty} I_k \cap [a,b] = \bigcup_{\infty}^{k=1} J_k \text{ and } \bigcup_{k=1}^{\infty} \subseteq \bigcup_{\infty}^{k=1} I_k.$$

Thus, we have

$$\lambda \leq \sum_{k=1}^{\infty} |J_k| \leq \sum_{k=1}^{\infty} |I_k| \leq \lambda(A) + \varepsilon.$$

Choose $m \in \mathbb{Z}^+$ so that

$$\sum_{k=m+1}^{\infty} |J_k| < \varepsilon.$$

Then we approximate $\mathcal{X}_a$ by the step function

$$\mathcal{X}_{\substack{m \\ \bigcup J_k \\ k=1}} = \sum_{m}^{k=1} \mathcal{X}_{J_k}.$$

We have

$$\begin{aligned}
\left\| \mathcal{X} - \sum_{m}^{k=1} \mathcal{X}_{J_k} \right\|_p^p &= \int_{[a,b]} \left| \mathcal{X}_A - \sum_{k=1}^{m} \mathcal{X}_{J_k} \right|^p \\
&= \lambda\left( A \setminus \bigcup_{m}^{k=1} J_k \right) + \lambda\left( \bigcup_{k=1}^{m} J_k \setminus A \right)
\end{aligned}$$

Where $\lambda\Big(A \setminus \bigcup_{m}^{k=1} J_k\Big)$ is the value of the integral when inside integral takes 1 and $\lambda\big(\bigcup_{k=1}^{m} J_k \setminus A\big)$ is the value of the integral when inside integral takes -1. Then,

$$\begin{aligned}
\left\|\mathcal{X} - \sum_{m}^{k=1} \mathcal{X}_{J_k}\right\|_p^p &\leq \lambda\left(\bigcup_{k=1}^{\infty} J_k \setminus \bigcup_{k=1}^{m} J_k\right) + \lambda\left(\bigcup_{k=1}^{\infty} J_k \setminus A\right) \\
&= \sum_{k=m+1}^{\infty} |J_k| + \lambda\left(\bigcup_{k=1}^{\infty} J_k\right) - \lambda(A) \\
&\leq \varepsilon + \varepsilon = 2\varepsilon
\end{aligned}$$

This proves claim 2.

<u>Claim 3:</u> Every step function on $[a,b]$ can be approximated arbitrarily closely by a continuous function on $[a,b]$ in $L_p[a,b]$.

---

To approximate

$$s = \sum_{k=1}^{m} c_k \mathcal{X}_{I_k}$$

where the $I_k$ are disjoint intervals in $[a,b]$ it suffices to approximate each $\mathcal{X}_{I_k}$ by a continuous function $f_k$ (as above). We can approximate $\mathcal{X}_I$ where $I = (c,d), [c,d), (c,d]$ or $[c,d]$ with $a \leq c \leq d \leq b$ using the picture above or

$$f(x) = \begin{cases} 0 & \text{for } a \leq x \leq c \\ m(x-c) & \text{for } c \leq x \leq c + \frac{1}{m} \\ 1 & \text{for } c + \frac{1}{m} \leq x \leq d - \frac{1}{m} \\ m(d-x) & \text{for } d - \frac{1}{m} \leq x \leq d \\ 0 & \text{for } d \leq x \leq b \end{cases}$$

So that,

$$\|\mathcal{X}_I - f\|_p^p = \int_I |\mathcal{X}_I - f|^p \leq \frac{2}{m}$$

which goes to zero as $m \to \infty$. More rigorously we have

$$\begin{aligned}
\|\mathcal{X}_I - f\|_p^p &= \int_I |\mathcal{X}_I - f|^p \\
&= 2\int_0^{\frac{1}{m}} (mx)^p \\
&= \frac{2m}{p+1}\left[x^{p+1}\right]_0^{\frac{1}{m}} \\
&= \frac{2m}{(p+1)m^{p+1}}
\end{aligned}$$

which goes to zero as $m \to \infty$. It follows that the set $C[a,b]$ of all continuous functions on $[a,b]$ is dense in $L_p([a,b])$.

Moreover, the set of polynomial functions $\mathbb{R}[x]$ on $[a,b]$ is dense in $C[a,b]$ in $L_\infty([a,b])$. This is by Weierstrass approximation theorem. [As an exercise prove this statement.] Also note that $\mathbb{Q}[x]$ is dense in $\mathbb{R}[x]$ in

$L_\infty([a,b])$ because for $c_k \in \mathbb{R}, r_k \in \mathbb{Q}$ we have

$$\left\| \sum_{\ell}^{k=1} c_k x^k - \sum_{\ell}^{k=1} r_k x^k \right\|_\infty = \left\| \sum_{k=1}^{\ell} (c_k - r_k) x^k \right\|_\infty \leq m \cdot M$$

$$\text{where } m = \max_{1 \leq k \leq \ell} |c_k - r_k| \text{ and } M = \{1, |a|^\ell, |b|^\ell\}$$

which goes to zero as $m \to \infty$. Since $\mathbb{Q}[x]$ is dense in $C[a,b]$ in $L_\infty$ it is also dense in $C[a,b]$ in $L_p([a,b])$ because $L_\infty([a,b] \subseteq L_p([a,b]))$ with $\|f\|_p \leq \|f\|_\infty (b-a)^{\frac{1}{p}}$.Rest of the proof will be included later.

**Remark 24.1:**

1. The set of simple functions is dense in $L_\infty[a,b]$
2. The set of step functions is not dense in the set of simple functions.
3. $C[a,b]$ is not dense in $L_\infty[a,b]$.

**End of Lecture 24**

**Start of Lecture 25**

The chapter Hilbert Space is covered in chapter 14 in the textbook.

**Definition 25.1:** For a field $\mathbb{F}$ and a vector space $V$ (which is over $\mathbb{F}$), we define **inner product** as a map $\langle \cdot, \cdot \rangle : V \times V \to F$ that satisfies the following properties:

1. (Positive definiteness) $\langle x, x \rangle \geq 0$ with $\langle x, x \rangle = 0 \iff x = \mathbf{0}_V \quad \forall\, x \in V$.
2. (Conjugate symmetrical) $\langle x, y \rangle = \overline{\langle y, x \rangle} \quad \forall\, x, y \in V$.
3. (Sesquilinear form) $\langle a_1 x + a_2 x_2, y \rangle = a_1 \langle x_1, y \rangle + a_2 \langle x_2, y \rangle$,
   $\langle x, b_1 y_1 + b_2 y_2 \rangle = \overline{b_1} \langle x, y_1 \rangle + \overline{b_2} \langle x, y_2 \rangle \quad \forall\, x_i, y_i \in V$ and $a_i, b_i \in \mathbb{F}$ where $i = 1, 2$.

**Remark 25.2:** Inner products satisfy Cauchy-Schwarz inequality.

$$\big|\langle x, y \rangle\big| = \|x\|\|y\| \quad \forall\, x, y \in V.$$

**End of Lecture 25**

**Start of Lecture 26**

**Theorem 26.1:**

1. $\ell_2 = \ell_2(\mathbb{C}) = \left\{ x = (x_1, x_2, \ldots, x_k) \in \mathbb{C} \;\middle|\; \sum_{k=1}^{\infty} |x_k| < \infty \right\}$ is a complex Hilbert space.
2. For $A \subseteq \mathbb{R}$ measurable and $f : A \subseteq \mathbb{R} \to \mathbb{C}$ given by $f = u + iv$ with $u, v : A \subseteq \mathbb{R} \to \mathbb{R}$, we say that $f$ is measurable when $u$ and $v$ are both measurable and we say that $f$ is (Lebesgue) integrable when $u$ and $v$ are both Lebesgue integrable and in this case $\int_A f = \int_A u + i \int_A v$. We define
$$L_p(A, \mathbb{C}) = \left\{ \text{measurable } f : A \to \mathbb{C} \;\middle|\; \int_A |f|^p < \infty \right\}$$
where $f \sim g \iff f = g$ a.e in $A$ with the given definitions above.

$L_2(A, \mathbb{C})$ is a complex Hilbert space using the inner product $\langle f, g \rangle = \int_A f\overline{g}$.

Proof of 1.

We skip most of the proof. Let us verify that when $f, g \in L_2(A, \mathbb{C})$ we have $\langle f, g \rangle = \int_A f\overline{g} \in \mathbb{C}$. Recall that for $f, g \in L_2(A, \mathbb{R})$ we showed that $\langle f, g \rangle = \int_A fg \in \mathbb{R}$. Indeed for $a, b \in \mathbb{R}$, we have $|ab| \leq \frac{1}{2}(|a|^2 + |b|^2)$. So
$$\left| \int_A fg \right| \leq \int_A |fg| \leq \int_A \frac{1}{2}(|f|^2 + |g|^2) = \frac{1}{2}(\|f\|_2^2 + \|g\|_2^2).$$
If we write $f = u + iv,\ g = p + iq$ then
$$\|f\|_2^2 = \int_A |f|^2 = \int_A |u|^2 + |v|^2.$$
We also have
$$\langle f, g \rangle = \int_A f\overline{g} = \int_A (u + iv)(p - iq) = \int_A up + vq + i \int_v p - uq \in \mathbb{C}.$$
Also, let us show that $L_2(A, \mathbb{C})$ is complete. Indeed, for $f_n = u_n + iv_n,\ f = u + iv$. $\{f_n\}$ is Cauchy if and only if $\{u_n\}$ and $\{v_n\}$ are both Cauchy and $f_n \to f$ in $L_2(A, \mathbb{C})$ if and only if $u_n \to u$ and $v_n \to v$ in $L_2(A, \mathbb{R})$.

Proof of 2.

Suppose $\{f_n\}$ is Cauchy in $L_2(A, \mathbb{C})$ given $\varepsilon > 0$ choose $m \in \mathbb{Z}^+$ so $k, \ell \geq m \implies \|f_k - f_\ell\| \leq \varepsilon$. Using the same $m \in \mathbb{Z}^+$,
$$k, \ell \geq m \implies \|u_k - u_\ell\|_2 \leq \|f_k - f_\ell\|_2 \leq \varepsilon$$
since $\|u\|^2 \leq \|u\|_2^2 + \|v\|_2^2 = \|f\|_2^2$ when $f = u + iv$. Thus, $\{u_n\}$ is Cauchy. Similarly, $\{v_n\}$ is Cauchy. Suppose $\{u_n\}$ and $\{v_n\}$ are Cauchy in $L_2(A, \mathbb{R})$. Given $\varepsilon > 0$ we choose $m \in \mathbb{Z}^+$ so that
$$k, \ell \geq m \implies \|u_k - u_\ell\| \leq \frac{\varepsilon}{\sqrt{2}} \text{ and } \|v_k - v_\ell\| \leq \frac{\varepsilon}{\sqrt{2}}.$$
Then using the same $m \in \mathbb{Z}^+$,
$$k, \ell \geq m \implies \|f_k - f_\ell\|_2^2 = \|u_k - u_\ell\|_2^2 + \|v_k - v_\ell\|_2^2 \leq \frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2} = \varepsilon^2 \implies \|f_k - f_\ell\|_2 \leq \varepsilon.$$

**Example 26.2:** If $V$ is a countable dimensional inner product space and $U \subseteq V$ is a subspace then if $U$ is finite dimensional then any orthonormal basis for $U$ can be extended by the Gram-Schmidt procedure to an orthonormal basis for $V$. When $U$ is countable dimensional, the procedure can break down.
For example, if

$$V = \mathbb{R}^\infty = \{x = (x_1, x_2, \ldots) \mid \exists\, n \in \mathbb{Z}^+ \;\forall\, k \geq n,\; x_k = 0\} \text{ with inner product} \langle x, y\rangle = \sum_{k=1}^{\infty} x_k y_k$$

and if

$$U = \left\{ x \in V \;\middle|\; \sum_{k=1}^{\infty} x_k = 0 \right\}.$$

Then $U$ has a basis $\mathcal{U}$ such that

$$\mathcal{U} = \{u_k \mid k \geq 2\} \text{ where } u_k = e_1 - e_k = (1, 0, \ldots, 0, -1, 0, \ldots)$$

and $\mathcal{U}$ can be extended to the basis $\mathcal{V} = \mathcal{U} \cup \{e_1\}$ for $V$. But $U^\perp = \{0\}$ since

$$\begin{aligned}
U^\perp &= \{x \in V \mid \langle x, u\rangle = 0 \forall\, u \in U\} \\
&= \{x \in V \mid \langle x, u\rangle = 0 \forall\, u \in \mathcal{U}\} \\
&= \{x \in V \mid \langle x, u_k\rangle = 0 \forall\, k \geq 2\} \\
&= \{x \in V \mid x_1 - x_k = 0 \forall\, k \geq 2\} \\
&= \{x = (x_1, x_2, \ldots) \in V \mid x_1 = x_2 = \ldots\} \\
&= \{0\}.
\end{aligned}$$

Note that $U \oplus U^\perp \neq U$.

**End of Lecture 26**

**Start of Lecture 27**

Hilbert spaces

**Example 27.1:**

$$W = \mathbb{R}^\infty = \left\{ x = (x_1, x_2, \dots) \;\middle|\; \begin{array}{c} \text{each } x_k \in \mathbb{R}, \exists\, n \in \mathbb{Z}^+ \\ \forall\, k \geq n,\; x_k = 0. \end{array} \right\}$$

$$\langle x, y \rangle = \sum_{k=1}^{\infty} x_k y_k$$

$$U = \left\{ x \in \mathbb{R}^\infty \;\middle|\; \sum_{k=1}^{\infty} x_k = 0 \right\}$$

$U$ has a basis $\mathcal{U} = \{u_1, u_2, \dots, \}$ where $u_k = e_1 - e_k = (1, 0, \dots, 0, -1, 0, \dots)$. Note that $U^\perp = \{0\}$ because for $x \in V = \mathbb{R}^\infty$,

$$\begin{aligned} x \in U^\perp &\implies x \cdot u = 0 \quad \forall\, u \in U \\ &\implies x \cdot u_k = 0 \quad \forall\, k \geq 2 \\ &\implies x_1 - x_k = 0 \quad \forall\, k \geq 2 \\ &\implies x_1 = x_2 = x_3 = \dots \\ &\implies x = 0 \text{ (since } \exists\, n x_n = 0) \end{aligned}$$

Note that $W \neq U \oplus U^\perp$, which means it is not the case that every $w \in W$ can be written uniquely as $w = u + v$ with $u \in U$, and $v \in U^\perp$. Given $w \in W$ there does not exist a (unique) nearest point $u \in U$ to $w$. For example, when $w = e_1$, there is no nearest point $u \in U$ to $w = e_1$:

For $u = e_1 - \sum_{k=1}^{n} \frac{1}{e} e_k = \left(1 - \frac{1}{n}, -\frac{1}{n}, -\frac{1}{n}, \dots, -\frac{1}{n}, 0, 0, \dots\right) = \sum_{k=1}^{\infty} u_k = 1 - n\frac{1}{n} = 0$. So $u \in U$. We also have

$$\|e_1 - u\|^2 = \left\| \left( \frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}, 0, 0, \dots \right) \right\| = n \cdot \frac{1}{n^2} = \frac{1}{n}$$

which goes to zero as $n \to \infty$.

**Theorem 27.2:** Let $H$ be a real or complex Hilbert space and $S \subseteq H$ be convex and closed. Then for every $a \in H$ there exists a unique point $b \in S$ such that $\|b - a\| \leq \|x - a\|$ for all $x \in S$.

Proof.

Recall that if for all $b, c \in S$ we have $b + t(c - b) \in S$ for all $t \in [0, 1]$, $S$ is **convex**.

When $W$ is an inner product space and $\|x\| = \sqrt{\langle x, x \rangle} \quad \forall\, x \in W$, over $\mathbb{R}$ we have,

$$\|x + y\|^2 = \|x\|^2 + 2\langle x, y \rangle + \|y\|^2$$
$$\|x - y\|^2 = \|x\|^2 - 2\langle x, y \rangle + \|y\|^2$$
$$\text{Polarization: } \langle x, y \rangle = \frac{1}{2}\left( \|x\|^2 + \|y\|^2 - \|x + y\|^2 \right)$$
$$\langle x, y \rangle = \frac{1}{4}\left( \|x + y\|^2 - \|x - y\|^2 \right)$$
$$\text{Pythagoras: } \|x + y\|^2 = \|x\|^2 + \|y\|^2 \iff \langle x, y \rangle = 0.$$

Over $\mathbb{C}$ we have

$$\begin{aligned}
\|x+y\|^2 &= \|x\| + 2\operatorname{Re}\{\langle x,y\rangle\} + \|y\| \\
\|x-y\|^2 &= \|x\| - 2\operatorname{Re}\{\langle x,y\}\rangle + \|y\| \\
\text{Polarization: } \langle x,y\rangle &= \frac{1}{8}\Big(\|x+y\|^2 + i\|x+iy\|^2 - \|x-y\|^2 - i\|x-iy\|^2\Big) \\
\text{Parallelogram law: } \|x+y\|^2 - \|x-y\|^2 &= 2(\|x\|^2 + \|y\|^2).
\end{aligned}$$

Let $\varnothing \neq S \subseteq H$ be closed and convex. Let $a \in H$. Translate by $-a$ and replace $a$ by 0. We need to show that there exists a unique point $b \in S$ with $\|b\| \leq \|x\|$ for all $x \in S$. Let $d = \operatorname{dist}(a,S)$. Then

$$d = \operatorname{dist}(a,S) = \inf\{\|a-x\| \ \mid \ x \in S\} = \inf\{\|x\| \ \mid \ x \in S\} \text{ since } a = 0.$$

Choose $x_n \in S$ so that $\|x_n\| \to d$ as $n \to \infty$. We claim that $\{x_n\}$ is Cauchy. Indeed

$$\begin{aligned}
\|x_k - x_\ell\|^2 + \|x_k + x_\ell\|^2 &= 2\|x_k\|^2 + 2\|x_\ell\|^2 \\
\|x_k - x_\ell\|^2 &= 2\|x_k\|^2 + 2\|x_\ell\|^2 - \left\|\frac{x_k + x_\ell}{2}\right\|^2.
\end{aligned}$$

So given $\varepsilon > 0$ we can choose $n \in \mathbb{Z}^+$ so $k,\ell \geq n \implies \|x_k\|^2 < d^2 + \varepsilon$, $\|x_\ell\|^2 < d^2 + \varepsilon$. Then for $k,\ell \geq n$,

$$\|x_k - x_\ell\|^2 < 2(d^2+\varepsilon) + 2(d^2+\varepsilon) - 4d^2 = 4\varepsilon.$$

Since $\frac{x_k+x_\ell}{2}$ then $\left\|\frac{x_k+x_\ell}{2}\right\| \geq d$. Since $H$ is complete, $\{x_n\}$ converges in $H$. Say $b = \lim\limits_{n\to\infty} x_n$ and since $S$ is closed, then $b \in S$. Since $x_n \to b$ we have $\|x_n\| \to \|b\|$. So $b = d = \operatorname{dist}(a=0,S)$. Finally, note that $b \in S$ is unique because if $b,c \in S$ with $\|b\| = \|c\| = d$. Then,

$$\begin{aligned}
\|b-c\|^2 &= 2\|b\| + 2\|c\|^2 - 4\left\|\frac{b+c}{2}\right\|^2 \\
&\leq 2d^2 + 2d^2 - 4d^2 \\
&= 0 \text{ since } \frac{b+c}{2} \in S \text{ so } \left\|\frac{b+c}{2}\right\| \geq d.
\end{aligned}$$

**End of Lecture 27**

**Start of Lecture 28**

A corollary to Theorem 27.2 is the following corollary.

**Corollary 28.1:** If $H$ is a Hilbert space and $U \subseteq H$ is a closed subspace, then

1. $\forall\, w \in H,\ \exists! u \in U$ such that $u$ is nearest to $w$.
2. $H = U \oplus U^{\perp}$, so for every $w \in H$ $\exists!\ u, v$ with $u \in U,\ v \in V,\ u + v = w$.

Moreover, when $w = u + v$ with $u \in U,\ v \in U^{\perp},\ u + v = w$ the point $u$ is the unique nearest point to $w$ and, in this case, we write $u = \mathrm{proj}_U(w)$ and $v = \mathrm{proj}_{U^{\perp}}(w)$

Proof.

Let $U$ be a closed subspace of a Hilbert space $H$. Note that $U$ is convex. Given $x \in H$, let $u \in U$ be the unique nearest point and let $v = x - u$. We claim that $v \in U^{\perp}$. Suppose, for contradiction, this is false. Choose $u' \in U$ such that $\langle v, u' \rangle \neq 0$. We may assume $\langle v, u' \rangle > 0$ (if not replace $u'$ by $e^{i\theta} u'$ for some $\theta$). Then $\|x - u\| = \|v\|$ and for $t \in \mathbb{R}$,

$$\begin{aligned}
\left\|x - (u + tu')\right\|^2 &= \left\|u + v - u - tu'\right\|^2 \\
&= \left\|v - tu'\right\|^2 \\
&= \|v\|^2 - 2t \,\mathrm{Re}\langle v, u' \rangle + t^2 \left\|u'\right\|^2 \\
&= \|v\|^2 - 2t \langle v, u' \rangle + t^2 \left\|u'\right\|^2 \\
&< \|v\|^2 \text{ for small } t > 0.
\end{aligned}$$

So for small $t > 0$ we get

$$\left\|x - (u + tu')\right\| < \|x - u\|$$

which contradicts the fact that $u$ is the point in $U$ nearest to $x$.

This proves existence. To prove uniqueness let $x \in H,\ u \in U,\ v \in U^{\perp}$ with $u + v = x$. We claim that $u$ is the point in $U$ nearest to $x$. Let $u' = U$ with $u' \neq u$. Then $u' - u \in U$. So $\langle v, u' - u \rangle = 0$. That is, $\langle x - u, u' - u \rangle = 0$. So,

$$\begin{aligned}
\left\|x - u'\right\|^2 &= \left\|(x - u) + (u - u')\right\|^2 \\
&= \|x - u\|^2 + 2 \,\mathrm{Re}\langle x - u, u - u' \rangle + \left\|u - u'\right\|^2 \\
&= \|x - u\|^2 + \left\|u - u'\right\|^2 \\
&> \|x - u\|^2 \text{ since } u \neq u'
\end{aligned}$$

**Remark 28.2:** In any inner product space $W$ there exists a maximal orthonormal set by Zorn's lemma.

When $W$ is finite dimensional, any maximal orthonormal set is a (Hamel) basis.

**Theorem 28.3:** Let $W$ be an inner product space, let $\mathcal{U} = \{u_1, u_2, \ldots, u_n\}$ be an orthonormal set, and let $U = \mathrm{span}_{\mathbb{F}} \mathcal{U}$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Then for $x \in U$ we have

$$x = \sum_{k=1}^{n} \langle x, u_k \rangle u_k \text{ and } \|x\|^2 = \sum_{k=1}^{n} \left\|\langle x, u_k \rangle\right\|^2$$

ad for $x \in W$, we have

$$\operatorname{proj}_U x = \sum_{k=1}^{n} \langle x, u_k \rangle u_k \text{ with } \|x\|^2 \geq \sum_{k=1}^{n} \left|\langle x, u_k \rangle\right|^2.$$

Proof.

Suppose $x \in U$, then $x = \sum_{k=1}^{n} c_k u_k$. Then,

$$\begin{aligned} \langle x, u_\ell \rangle &= \left\langle \sum c_k u_k, u_\ell \right\rangle \\ &= \sum c_k \langle u_k, u_\ell \rangle && \text{(by linearity)} \\ &= \sum c_k \delta_{k\ell} \\ &= c_\ell \end{aligned}$$

We also have

$$\begin{aligned} \|x\|^2 &= \left\langle \sum_k \langle x, u_k \rangle u_k, \sum_\ell \langle x, u_\ell \rangle u_\ell \right\rangle \\ &= \sum_{k,\ell} \langle x, u_k \rangle \overline{\langle x, u_\ell \rangle} \langle u_k, u_\ell \rangle \\ &= \sum_{k,\ell} \langle x, u_k \rangle \overline{\langle x, u_\ell \rangle} \delta_{k,\ell} \\ &= \sum_k \langle x, u_k \rangle \overline{\langle x, u_k \rangle} \\ &= \sum_k \left|\langle x, u_k \rangle\right|^2 \end{aligned}$$

**Exercise 28.4:** For $u = \sum_{k=1}^{n} \langle x, u_k \rangle u_k \in U$. let $v = x - u$. Verify that $v \in U^\perp$.

**Theorem 28.5:** Let $H$ be a Hilbert space and let $\mathcal{U}$ be a maximal orthonormal set and let $U = \operatorname{span}_{\mathbb{F}} \mathcal{U}$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Then $U$ is dense in $H$.

Proof.

Note that $\overline{U}$ is a closed vector space in $H$ ($\overline{U}$ is a vector space because if $x_n \to a$ and $y_n \to b$ in $H$ then $x_n + y_n \to a + b$ and $cx_n \to ca$). It follows that $H = \overline{U} \oplus \overline{U}^\perp$. But $\overline{U}^\perp = \{0\}$ because if $0 \neq v \in \overline{U}^\perp$ with $|v| = 1$ then $\mathcal{U} \cup \{v\}$ would be a larger orthonormal basis ($v \neq \mathcal{U}$ since if $v \in \mathcal{U}$ then $|v|^2 = \langle v, v \rangle = 0$ ). Thus,

$$H = \overline{U} \oplus \overline{U}^\perp = \overline{U} \oplus \{0\} = \overline{U}.$$

**Theorem 28.6:** Let $H$ be a Hilbert space and let $\mathcal{U}$ be a maximal orthonormal set. Then $H$ is separable (meaning that it has a countable dense subset) if and only if $\mathcal{U}$ is at most countable.

Proof.

Suppose $\mathcal{U}$ is uncountable. Let $S \subseteq H$ be a dense set in $H$. For each $u \in \mathcal{U}$, choose $s_k \in S$ with $|s_k - u| < \frac{\sqrt{2}}{2}$.

Then for $u, v \in \mathcal{U}$ with $u \neq v$,

$$\|u-v\|^2 = \|u\|^2 - 2\operatorname{Re}\langle u, v\rangle + \|v\|^2 = \|u\|^2 + \|v\|^2 = 1 + 1 = 2.$$

So, $\|u - v\| = \sqrt{2}$ and so

$$\|s_u - s_v\| = \|s_u - u + u - v + v - s_v\| \geq \|u - v\| - \big(\|s_u - u\| + \|v - s_v\|\big) > \sqrt{2} - \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) = 0.$$

hence, $s_u \neq s_v$. Then $H$ is not separable. Hence, $S$ is uncountable. Suppose $\mathcal{U}$ is countable. Let $U = \operatorname{span}_{\mathbb{F}} \mathcal{U}$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. We know that $\mathcal{U}$ is dense in $H$. When $\mathbb{F} = \mathbb{R}$, $\operatorname{span}_{\mathbb{Q}} \mathcal{U}$ is dense in $U = \operatorname{span}_{\mathbb{R}} \mathcal{U}$ and when $\mathbb{F} = \mathbb{C}$ then $\operatorname{span}_{\mathbb{Q}[i]} \mathcal{U}$ is dense in $U = \operatorname{span}_{\mathbb{C}} \mathcal{U}$ where $\mathbb{Q}[i] = \{a + ib \mid a, b \in \mathbb{Q}\}$ because if $c_1, c_2, \ldots, c_n \in \mathbb{C}$ and $r_1, r_2, \ldots, r_n \in \mathbb{Q}[i]$ then

$$\begin{aligned}
\left\|\sum_{k=1}^n c_k u_k - \sum_{k=1}^n r_k u_k\right\| &= \left\|\sum_{k=1}^n (c_k - r_k) u_k\right\| \\
&\leq \sum_{k=1}^n \big\|(c_k - r_k) u_k\big\| \\
&= \sum_{k=1}^n |c_k - r_k| \|u_k\| \\
&= \sum_{n}^{k=1} |c_k - r_k|
\end{aligned}$$

When $\mathcal{U}$ is countable, so is $\operatorname{span}_{\mathbb{Q}[i]} \mathcal{U}$.

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**Remark 28.7:** The map $F : \bigcup_{n=0}^{\infty} \mathbb{Q}[i]^n \to \operatorname{span}_{\mathbb{Q}[i]} \mathcal{U}$.

$F((r_1, r_2, \ldots, r_n)) = \sum_{k=1}^n r_k u_k$ is surjective (even bijective).

**End of Lecture 28**

**Start of Lecture 29**

Solutions for the midterm are posted on course website. Today's notes aren't fully complete. It will be completed at a later date.

**Theorem 29.1:** Let $H$ be an inner product space. Let $\mathcal{U} = \{u_1, u_2, \dots, u_n\}$ be a finite orthonormal set then,

$$\text{for } x \in U = \operatorname{span}_{\mathbb{F}} \mathcal{U}$$
$$x = \sum_{k=1}^{n} \langle x, u_k \rangle u_k \text{ and } \|x\|^2 = \sum_{k=1}^{n} \big\|\langle x, u_k \rangle\big\|^2$$
$$\text{where } U \text{ is closed and } H = U \oplus U^{\perp}$$

For $x \in H$,

$$\operatorname{proj}_U x = \sum_{k=1}^{n} \langle x, u_k \rangle u_k \text{ and } \|x\|^2 \geq \|\operatorname{proj}_U x\|^2 = \sum_{k=1}^{n} \big|\langle x, u_k \rangle\big|^2$$

The inequality $\|x\|^2 \geq \sum_{k=1}^{n} \big|\langle x, u_k \rangle\big|^2$ is called the **Bessel's inequality**.

**Exercise 29.2:** Prove this theorem.

**Definition 29.3:** When the statements in Theorem 29.4 (which are all equivalent) holds, we say that $\mathcal{U}$ is a **Hilbert basis** for $H$.

**Theorem 29.4:** Let $H$ be a separable Hilbert space. Let $\mathcal{U} = \{u_1, u_2, \dots\}$ be a countable orthonormal set in $H$. Let $U = \operatorname{span}_{\mathbb{F}} \mathcal{U}$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Then the following are equivalent:

1. $\mathcal{U}$ is maximal.
2. $U$ is dense in $H$.
3. For all $x \in H$, $x = \sum_{k=1}^{\infty} \langle x, u_k \rangle u_k = \lim_{n\to\infty} \sum_{k=1}^{n} \langle x, u_k \rangle u_k$ in $H$.
4. For all $x \in H$, $\|x\|^2 = \sum_{k=1}^{\infty} \big|\langle x, u_k \rangle\big|^2 = \lim_{n\to\infty} \sum_{k=1}^{n} \big|\langle x, u_k \rangle\big|^2$ in $\mathbb{R}$.
5. For all $x, y \in H$, $\langle x, y \rangle = \sum_{k=1}^{\infty} \langle x, u_k \rangle \overline{\langle y, u_k \rangle}$ in $\mathbb{C}$.

The inequality $\|x\|^2 = \sum_{k=1}^{n} \big|\langle x, u_k \rangle\big|^2$ is called the **Parseval's inequality**.

$1 \implies 2.$

This was already proven.

$2 \implies 1.$

Suppose $\mathcal{U}$ is not maximal. Then we can choose $v$ with $\langle , u_k \rangle = 0$ for all $k$ and $\|v\| = 1$. Then, $\langle v, x \rangle = 0$ for all $x \in U$. Note that if $x = \sum_{k=1}^{n} c_k u_k$ then $\langle v, x \rangle = \sum_{k=1}^{n} \overline{c_k} \langle v, u_k \rangle = 0$. So that $v \in U^{\perp}$. We cannot find $u \in U$

with $\|u-v\|<1$ because for all $u \in U$

$$\begin{aligned}\|u-v\|^2 &= \|u\|^2 - 2\operatorname{Re}\langle u, v\rangle + \|v\|^2 \\ &= \|u\|^2 + \|v\|^2 \\ &\geq \|v\|^2 = 1\end{aligned}$$

$2 \implies 3$ .

Suppose $U = \operatorname{span}_{\mathbb{F}} \mathcal{U}$ is dense in $H$. Let $x \in H$. We need to show that $x = \sum_{k=1}^{\infty} \langle x, u_k\rangle u_k$. Let $\varepsilon > 0$. Choose $u \in U$ with $\|u - x\| < \varepsilon$. Say $u = \sum_{k=1}^{m} c_k u_k$. Let $n \geq m$. Then $u \in \operatorname{span}_{\mathbb{F}}\{u_1, u_2, \ldots, u_n\}$. Since $\sum_{k=1}^{n} \langle x, u_k\rangle u_k$ is the point in $U_n = \operatorname{span}_{\mathbb{F}}\{u_1, u_2, \ldots, u_n\}$ to $x$ and $u \in U_n$, we have

$$\left\| x - \sum_{k=1}^{n} \langle x, u_k\rangle \right\| \leq \|x - u\| < \varepsilon.$$

Given $\varepsilon > 0$ we chose $m \in \mathbb{Z}^+$ so that for all $n \geq m$ we have $\left\| x - \sum_{k=1}^{n} \langle x, u_k\rangle u_k \right\| < \varepsilon$.

Hence we get $\lim_{n\to\infty} \sum_{k=1}^{n} \langle x, u_k\rangle u_k = x$ in $H$ as required.

$3 \implies 4$.

We suppose that for all $x \in H$, $x = \sum_{k=1}^{\infty} \langle x, u_k\rangle u_k$. We need to show that for all $x \in H$ we have $\|x\|^2 = \sum_{k=1}^{\infty} \left|\langle x, u_k\rangle\right|^2$. Write $w_n = \sum_{k=1}^{n} \langle x, u_k\rangle u_k$. Then we have $\|x - w_n\| \to 0$ and, by triangle inequality

$$\big|\|x\| - \|w_n\|\big| \leq \|x - w_n\| \to 0.$$

Thus $\|w_n\| \to \|x\|$. Thus $\|w_n\|^2 \to \|x\|^2$. We also have

$$\begin{aligned}\|w_n\|^2 &= \left\langle \sum_{k=1}^{n} \langle x, u_k\rangle u_k, \sum_{\ell=1}^{n} \langle x, u_\ell\rangle u_\ell \right\rangle \\ &= \sum_{k,\ell} \langle x, u_k\rangle \overline{\langle x, u_\ell\rangle} \delta_{k,\ell} \\ &= \sum_{k} \left|\langle x, u_k\rangle\right|^2\end{aligned}$$

**End of Lecture 29**

**Start of Lecture 30**

From last time

thm: Let $H$ be a separable Hilbert space with countable Hilbert basis $\mathcal{U} = \{u_1, u_2, \ldots\}$

1. If $x \in \sum_{k=1}^{\infty} a_k u_k = \sum_{k=1}^{\infty} b_k u_k$ in $H$ then $a_k = b_k = \langle x, u_k \rangle$.

2. For $c_1, c_2, \ldots \in F$, $\sum_{k=1}^{\infty} c_k u_k$ converges in $H$ if and only if $\sum |c_k|^2$ converges in $\mathbb{R}$.

3. The map $\phi : H \to \ell_2$ given by

$$\phi\left(\sum_{k=1}^{\infty} c_k u_k\right) = (c_1, c_2, \ldots) \text{ or by } \phi(x) = (\langle x, u_1 \rangle, \langle x, u_2 \rangle, \ldots)$$

is an isomorphism of inner product spaces.

**End of Lecture 30**

**Start of Lecture 31**

A $2\pi-$periodic function $f : \mathbb{R} \to [-\infty, \infty]$ determines and is determined by a function $f : [0, 2\pi) \to [-\infty, \infty]$ or by a function $f : [-\pi, \pi) \to [-\infty, \infty]$ or by a function $f : [-\infty, \infty] \to [-\infty, \infty]$ with $f(-\pi) = f(\pi)$ or by a function $f : T \to [-\infty, \infty]$ where $T = \mathbb{R}/2\pi\mathbb{Z}$ ($x \sim y \iff x - y \in 2\pi\mathbb{Z}$).

Also when such a function $f : \mathbb{R} \to [-\infty, \infty]$ satisfies $\int_{-\pi}^{\pi} |f|^p < \infty$, it determines and is determined a.e in $\mathbb{R}$ (or in $[-\pi, \pi]$) by an element $f \in L_p[-\pi, \pi]$. We shall write $L_p[-\pi, \pi]$ or $L_p(T)$ for the set of such periodic functions with $f = g$ in $L_p[-\pi, \pi]$ when $f = g$ a.e in $\mathbb{R}$ (or when $f = g$ a.e in $[-\pi, \pi]$).

**Definition 31.1:** A (real) **trigonometric polynomial** is a function of the form

$$f(x) = a_0 + \sum_{n=1}^{m} a_n \cos nx + \sum_{n=1}^{m} b_n \sin nx.$$

**Remark 31.2:** By the Stone-Weierstrass theorem the trigonometric polynomials are dense in the space $C(T)$ of continuous functions $f : T \to \mathbb{R}$ using the supremum norm $\|f\|_\infty$ but not in $C[-\pi, \pi]$. Note that since the trigonometric polynomials form a sub-algebra of $C(T)$ which contains the identity and separates points.

**Example 31.3:** $\cos nx \cdot \sin mx \overset{?}{=} \frac{1}{2}(\sin(n+m)x - \sin(n-m)x$?

**Example 31.4:** If $x, y \in [-\pi, \pi]$ with $x \neq y$ then

$$\begin{aligned} \cos x = \cos y \text{ and } \sin x = \sin y &\implies x = y \mod 2\pi \\ &\implies x = \pm\pi \text{ and } y = -x. \end{aligned}$$

Hence the trigonometric polynomials are dense in $C[-\pi, \pi]$ in $L_\infty[-\pi, \pi]$ hence also in $L_p[-\pi, \pi]$ (since $\|f\|_p \leq (2\pi)^{1/p} \|f\|_\infty$).
When $p = 2$, $L_2[-\pi, \pi]$ is a Hilbert space.

$$\{1, \cos nx, \sin nx \mid n \in \mathbb{Z}^+\}$$

is an orthogonal set. We have

$$\begin{aligned} \|1\|_2^2 = \langle 1, 1 \rangle &= \int_{-\pi}^{\pi} 1^2 = 2\pi \\ \|\cos nx\|_2^2 &= \int_{-\pi}^{\pi} \cos^2 nx \, \mathrm{d}x = \pi \\ \|\sin nx\|_2^2 &= \int_{-\pi}^{\pi} \sin^2 nx \, \mathrm{d}x = \pi \\ \langle 1, \cos nx \rangle &= \int_{-\pi}^{\pi} \cos nx \, \mathrm{d}x = 0 \\ \langle 1, \sin nx \rangle &= \int_{-\pi}^{\pi} \sin nx \, \mathrm{d}x = 0 \\ \langle \cos nx, \sin mx \rangle &= \int_{-\pi}^{\pi} \cos nx \sin mx \, \mathrm{d}x \\ &= \frac{1}{2} \int_{-\pi}^{\pi} \sin(n+m)x + \sin(n-m)x \, \mathrm{d}x \\ &= 0 \end{aligned}$$

Hence $\left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos nx, \frac{1}{\sqrt{\pi}} \sin nx \;\middle|\; n \in \mathbb{Z}^+ \right\}$ is an orthonormal set in $L_2[-\pi, \pi]$, which is separable, and its span is dense, so it is a Hilbert basis for $L_2[-\pi, \pi]$. Thus, for every $f \in L_2[-\pi, \pi]$ we have

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

in $L_2[-\pi, \pi]$ with

$$a_0 = \frac{\langle f, 1 \rangle}{\langle 1 \rangle^2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, \mathrm{d}x$$

$$a_n = \frac{\langle f, \cos nx \rangle}{\langle \cos nx \rangle^2} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, \mathrm{d}x$$

$$b_n = \frac{\langle f, \sin nx \rangle}{\langle \sin nx \rangle^2} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, \mathrm{d}x$$

**End of Lecture 31**

**Start of Lecture 32**

Some examples involving Fourier series.

**Example 32.1** (Forced Damping String)**:** Consider a spring with a mass with $m$ is attached on it. The spring is resting at its equilibrium point initially. We have $F_{\text{spring}} = -ky$. This is also known as the Hooke's law. We also have $F_{\text{damping}} = -cy'$ and $F_{\text{applied}} = g(t)$. By Newton's law we have $F_{\text{total}} = my''$. Hence

$$-ky - cy' + g(t) = my'' \implies my'' + cy' + ky = g(t).$$

For the sake of example, let $m = 1, c = 2,$ and $k = 10$. Then we have

$$y'' + 2y' + 10y = g(t)$$

where $g(t)$ is the $2\pi-$periodic function with

$$g(t) = \begin{cases} \frac{\pi}{2} + t & -\pi \leq t \leq 0 \\ \frac{\pi}{2} - t & 0 \leq t \leq \pi \end{cases}.$$

To solve $y'' + 2y' + 10y = 0$ try $y = e^{rx}, y' = re^{rx}, y'' = r^2 e^{rx}$. Then, the DE becomes

$$r^2 e^{rx} + 2re^{rx} + 10e^{rx} = 0 \implies r^2 + 2r + 10 = 0 \implies r = -1 \pm 3i.$$

Then we have

$$y_1 = e^{(-1+3i)t} = e^{-t}(\cos 3t + i \sin 3t) \text{ and } y_2 = e^{(-1-3i)t} = e^{-t}(\cos 3t - i \sin 3t).$$

This gives real solutions

$$\frac{y_1 + y_2}{2} = e^{-t} \cos 3t \text{ and } \frac{y_1 - y_2}{2i} = e^{-t} \sin 3t.$$

The general solution to $y'' + 2y' + 10y = 0$ is

$$y = Ae^{-t} \cos 3t + Be^{-t} \sin 3t.$$

Note that this goes to 0 as $t \to \infty$.
To solve $y'' + 2y' + 10y = \cos nt$ try $y = A_n \cos nt + B_n \sin nt$, where

$$\begin{aligned} y &= A_n \cos nt + B_n \sin nt \\ y' &= -nA_n \sin nt + nB_n \cos nt \\ y'' &= -n^2 A_n \cos nt - n^2 B_n \sin nt \end{aligned}$$

Substitute these in the DE t obtain

$$\begin{aligned} y'' + 2y' + 10y = \cos nt &\implies -n^2 A_n \cos nt - n^2 B_n \sin nt - 2nA_n \sin nt + 2nB_n \cos nt + 10A_n \cos nt + 10B_n \sin nt = \cos \\ &\implies (-n^2 A_n + 2nB_n + 10A_n) \cos nt + (n^2 B_n - 2nA_n + 10B_n) \sin nt = \cos nt. \end{aligned}$$

We need,

$$\begin{aligned} \begin{bmatrix} 10 - n^2 & 2n \\ -2n^1 0 - n^2 & \end{bmatrix} \begin{bmatrix} A_n \\ B_n \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \implies \begin{bmatrix} A_n \\ B_n \end{bmatrix} &= \frac{1}{(10 - n^2)^2 + 4n^2} \begin{bmatrix} 10 - n^2 & -2n \\ 2n & 10 - n^2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{aligned}$$

This gives the solution

$$y_n = \frac{10-n^2}{(10-n^2)+4n^2}\cos nt + \frac{2n}{(10-n^2)^2+4n^2}\sin nt.$$

To solve $y''+2y'+10y=g(t)$, we find the Fourier series for $g(t)$, that is, we write

$$g(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt$$

where

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)\,\mathrm{d}t = 0 \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\sin nt\,\mathrm{d}t = 0 \text{ because } g(t) \text{ is even and } \sin nt \text{ is odd} \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\cos nt\,\mathrm{d}t \\
&= \frac{2}{\pi}\int_{0}^{\pi}\left(\frac{\pi}{2}-t\right)\cos nt\,\mathrm{d}t \\
&= -\frac{2}{\pi}\int_{0}^{\pi} t\cos nt\,\mathrm{d}t \\
&= -\frac{2}{\pi}\left[\frac{1}{n}t\sin nt + \frac{1}{n^2}\cos nt\right]_0^{\pi} \\
&= \begin{cases} \frac{4}{\pi n^2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}.
\end{aligned}$$

Thus we have

$$g(t) = \sum_{n=1}^{\infty} \frac{4}{\pi n^2}\cos nt$$

So the general solution to the original DE $y''+2y'+10y = \sum_{n=1}^{\infty}\frac{4}{\pi n^2}\cos nt$ is

$$y = Ae^{-t}\cos 3t + Be^{-t}\sin 3t + \sum_{n=1}^{\infty}\frac{4}{\pi n^2}(A_n\cos nt + B_n \sin nt)$$

where

$$A_n = \frac{10-n^2}{(10-n^2)^2+4n^2} \text{ and } B_n = \frac{2n}{(10-n^2)^2+4n^2}$$

We have

$$A_n\cos nt + B_n\sin nt = C_n\sin(nt+\phi_n) \text{ where } C_n = \sqrt{A_n^2+B_n^2} = \frac{1}{(10-n^2)^2+4n^2}.$$

**Example 32.2** (Vibrating String)**:** A vibrating string of length $\pi$ with fixed endpoints approximately satisfies the DE

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

with the boundary and initial conditions

$$u(0,t) = u(\pi,t) = 0 \text{ and } u(x,0) = g(x) \text{ and } u_t(x,0) = h(x)(\text{ often } h(x)=0).$$

We use the method of separation of variables. For more on this, refer to AMATH 353 - Partial Differential Equations notes. Try a solution of the form $u(x,t) = X(x)T(t)$. The DE becomes

$$XT'' = c^2X''T \implies \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T}.$$

Since this is true for all $x$ and $t$, then we must have

$$XT'' = c^2X''T \implies \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = k \text{ where } k \text{ is a constant.}$$

So we obtain two separate DEs in the form

$$X''(x) = kX(x) \text{ and } T''(t) = kc^2T(t).$$

When we apply the boundary conditions [$u(0,t) = u(\pi,t) = 0$ we get

$$X(0)T(t) = X(\pi)T(t) = 0 \quad \forall\, t \implies \text{ either } T(t) = 0 \quad \forall\, t \text{ or } X(0) = X(\pi) = 0.$$

If $T(t) = 0$ then $u(x,t) = 0 \quad \forall\, x,t$. If $X(0) = X(\pi) = 0$ then

$$X''(x) = kX(x), \quad X(0) = X(\pi) = 0.$$

When $k = 0$ we get $X''(x) = 0$ then $X(x) = ax + b$. Since $X(0) = X(\pi) = 0$ then $a = 0 = b$. Then $X = 0$. Then $u(x,t) = 0 \quad \forall\, x,t$.
When $k = p^2 > 0$ we get $X'' = p^2X$ (we try $y = e^{rx}$ as our solution). We get

$$X(x) = Ae^{px} + Be^{-px}.$$

Since the boundary conditions implies $X(0) = X(\pi) = 0$ then $A = B = 0$. Then $X = 0$. Then $u(x,t) = 0$ for all $x,t$.
When $k = -p^2 < 0$ we get $X'' = -p^2X$ (we try $y = e^{rx}$ as our solution). We get

$$X(x) = A\cos px + B\sin px.$$

Since $X(0) = 0$ then $A = 0$. Then $X(x) = B\sin px$. Since $X(\pi) = 0$ then $B\sin \pi p = 0$. Then either $B = 0$ or $p = n \in \mathbb{Z}$. We obtain the solution

$$X(x) = B\sin nx \text{ where } n \in Z \text{ and } k = -n^2.$$

Consider the other DE when $k = -n^2$, $n \in \mathbb{Z}^+$ and $X_n(x) = \sin nx$. We have

$$T''(t) = kc^2T(t) \implies T'' + n^2c^2T = 0.$$

Which has the solution

$$T_n(t) = A_n\cos nct + B_n\sin nct$$

which gives us

$$u_n(x,t) = \sin nx(A_n\cos nct + B_n\sin nct).$$

Then we look for as solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} \sin nx(A_n\cos cnt + B_n\sin cnt).$$

Note that any solution in this form satisfies the boundary conditions. So we look for solutions in the form that satisfy the given initial condition where $u(x, 0) = g(x)$ and $u_t(x, 0) = h(x)$. We have

$$u_t(x, t) = \sum_{n=1}^{\infty} \sin nx(-ncA_n \sin nct + ncB_n \cos nct).$$

We need

$$\begin{aligned} u(x, 0) = g(x) &\implies \sum_{n=1}^{\infty} A_n \sin nx = g(x) \text{ and,} \\ u_t(x, 0) = h(x) &\implies \sum_{n=1}^{\infty} ncB_n \sin nx = h(x) \end{aligned}$$

We take the $A_n$ to be the coefficients in the Fourier series for the odd $2\pi$ periodic function which agrees with $g(x)$ for $0 \le x \le r$. We take $ncB_n$ in a similar way.

**End of Lecture 32**

**Start of Lecture 33**

**Definition 33.1:** A (complex) **trigonometric polynomial** is a function $f : \mathbb{R} \to \mathbb{C}$ of the form

$$f(x) = \left(a_0 + \sum_{n=1}^{\ell} a_n \cos nx + \sum_{n=1}^{\ell} b_n \sin nx\right) + i\left(a_0' + \sum_{n=1}^{\ell} a_n' \cos nx + \sum_{n=1}^{\ell} b_n' \sin nx\right)$$

for some $a_n, b_n, a_n', b_n' \in \mathbb{R}$. Equivalently a function $f : \mathbb{R} \to \mathbb{C}$ of the form

$$f(x) = \sum_{-\ell}^{\ell} c_n e^{inx} \text{ for some } c_n \in \mathbb{C}.$$

Where

$$\begin{aligned} \cos nx &= \frac{e^{inx} + e^{-inx}}{2} \text{ where } e^{inx} = \cos nx + i \sin nx, \\ \sin nx &= \frac{e^{inx} - e^{-inx}}{2i} \text{ where } e^{-inx} = \cos nx - i \sin nx. \end{aligned}$$

**Remark 33.2:** The set of all trigonometric polynomials is dense in

$$C(T) = C(T, \mathbb{C}) = \{\text{continuous } 2\pi - \text{periodic functions } f : \mathbb{R} \to \mathbb{C}\}$$

using the $\infty-$norm. Hence also in $L_p(T) = L_p(T, \mathbb{C})$ using the $p-$norms. The set $\{e^{inx} \mid n \in \mathbb{Z}\}$ is orthogonal in $L_2(T)$. Since

$$\langle e^{inx}, e^{imx}\rangle = \int_{-\pi}^{\pi} e^{inx}\overline{e^{inx}}\,\mathrm{d}x = \int_{-\pi}^{\pi} e^{i(n-m)x}\,\mathrm{d}x = \begin{cases} \int_{-\pi}^{\pi} 1\,\mathrm{d}x = 2\pi & \text{if } n = m \\ \left[\frac{1}{(n-m)i} e^{i(n-m)x}\right]_{-\pi}^{\pi} = 0 & \text{if } n \neq m \end{cases}.$$

So the set $\mathcal{U} = \left\{\frac{1}{\sqrt{2\pi}} e^{inx} \;\middle|\; n \in \mathbb{Z}\right\}$ is a Hilbert basis for $L_2(T) = L_2(T, \mathbb{C}) = L_2[-\pi, \pi]$. For every $f \in L_2(T)$, we have

$$f(x) = \sum_{n=-\infty}^{n=\infty} c_n e^{inx} = \lim_{\ell \to \infty} \sum_{n=-\ell}^{\ell} c_n e^{inx} \text{ in } L_2(T) \text{ where } c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) e^{-int}\,\mathrm{d}t.$$

**Remark 33.3:** Recall that for $s_\ell \sum_{n=1}^{\ell} a_n$ with $a_n \in \mathbb{R}$,

$$\sum_{n=1}^{\infty} a_n \text{ converges } \iff \{s_\ell\} \text{ and } \{\sigma_\ell\} \text{ converges where } \sigma_\ell = \frac{s_1 + s_2 + \ldots + s_\ell}{\ell}.$$

Note that the converse is not necessarily true ($a_n = (-1)^n$).

**Remark 33.4:** For the assignment, use the sequential characterization of compactness. $S$ is compact if and only if $S$ is sequentially compact. Every sequence in $S$ has a subsequence that converges to element which is in $S$.

**End of Lecture 33**

**Start of Lecture 34**

**Notation 34.1:** For $f \in L_1(T) = L_1(T, \mathbb{C})$ (or $f : \mathbb{R} \to \mathbb{C}$ is $2\pi-$ periodic and measurable with $\int_{-\pi}^{\pi} |f| < \infty$) we have

$$c_n(f) = \hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} \, \mathrm{d}t.$$

as the **(complex) Fourier coefficients** of $f$.

**Notation 34.2:** We have the partial sums of the Fourier series of $f$ as

$$S_\ell(f)(x) = \sum_{-\ell}^{\ell} c_n(f) e^{inx}$$

and the Cesàro means of the Fourier series of $f$ as

$$\sigma_m(f) = \frac{S_0(f) + S_1(f) + \ldots + S_m(f)}{m+1} = \frac{1}{m+1} \sum_{\ell=0}^{m} S_\ell f(x).$$

**Question:** If $f \in L_p(T)$ then,

$$\begin{aligned}
&\text{do we have } \lim_{\ell \to \infty} S_\ell(f) && = f \text{ in } L_p(T), \\
&\text{or do we have } \lim_{m \to \infty} \sigma_m(f) && = f \text{ in } L_p(T), \\
&\text{or do we have } \lim_{\ell \to \infty} S_\ell(f)(x) && = f(x) \quad \forall\, x \in T \text{ (or for a.e } x \in T), \\
&\text{or do we have } \lim_{m \to \infty} \sigma_m(f)(x) && = f(x) \text{ in } L_p(T)?
\end{aligned}$$

Solution.

For $f \in L_1(T)$, we have

$$\begin{aligned}
S_\ell(f)(x) &= \sum_{n=-\ell}^{\ell} c_n(f) e^{inx} \\
&= \sum_{n=-\ell}^{\ell} \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} \right) e^{inx} \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \frac{1}{2} \sum_{n=-\ell}^{\ell} e^{in(x-t)} \, \mathrm{d}t \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) D_\ell(x-t) \, \mathrm{d}t \quad \text{where ,} \\
D_\ell(u) &= \frac{1}{2} \sum_{n=-\ell}^{\ell} e^{inu} \\
&= \frac{1}{2} e^{-i\ell u} \frac{e^{i(2\ell+1)u} - 1}{e^{iu} - 1} \quad \text{if } u \neq 0
\end{aligned}$$

Since $e^{i\theta}-1=e^{i\theta/2}(e^{i\theta/2}-e^{-i\theta/2})=e^{i\theta/2}\cdot 2i\sin\frac{\theta}{2}$, then

$$\begin{aligned}
&= \frac{1}{2}e^{-i\ell u}\frac{2i\exp\Big(i(\ell+\frac{1}{2})u\Big)\sin\Big(\ell+\frac{1}{2}u\Big)}{2i\exp\big(i\frac{u}{2}\big)\sin\frac{u}{2}}\\
&= \frac{\sin\Big(\ell+\frac{1}{2}\Big)u}{2\sin\frac{1}{2}u}.
\end{aligned}$$

So we have

$$S_\ell(f)(x)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)D_\ell(x-t)\,\mathrm{d}t$$

where

$$D_\ell(u)=\frac{1}{2}\sum_{n=-\ell}^{\ell}e^{inu}=\begin{cases}\frac{\sin\left(\ell+\frac{1}{2}\right)u}{2\sin\frac{1}{2}u} & \text{if } u\neq 0\\ \ell+\frac{1}{2} & \text{if } u=0.\end{cases}.$$

$D_\ell(u)$ is called the **Dirichlet kernel**.

Note that $D_\ell(u)$ is real-valued and $2\pi-$periodic and even. Also,

$$\int_{-\pi}^{\pi}D_\ell(u)\,\mathrm{d}u=\frac{1}{2}\sum_{n=-\ell}^{\ell}\int_{-\pi}^{\pi}e^{inu}\,\mathrm{d}u=\frac{1}{2}\int_{-\pi}^{\pi}e^0=\pi.$$

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**Exercise 34.3:** Show $\int_{-\pi}^{\pi}\big|D_\ell(u)\big|$.

**Remark 34.4:** Also, for $f\in L_1(T)$, we have

$$\begin{aligned}
\sigma_m(f)(x) &= \frac{1}{m+1}\sum_{\ell=0}^{m}S_\ell(f)(x)\\
&= \frac{1}{m+1}\sum_{\ell=0}^{m}\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)D_\ell(x-t)\,\mathrm{d}t\\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}f(t)K_m(x-t)\,\mathrm{d}t,\ \text{where}\\
K_m(u) &= \frac{1}{m+1}\sum_{\ell=0}^{m}D_\ell(u)\\
&= \frac{1}{m+1}\sum_{\ell=0}^{m}\frac{\sin\Big(\ell+\frac{1}{2}\Big)u}{2\sin}\\
&= \frac{1}{2(m+1)\sin\frac{u}{2}}\operatorname{Im}\left(\sum_{\ell=0}^{m}e^{i\left(\ell+\frac{1}{2}\right)u}\right)\\
&= \frac{1}{2(m+1)\sin\frac{u}{2}}\operatorname{Im}\left(\frac{e^{iu/2}\Big(e^{i(m+1)u}-1\Big)}{e^{iu}-1}\right).
\end{aligned}$$

Since $e^{i\theta} - 1 = e^{i\theta/2}(e^{i\theta/2} - e^{-i\theta/2}) = e^{i\theta/2} \cdot 2i \sin\frac{\theta}{2}$, then

$$\begin{aligned} &= \frac{1}{2(m+1)\sin\frac{u}{2}} \operatorname{Im}\left( e^{iu/2} \frac{2ie^{i\frac{m+1}{2}u} \sin\frac{m+1}{2}u}{2ie^{iu/2}\sin\frac{u}{2}} \right) \\ &= \frac{1}{2(m+1)} \frac{\sin^2(\frac{(m+1)u}{2})}{\sin^2(\frac{u}{2})} \text{ when } u \neq 0. \end{aligned}$$

Thus we have

$$\sigma_m(f)(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) K_m(x-t)\,\mathrm{d}t$$

where

$$K_m(u) = \frac{1}{m+1}\sum_{\ell=0}^{m} D_\ell(u) = \begin{cases} \frac{1}{2(m+1)}\frac{\sin^2(\frac{(m+1)u}{2})}{\sin^2(\frac{u}{2})} & \text{if } u \neq 0 \\ \frac{m+1}{2} \text{ if } u = 0 \end{cases}$$

Since

$$K_m(0) = \frac{1}{m+1}\sum_{\ell=0}^{m}\left(\ell + \frac{1}{2}\right) = \frac{1}{m+1}\left(\frac{m(m+1)}{2} + \frac{m+1}{2}\right) = \frac{m+1}{2}.$$

The function $K_m(u)$ is called the **Fejer kernel**.

**Remark 34.5:** $K_m(u)$ is real-valued, even and $2\pi-$ periodic.

**Exercise 34.6:** Show that $\max\left|K_m(u)\right| = K_m(0) = \dfrac{m+1}{2}$.

**Remark 34.7:** Also,

$$\int_{-\pi}^{\pi} K_m(u)\,\mathrm{d}u = \frac{1}{m+1}\sum_{\ell=0}^{m}\int_{-\pi}^{\pi} D_\ell(u)\,\mathrm{d}u = \frac{1}{m+1}\sum_{\ell=0}^{m}\pi = \pi.$$

**Definition 34.8:** The **convolution** of $f$ with $g$ is defined as

$$(f * g)(x) = \int_A f(t)g(x-t)\,\mathrm{d}t.$$

**Remark 34.9:** Note that

$$\sigma_m(f)(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)K_m(x-t)\,\mathrm{d}t.$$

Letting $s = x - t$ gives

$$\begin{aligned} \sigma_m(f)(x) &= \frac{1}{\pi}\int_{x+\pi}^{x-\pi} -f(x-s)K_m(s)\,\mathrm{d}s \\ &= \frac{1}{\pi}\int_{x-\pi}^{x+\pi} f(x-s)K_m(s)\,\mathrm{d}s \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-t)K_m(t)\,\mathrm{d}t \end{aligned}$$

Then letting $s = -t$ gives us

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)K_m(t)\,\mathrm{d}t \qquad \text{since } K_m(t) \text{ is even.}$$

**End of Lecture 34**

**Start of Lecture 35**

Recall the definitions of Fourier coefficients and Cesàro means.

**Theorem 35.1** (Riemann-Lebesgue lemma)**:** Let $f \in L_1(T)$. Then $\lim_{n\to\infty} c_n(f) = 0$.

Proof.

Let $\varepsilon > 0$. Choose a trigonometric polynomial $g(x)$ with $\|f-g\|_1 < \varepsilon \cdot 2\pi$. Say $g(x) = \sum_{-\ell}^{\ell} a_n e^{inx}$. Then, for $n > \ell$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)e^{-int}\,\mathrm{d}t = 0.$$

So,

$$\begin{aligned}
\left|c_n(f)\right| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,\mathrm{d}t\right| \\
&\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} (f(t)-g(t))e^{-int}\,\mathrm{d}t\right| \\
&\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|f(t)-g(t)\right|\mathrm{d}t \\
&= \frac{1}{2\pi}\|f-g\|_1 \\
&= \varepsilon
\end{aligned}$$

Thus $\lim\limits_{n\to\infty} c_n(f) = 0$.

**Theorem 35.2** (Fejer)**:** Let $f \in L_1(T)$. Let $a \in T$. Suppose $f(a^+)$ and $f(a^-)$ exist in $\mathbb{C}$ where

$$\begin{aligned}
f(a^+) &= \lim_{x\to a^+} f(x) = \lim_{t\to 0^+} f(a+t) \\
f(a^-) &= \lim_{x\to a^-} f(x) = \lim_{t\to 0^-} f(a-t).
\end{aligned}$$

Then,

$$\lim_{m\to\infty} \sigma_m(f)(a) = \frac{f(a^+)+f(a^-)}{2} \text{ in } \mathbb{C}.$$

Moreover, if $f$ is continuous in a closed interval $I$ then the convergence is uniform.

**Exercise 35.3:** Show that

$$\begin{aligned}
\sigma_m(f)(x) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)K_m(x-t)\,\mathrm{d}t \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-t)K_m(t)\,\mathrm{d}t \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)K_m(t)\,\mathrm{d}t.
\end{aligned}$$

Proof.

Recall the following:

$$S_\ell(f)(x) = \sum_{-\ell}^{\ell} c_n(f)e^{inx} = \sum_{n=-\ell}^{\ell}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,\mathrm{d}t\right)e^{inx} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)D_\ell(x-t)\,\mathrm{d}t$$

$$\text{where } D_\ell(u) = \frac{1}{2}\sum_{-\ell}^{\ell} e^{-inu} = \frac{\sin\left(\ell+\frac{1}{2}\right)u}{2\sin\frac{1}{2}u}$$

$$\text{and } \sigma_m(f)(x) = \frac{1}{m+1}\sum_{\ell=0}^{m} S_\ell(f)(x) = \frac{1}{m+1}\sum_{\ell=0}^{m}\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)D_\ell(x-t)\,\mathrm{d}t = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)K_m(x-t)\,\mathrm{d}t$$

$$\text{where } K_m(u) = \frac{1}{m+1}\sum_{\ell=0}^{m} D_\ell(u) = \frac{1}{2(m+1)}\frac{\sin^2\left(\frac{m+1}{2}u\right)}{\sin^2\left(\frac{u}{2}\right)}$$

$$\text{and } \int_{-\pi}^{\pi} D_\ell(u)\,\mathrm{d}u = \pi \text{ and } \int_{-\pi}^{\pi} K_m(u)\,\mathrm{d}u = \pi$$

$$\sigma_m(f)(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)K_m(x-t)\,\mathrm{d}t = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-t)K_m(t)\,\mathrm{d}t = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)K_m(t)\,\mathrm{d}t$$

(by Exercise 35.3)

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Given $\varepsilon > 0$ we can choose $\delta > 0$ so that $\left|f(a+t) - f(a^+)\right|$ and $\left|f(a-t) - f(a^-)\right|$.
Hence $\left|(f(a+t) - f(a-t)) - (f(a^+) - f(a^-))\right|$ is small. – Wrong
Hence $\left|(f(a+t) + f(a-t)) - (f(a^+) + f(a^-))\right|$ is small. We have

$$\begin{aligned}
\left|\sigma_m(f)(a) - \frac{f(a^+)+f(a^-)}{2}\right| &= \frac{1}{2}\left|\int_{-\pi}^{\pi}\Big((f(a+t) - f(a-t)) - f(a^+) - f(a^-)\Big)K_m(t)\,\mathrm{d}t\right| \\
&\le \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big|(f(a+t) + f(a-t)) - (f(a^+) + f(a^-))\Big|K_m(t)\,\mathrm{d}t \\
&= \frac{1}{\pi}\int_{0}^{\pi}\Big|(f(a+t) + f(a-t)) - (f(a^+) + f(a^-))\Big|K_m\,\mathrm{d}t \\
&= I + J \\
\text{where } I &= \frac{1}{\pi}\int_{0}^{\delta}\Big|(f(a+t) + f(a-t)) - (f(a^+) + f(a^-))\Big|K_m(t)\,\mathrm{d}t \\
\text{and } J &= \frac{1}{\pi}\int_{\delta}^{\pi}\Big|(f(a+t) + f(a-t)) - (f(a^+) + f(a^-))\Big|K_m(t)\,\mathrm{d}t.
\end{aligned}$$

Given $\varepsilon > 0$ choose $\delta > 0$ so that $0 < t < \delta \implies \left|(f(a+t) + f(a-t)) - (f(a^+) + f(a^-))\right| \le \varepsilon$. Then,

$$\begin{aligned}
I &\le \frac{1}{\pi}\varepsilon\int_{0}^{\delta} K_m(t)\,\mathrm{d}t \le \frac{1}{\pi}\varepsilon\int_{0}^{\pi} K_m(t)\,\mathrm{d}t = \frac{1}{\pi}\varepsilon\frac{\pi}{2} = \frac{\varepsilon}{2} \\
\text{Also, } J &\le \frac{1}{\pi}\max_{\delta t\le\pi} K_m(t)\int_{\delta}^{\pi}\Big|(f(a+t) + f(a-t)) - (f(a^+) + f(a^-))\Big|\,\mathrm{d}t \\
&\le \frac{1}{\pi}\max_{\delta t\le\pi} K_m(t)\int_{0}^{\pi}\left|f(a+t)\right| + \left|f(a-t)\right| + \Big|f(a^+)\Big| + \Big|f(a^-)\Big|\,\mathrm{d}t \\
&\le \frac{1}{\pi}\max_{\delta\le t\le\pi} K_m(t)\Big(\|f\|_1 + \pi\Big(f(a^+) + f(a^-)\Big)\Big)
\end{aligned}$$

Since we have $K_m(t) = \frac{1}{2(m+1)} \frac{\sin^2\left(\frac{m+1}{2}t\right)}{\sin^2 \frac{t}{2}} \leq \frac{1}{2(m+1)} \frac{1}{\sin^2 \frac{t}{2}} \leq \frac{1}{2(m+1)} \frac{\pi^2}{t^2}$, then

$$J \leq \frac{1}{\pi} \frac{1}{2(m+1)} \frac{\pi^2}{\delta^2} \left( \|f\|_1 + \pi\left(f(a^+) + f(a^-)\right)\right) \to 0 \text{ as } m \to \infty.$$

So we can choose $m \in \mathbb{Z}^+$ large enough so that $J \leq \frac{\varepsilon}{2}$.

**Corollary 35.4:** Let $f \in L_1(T)$. If all three

$$\begin{aligned} f(a^+) &= \lim_{x \to a^+} f(x) = \lim_{t \to 0^+} f(a+t) \\ f(a^-) &= \lim_{x \to a^-} f(x) = \lim_{t \to 0^-} f(a-t) \\ &\lim_{\ell \to \infty} S_\ell(f)(a) \end{aligned}$$

exist in $\mathbb{C}$, then $\lim\limits_{\ell \to \infty} S_\ell(f)(a) = \dfrac{f(a^+) + f(a^-)}{2}$.

**Example 35.5:** Recall that when $g(x)$ is the $2\pi-$periodic function,

$$g(x) = \begin{cases} \dfrac{\pi}{2} + x \text{ for } -\pi \leq x \leq 0 \\ \dfrac{\pi}{2} - x \text{ for } 0 \leq x \leq \pi \end{cases}$$

we found that

$$g(x) = \frac{4}{\pi} \sigma_{n \text{ odd}} \frac{1}{n^2} \cos nx = \frac{4}{\pi}\left(\frac{1}{1^2}\cos x + \frac{1}{3^1}\cos 3x + \frac{1}{5^2}\cos 5x\right).$$

Put in $x = 0$ to get

$$\frac{\pi}{2} = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \text{ where } \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}.$$

Another method is done by letting $S = \sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$. Since every term is positive, then we can rearrange the sum as

$$S = \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right) + \left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \dots\right) = \frac{\pi^2}{8} + \frac{1}{4}S.$$

Then, $\dfrac{3}{4}S = \dfrac{\pi^2}{8}$. Then $S = \dfrac{\pi^2}{6}$. Hence

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

**End of Lecture 35**

**Start of Lecture 36**

**Exercise 36.1:** Find $\|g\|^2$ where $g$ is defined as

$$g(x) = \begin{cases} \frac{\pi}{2} + x \text{ for } -\pi \le x \le 0 \\ \\ \frac{\pi}{2} - x \text{ for } 0 \le x \le \pi \end{cases}$$

in Example 35.5 We have

$$\begin{aligned} \|g\|^2 &= \int_{-\pi}^{\pi} g^2 = 2\int_0^{\pi} \left(\frac{\pi}{2} - x\right)^2 \mathrm{d}x \\ &= 2\left[\frac{1}{3}\left(x - \frac{\pi}{2}\right)^3\right]_0^{\pi} \\ &= \frac{2}{3}\left(\frac{\pi^2}{2^x} + \frac{\pi^3}{3}\right) \\ &= \frac{\pi^3}{6}. \end{aligned}$$

**Remark 36.2:** Working in $L_2(T)$ using the orthonormal basis $\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos nx, \frac{1}{\sqrt{\pi}}\sin nx\}$, we have

$$\begin{aligned} g(x) &= \sum_{n(\text{ odd})} \frac{4}{\sqrt{\pi}n^2}\frac{1}{\sqrt{\pi}}\cos nx \\ \|g\|^2 &= \sum_{n(\text{ odd})} \left|\frac{4}{\sqrt{\pi}n^2}\right|^2 = \sum_{n(\text{ odd})} \frac{16}{\pi n^4}. \end{aligned}$$

Thus,

$$\sum_{n(\text{ odd})} = \frac{1}{n^4} = \frac{\pi^3}{6}\frac{\pi}{16} = \frac{\pi^4}{96}.$$

For $S = \sum_{n=1}^{\infty} \frac{1}{n^4}$ we have

$$S = \sum_{n(\text{ odd})} \frac{1}{n^4} + \sum_{n(\text{ even})} \frac{1}{n^4} = \frac{\pi^4}{96} + \frac{1}{16}S.$$

Then $\frac{15}{16}S = \frac{\pi^4}{96} \implies S = \frac{\pi^4}{90}$.

**Theorem 36.3:** Let $f \in L_1(T)$ and let $a \in T$. If

$$\lim_{h\to 0} \frac{1}{h}\int_0^h \left|f(a+t) - f(a)\right| \mathrm{d}t = 0$$

then $\lim_{m\to\infty} \sigma_m(f)(a) = f(a)$.

**Corollary 36.4:** $\lim\limits_{m\to\infty} \sigma_m(f)(x) = f(x)$ for a.e $x \in T$.

**Theorem 36.5:** If $f \in L_p(T)$ then $\lim\limits_{m\to\infty} \sigma_m(t) = f$ in $L_p(T)$.

**Corollary 36.6:** If $f, g \in L_1(T)$ with $c_n(f) = c_n(g)$ for all $n \in \mathbb{Z}$, then $f = g$ in $L_1(T)$ (that is, $f = g$ a.e in $T$).

Proof.

Suppose $c_n(f) = c_n(g)$ for all $n \in \mathbb{Z}$. Then, $\sigma_m(f) = \sigma_m(g)$ for all $m \in \mathbb{Z}^+$. So we have $\sigma_m(f) \to f$ in $L_1(T)$ and $\sigma_m(f) = \sigma_m(g) \to g$ in $L_1(T)$. So $f = g$ in $L_1(T)$.

Some harder theorems that are not covered in the textbook.

**Theorem 36.7:** For all $f \in L_p(T)$ where $1 < p < \infty$, $\lim\limits_{\ell\to\infty} S_\ell(f)(x) = f(x)$ for a.e $x \in T$ $\quad \exists\, f \in L_1(T)$ such that $\{S_\ell(f)(x)\}$ diverges for all $x \in T$.

**Remark 36.8:** By Fejer's theorem, for $f \in L_1(T)$, $a \in T$, if $f(a^+)$ and $f(a^-)$ exist and are finite and if $\{S_\ell(f)(a)\}$ converges then $\lim\limits_{\ell\to\infty} S_\ell(f)(a) = \dfrac{f(a^+) + f(a^-)}{2}$.

**Example 36.9:** For $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}\cos nx$ we use Dirichlet's test, that is given sequences $\{a_n\}$ and $\{b_n\}$ of real numbers, if $\{S_n\}$ is bounded where $S_n = \displaystyle\sum_{k=1}^{n} a_k$ and if $b_n \searrow 0$ (decreases and converges to 0) then $\displaystyle\sum_{n=1}^{\infty} a_n b_n$ converges.

**Remark 36.10:** Alternating series test follows as a special case of Dirichlet's test.

**Definition 36.11:** For the partition $a < x_0 < x_1 < \ldots < x_n < b$ (that is, $P = \{x_0, x_1, \ldots, x_k\}$), we define the **variation of $f$ with respect to $P$** as

$$V(f,P) = \sum_{k=1}^{n} \left| f(x_k) - f(x_{k-1}) \right|$$

and **total variation** of the function $f$ as $V(f) = \sup_P V(f,P)$. We say $f$ is of **bounded variation** when $V(f) < \infty$. Some supplementary information regarding variation can be found from: https://faculty.etsu.edu/gardnerr/5210/notes/6-3.pdf

**Definition 36.12:** Let $f \in L_1(T), a \in T$. If $f$ is of bounded variation in a closed interval with $a$ in its interior, then $S_\ell(f)(a) =$.

**Theorem 36.13** (Dini's Criterion)**:** https://en.wikipedia.org/wiki/Dini_criterion

**Corollary 36.14:** If $f$ is differentiable, then $\lim_{\ell\to\infty} S_\ell(f)(a) = f(a)$. More generally, if $f'(a^+)$ and $f'(a^-)$ exist and are finite, then $f(a^+)$ and $f(a^-)$ exist and are finite and $\lim_{\ell\to\infty} S_\ell(f)(a) = \frac{f(a^+)+f(a^-)}{2}$.

This concludes the final lecture for PMATH 450/650 - Summer 2018.

**End of Lecture 36**

Final exam information is posted in the course website as follows: http://www.math.uwaterloo.ca/~snew/pmath450-2018-S/index.html

PMATH 450/650 Lebesgue Integration and Fourier Analysis, Spring 2018

Note: the PMATH 450/650 final examination will be held on Thursday August 2, from 9:00-11:30 am, in PAC 9.
The exam will cover all of the course material.
There will be 5 questions, each with two parts.
In some of the problems you will be asked to state definitions.
You will also be asked to prove 2 of the following 5 theorems:

- Theorem 1.18 (Existence of Non-Measurable Sets)
- Theorem 2.31 (Fatou's Lemma)
- Theorem 3.18 Part (2) (Hölder's Inequality)
- Theorem 4.23 (Closed Convex Sets in a Hilbert Space)
- Theorem 15.4 from the textbook (Fejér's Theorem)