Chapter Based Lecture Notes

AMATH 353: Partial Differential Equations 1

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Preface and Notation

This PDF document includes lecture notes for AMATH 353 - Partial Differential Equations 1 taught by Francis J. Poulin in Winter 2019.

For any questions contact me at `c2kent(at)uwaterloo(dot)ca`.
Thanks to Zeqi Wang for notifying me of the typos.

Notation

Throughout the course and the notes, unless otherwise is explicitly stated, we adopt the following conventions and notations.

- The university logo is used as a place holder.
- Chapter numbering follows the order in the official course notes.
- Lectures are numbered according to the scheduling of classes. In order to be consistent with the lecture numbers from the instructor's notes, student should compare the notes with the lecture date instead.
- If an expression is said to be "$\equiv 0$" then it means it is **identically zero**.
- We use round brackets $(\cdot, \cdot)$ to denote the inner product (instead of angular brackets).
- We use $\mathcal{L}$ (mathcal L) to denote the Sturm-Liouville operator in Notation 4.1.1 and use $\mathscr{L}$ (curvy L) to denote the Laplace transform that we introduced in Definition 4.6.1.

Calvin KENT

Chapter 1 – Modeling with PDEs

1.1 Introduction

Definition 1.1.1: A ***partial differential equation*** (PDE) is an equation that relates a function of two or more variables with its partial derivatives. eg. If the variable is $u(x,t)$ then the PDE is of the form

$$F\left(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial t}, \frac{\partial^2 u}{\partial t^2}, \ldots\right) = 0.$$

PDEs are often derived from *conservation laws.* ◁

For this course, in lectures we solve PDEs exactly and in tutorials we solve PDEs computationally.

Given a solution we need to interpret the results.

Remark 1.1.2: We define some general concepts as follows.

- A ***scalar PDE*** is a single PDE.
- A ***system of PDEs*** is two or more PDEs.
- The ***order of a PDE*** is set by the highest partial derivative. Most of the problems in this course will be in first order or second order.
- To find a unique solution to a PDE where x is space t is time requires *boundary conditions* and *initial conditions.*
- The vast majority of PDEs we consider are the ones where F has a linear dependency. This is said to be a ***linear PDE***. Otherwise, we say the PDE is ***non-linear***. ◁

1.2 Conservation Laws

In today's lecture, we performed an experiment by adding a drop of green dye into a small bucket of water. We observed how the dye spread in water due diffusion. How do we describe this in mathematical sense? We know that

- there is no spontaneous generation of dye,
- if at a given location, the concentration of dye increases, this must be due to
  - transport across space,
  - adding extra dye.

Consider a narrow 3-D tube

Figure 1.2.1: Narrow 3-D tube.

We choose container to be narrow so that the concentration only depends on x (space) and on t (time). We define our variables as

$$\begin{aligned} u(x,t): &\quad \text{mass density of dye } [\mathrm{kg\,m^{-3}}] \\ A(x): &\quad \text{cross sectional area } [\mathrm{m^2}] \\ f(x,t,u): &\ \text{the rate at which the dye is added } [\mathrm{kg\,m^{-3}\,s^{-1}}] \\ \phi(x,t,u): &\ \text{flux (transport) of dye } [\mathrm{kg\,m^{-2}\,s^{-1}}] \end{aligned}$$

Now consider two points $x = a$ and $x = b$.

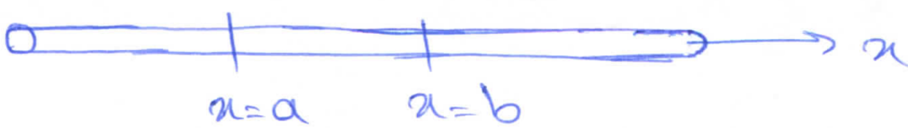


Figure 1.2.2: 3-D tube with bounds $x = a$ and $x = b$.

Total mass in $[a, b]$ is given by

$$\int_a^b u(x,t)A(x)\,\mathrm{d}x.$$

Rate of change of mass in $[a, b]$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b u(x,t)A(x)\,\mathrm{d}x = \int_a^b \frac{\partial u(x,t)}{\partial t}A(x)\,\mathrm{d}x.$$

Total change of mass by the source is given by

$$\int_a^b f(x,t,u)A(x)\,\mathrm{d}x.$$

Total change of mass by the flux (transport) is given by

$$\phi(a,t)A(a) - \phi(b,t)A(b) = -\int_a^b \frac{\partial}{\partial x}(\phi(x,t,u)A(x))\,\mathrm{d}x.$$

Hence, from the above we obtain

Net rate of change of mass in $[a, b]$	$=$	Net rate of change of mass in $[a, b]$ by source	$+$	Net rate of change of mass in $[a, b]$ by flux.

This is referred as the **conservation law**.In mathematics, we express the conclusion we found above as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b uA\,\mathrm{d}x = \int_a^b fA\,\mathrm{d}x - \int_a^b \frac{\partial}{\partial x}(\phi A)\,\mathrm{d}x. \tag{1.2.1}$$

Note that this is a global property.

Recall 1.2.1: In the previous lecture we found a mathematical expression for conversation law in Equation 1.2.1. We want to combine the terms to make things simpler, we assume A is constant. We obtain

$$\int_a^b \frac{\partial u}{\partial t}A\,\mathrm{d}x - \int_a^b fA\,\mathrm{d}x + \int_a^b \frac{\partial \phi}{\partial x}A\,\mathrm{d}x = 0 \implies \int_a^b \left[\frac{\partial u}{\partial t} + \frac{\partial \phi}{\partial x} - f\right]\mathrm{d}x = 0.$$

Hence, by contradiction, we deduce

$$\frac{\partial u}{\partial t} + \frac{\partial \phi}{\partial x} - f = 0 \quad \text{(identically zero).}$$

This is the conservation law in 1-D.This is a local property and is an example of a PDE. This PDE cannot be solved without knowing f and ϕ. ◁

1.2.1 Higher Dimensional Conservation Law

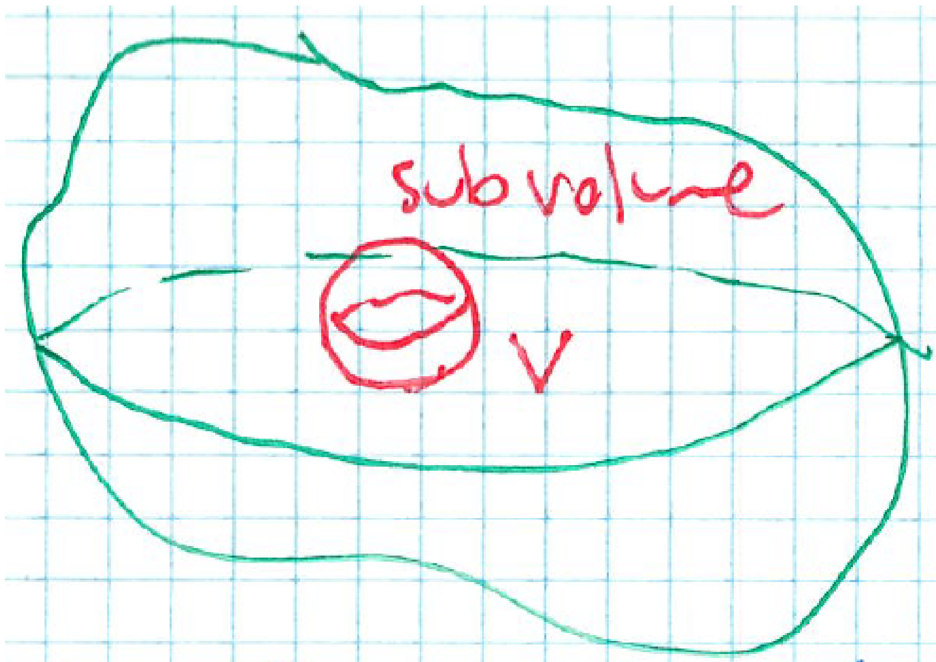


Figure 1.2.3: 3-D region.

We define

- $u(\mathbf{x}, t)$: mass density [$\mathrm{kg}\,\mathrm{m}^{-3}$].
- $f(\mathbf{x}, t, u)$: rate of mass added by source [$\mathrm{kg}\,\mathrm{m}^{-3}\,\mathrm{s}^{-1}$].
- $\boldsymbol{\phi}(\mathbf{x}, t, u)$: flux [$\mathrm{kg}\,\mathrm{m}^{-2}\,\mathrm{s}^{-1}$].
- $\hat{\mathbf{n}}$: unit outward normal.

⋆ Total mass is equivalent to $\iiint\limits_V u \, \mathrm{d}V$.

⋆ Rate of change of mass in V in time is equivalent to $\dfrac{\mathrm{d}}{\mathrm{d}t} \iiint\limits_V u \, \mathrm{d}V = \iiint\limits_V \dfrac{\partial u}{\partial t} \, \mathrm{d}V$.

⋆ Rate of change of mass by source is equivalent to $\iiint\limits_V f \, \mathrm{d}V$.

⋆ Rate of change of mass by flux is equivalent to $-\iint\limits_{\partial V} \boldsymbol{\phi} \cdot \hat{\mathbf{n}} \, \mathrm{d}S = -\iiint\limits_V \nabla \cdot \boldsymbol{\phi} \, \mathrm{d}V$.

Hence, the global conservation law becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint\limits_V u\,\mathrm{d}V = \iiint\limits_V f\,\mathrm{d}V - \oiint\limits_{\partial V} \boldsymbol{\phi}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$$

Using both FTCs we obtain

$$\begin{aligned}
&\iiint\limits_V \frac{\partial u}{\partial t}\,\mathrm{d}V - \iiint\limits_V f\,\mathrm{d}V + \iiint\limits_V \nabla\cdot\boldsymbol{\phi}\,\mathrm{d}V = 0\\
\implies &\iiint\limits_V \left[\frac{\partial u}{\partial t} + \nabla\cdot\boldsymbol{\phi} - f\right]\mathrm{d}V = 0.
\end{aligned}$$

True for any subvolume V. Hence, we deduce

$$\frac{\partial u}{\partial t} + \nabla\cdot\boldsymbol{\phi} - f = 0. \quad \text{(local quantitiy)} \tag{1.2.2}$$

Recall 1.2.2: For $n-$dimensions we have the conservation law as $\frac{\partial u}{\partial t} + \nabla\cdot\boldsymbol{\phi} - f = 0$. So, in 1-D we have

$$\frac{\partial u}{\partial t} + \frac{\partial \phi}{\partial x} - f = 0. \tag{1.2.3}$$

◁

1.3 Constitutive Relations

Definition 1.3.1: In general we must specify $\phi\big(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}, \ldots\big) = 0$. We denote this algebraic relation as the ***personality of the equation***. We need to make observations to find reasonable ϕ's. ◁

We will now drive and define some PDEs from the conservation law. For higher dimensions we will refer to Equation 1.2.2 and for 1-D we will refer to Equation 1.2.3.

1.3.1 Diffusion Equation

We assume the conservation law holds. Moreover, we assume

1. $n = 1$, so we are in 1-D.
2. No sources or sinks, so $f = 0$. In this case our conservation law becomes $\dfrac{\partial u}{\partial t} + \dfrac{\partial \phi}{\partial x} = 0$.
3. **Fick's Law:** Diffusion is proportional to the gradient of density it is down gradient. Fick's law assumes that

$$\phi(x,t) \underset{\text{def}}{=} -D\frac{\partial u}{\partial x}(x,t) \quad \text{where} \quad D \text{ is the diffusion constant with units } [D] = \frac{\text{length}^2}{\text{time}}.$$

We sub in our assumptions into (1.2.3) and obtain

$$\frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial x} = D\frac{\partial^2 u}{\partial x^2} \tag{Diffusion Equation}$$

Remark 1.3.2: If density has a length scale of L then what is the time scale of diffusion?

D is the diffusion coefficient with units $[D] = \frac{L^2}{T}$. Hence $T = \frac{L^2}{D}$. Which gives the intuition for the fact that large features diffuse more slowly. Large D has faster time scales. We will confirm this later on the course. ◁

1.3.2 Reaction-Diffusion Equation

We assume the conservation law holds. Moreover, we assume

1. $n = 1$, so we are in 1-D.
2. Fick's law: $\phi = -D\dfrac{\partial u}{\partial x}$.
3. Sources are present, so $f \neq 0$.

This is same as Diffusion Equation except we have $f \neq 0$. Hence we obtain

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + f. \tag{Reaction-Diffusion Equation}$$

1.3.3 Fisher's Equation

We have two choices for reactions:

$$f = \begin{cases} ru & \text{exponential growth} \\ ru\Big(1 - \dfrac{u}{K}\Big) & \text{logistic growth} \end{cases}$$

We sub logistic growth into Reaction-Diffusion Equation and obtain Fisher's equation as

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + ru\Big(1 - \frac{u}{K}\Big) \quad \text{where} \quad \begin{array}{r} r : \text{growth rate} \\ K : \text{carrying capacity} \end{array} \tag{Fisher's Equation}$$

1.3.4 Advection (Transport) Equation

We assume the conservation law holds. Moreover, we assume

1. $n = 1$, so we are in 1-D.
2. No sources, so $f = 0$.

③ Flux is proportional to concentration, so $\phi = cu$ where c is constant and has units of speed.

We sub this into (1.2.3) and get $\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(cu) - 0 = 0$. Hence we obtain

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \qquad \text{(Linear Advection (Transport) Equation)}$$

This describes the transport of solution at speed c.

1.3.5 Burger's Equation

We assume the conservation law holds. Moreover, we assume

① $n = 1$, so we are in 1-D.

② No sources, so $f = 0$.

③ Flux obeys the Fick's law and has a component that's proportional to concentration, that is

$$\phi = -D\frac{\partial u}{\partial x} + Q(u).$$

So, flux has properties of combination of Fick's law and advection.

We sub in our assumptions into (1.2.3) and obtain

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(-D\frac{\partial u}{\partial x} + Q(u)\right) = 0.$$

After rearranging we obtain

$$\frac{\partial u}{\partial t} + \frac{\partial Q}{\partial x} = D\frac{\partial^2 u}{\partial x^2}.$$

A particular choice for $Q(u)$ is $Q(u) = \frac{1}{2}u^2$. After subbing this in we obtain the Burger's equation as

$$\underbrace{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}}_{\text{non-linear advection}} = \underbrace{D\frac{\partial^2 u}{\partial x^2}}_{\text{diffusion}}. \qquad \text{(Burger's Equation)}$$

For the case where $D = 0$, we obtain the inviscid Burger's equation

$$\frac{\partial u}{\partial t} + u\frac{\mathrm{d}u}{\mathrm{d}x} = 0. \qquad \text{(inviscid Burger's equation)}$$

1.3.6 Diffusion Equation in 3-D

We assume the conservation law holds. Moreover, we assume

① $n = 3$, so we are in 3-D.

② No sources, so $f = 0$.

③ Fick's law (in 3-D): $\boldsymbol{\phi} = -D\nabla u$.

We sub this into (1.2.2) and obtain

$$\frac{\partial u}{\partial t} + \nabla \cdot (-D\nabla u) = 0.$$

Remark 1.3.3: We denote the divergence of the gradient as

$$\nabla \cdot \nabla = \nabla^2 \underset{\text{def}}{=} \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \qquad \text{(Laplacian)}$$

Older books might also use the notation $\nabla^2 = \Delta$. ◁

Hence, our equation becomes

$$\boxed{\frac{\partial u}{\partial t} = D\nabla^2 u \qquad \text{(diffusion equation in 3-D)}}$$

If we want a steady solution to this equation we assume $u(\mathbf{x})$ satisfies

$$\nabla^2 u = 0. \qquad \text{(Laplace's equation)}$$

Remark: We start the lecture by watching a short segment of a video featuring bowed violin string in slow motion: https://www.youtube.com/watch?v=6JeyiM0YNo4

Note that in this experiment the string eventually stops due to drag (air resistance). ◁

1.4 Wave Equation

We can use other tools to derive PDEs eg. (Newton's 2$^{\text{nd}}$ law). Motion of particle yields an ODE. Motion of string (continuum) yields a PDE. Newton's 2$^{\text{nd}}$ law can be rewritten as $\mathbf{F} = m\mathbf{a}$.

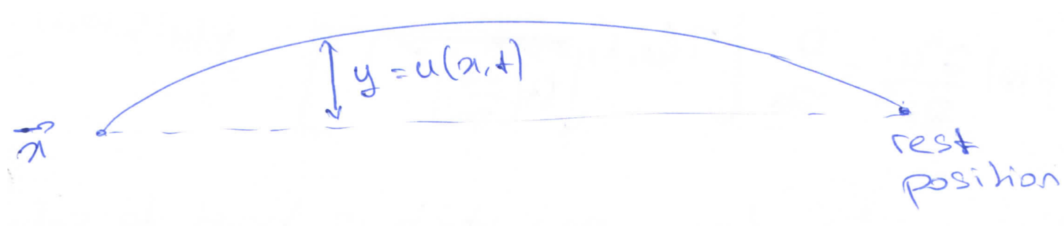


Figure 1.4.1: Vibrating string.

We define,

$$\begin{aligned} \rho(x) &: \text{ mass density } [\mathrm{kg\,m^{-3}}]. \\ A &: \text{ cross sectional area } [\mathrm{m^2}] \text{ (constant)}. \\ T &: \text{ string tension } (F/A)\ [\mathrm{N\,m^{-2}}]. \\ u(x,t) &: \text{ vertical displacement from rest.} \end{aligned}$$

We assume that motion is purely vertical and slope $\frac{\partial u}{\partial x}$ is small. We need to know

1. Acceleration: $\dfrac{\partial^2 u}{\partial t^2}$ (vertical)
2. Mass: $\rho(x)A(2\Delta x)$ (mass of subinterval)
3. Sum of forces; described in Figure 1.4.2 below.

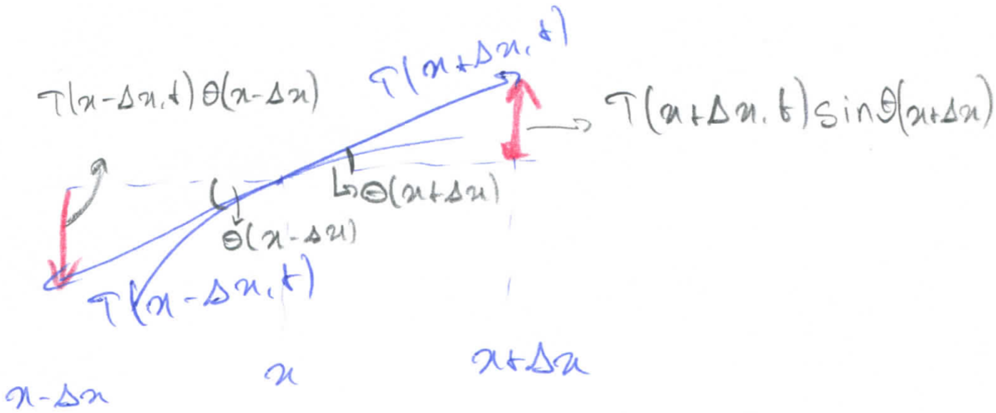


Figure 1.4.2: Sum of forces on a vibrating string.

Note that in here the two tensions are parallel and they face opposite directions. We define the tension (T) as force divided by the area (F/A). We have the total forces in vertical as

$$F = AT(x+\Delta x,t)\sin\theta(x+\Delta x,t) - AT(x-\Delta x,t)\sin\theta(x-\Delta x,t) - \rho(x)A2\Delta x Q(x,t),$$

where $Q(x,t)$ is the acceleration due to gravity. We sub this into the Newton's 2$^{\text{nd}}$ law. We obtain

$$[\rho(x)A2\Delta x]\frac{\partial^2 u}{\partial t^2} = AT(x+\Delta x,t)\sin\theta(x+\Delta x,t) - AT(x-\Delta x,t)\sin\theta(x-\Delta x,t) - \rho(x)A2\Delta x Q(x,t).$$

We divide the above equation by $2\Delta x A$ and take the limit as $\Delta x \to 0$. We have

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \lim_{\Delta x\to 0}\frac{T(x+\Delta x,t)\sin\theta(x+\Delta x,t) - T(x-\Delta x,t)\sin\theta(x-\Delta x,t)}{2\Delta x A} - \rho(x)Q(x,t).$$

Hence we obtain

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}[T(x,t)\sin\theta(x,t)] - \rho(x)Q(x,t).$$

Note that ρ, T and Q must be specified. We now need to find θ in terms of u. From geometry we have $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\partial u}{\partial x}$. Hence, we get

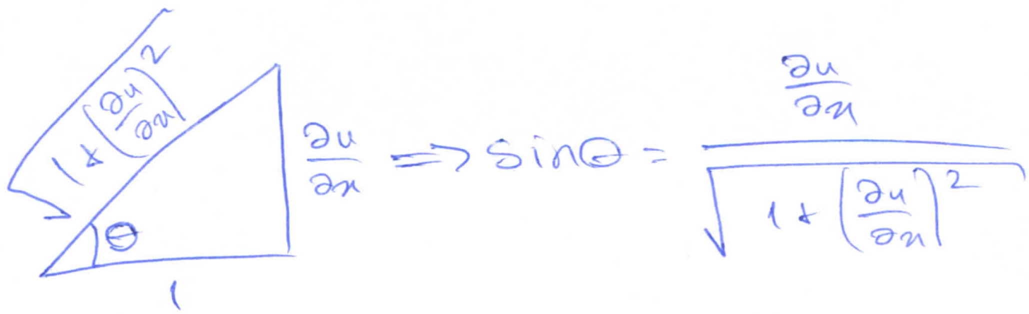


Figure 1.4.3: Trigonometric identities.

So we can write

$$\sin\theta = \frac{\frac{\partial u}{\partial x}}{\sqrt{1+\left(\frac{\partial u}{\partial x}\right)^2}}.$$

We obtain a closed PDE of the form

$$\rho\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left[T(x,t)\frac{\frac{\partial u}{\partial x}}{\sqrt{1+\left(\frac{\partial u}{\partial x}\right)^2}}\right] - \rho(x)Q(x,t).$$

We assume ρ, T are constant with slopes being small and we neglect gravity. So $\frac{\partial u}{\partial x} \ll 1$. Then $\sqrt{1+\left(\frac{\partial u}{\partial x}\right)^2} \approx 1$. We divide the above expression by ρ and obtain

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 u}{\partial x^2}.$$

Since $\left[\frac{T}{\rho}\right] = \frac{\mathrm{m}^2}{\mathrm{s}^2}$, we denote $c^2 \underset{\text{def}}{=} \frac{T}{\rho}$ and obtain the wave equation as

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \qquad \text{(Wave Equation)}$$

Note the comparison of wave equation with the Diffusion Equation in 1-D. The difference of 1st and 2nd derivatives yields completely different solutions.

1.4.1 Boundary Conditions

To find unique solutions to PDEs we must impose boundary conditions (BCs) and initial conditions (ICs) that describe the physical state of the ends. The number is set by the order of time derivative.

1.4.1.1 Dirichlet

e.g. String is clamped.

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0. \qquad \text{(Homogeneous)}$$

e.g. String moving up and down.

$$u(0,t) = f(t). \qquad \text{(Inhomogeneous)}$$

1.4.1.2 Neumann

e.g. Insulated boundaries

$$\frac{\partial u}{\partial x}(0,t) = 0. \qquad \text{(Homogeneous)}$$

1.4.1.3 Mixed/Robin

A mixed or Robin boundary condition is of the form,

$$a\frac{\partial u}{\partial x}(0,t) + bu(0,t) = 0. \qquad \text{(Homogeneous)}$$

1.4.1.4 Periodic

Periodic boundary conditions. That is to say the solution is equal at the left and right ends of the domain.

$$u(0,t) = u(L,t)$$
$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t).$$

1.5 Vibrating Membrane

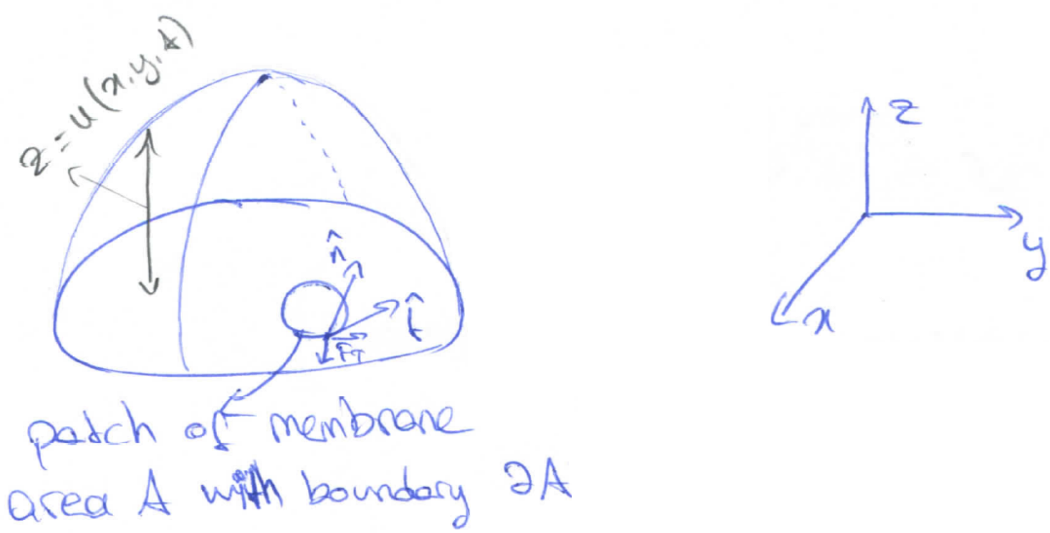


Figure 1.5.1: Visualization of a vibrating drum.

We define,

$$\begin{aligned}
u(x,y,t) &: \text{ vertical displacement [m]}\\
\rho(x,y) &: \text{ mass density [kg m}^{-2}]\\
\hat{\mathbf{t}} &: \text{ unit tangent vector}\\
\hat{\mathbf{n}} &: \text{ unit normal vector to surface (upwards)}\\
\mathbf{F}_T &: \text{ tensile (line) force, tangent to membrane[N m}^{-2}]
\end{aligned}$$

We assume that

1. $u(x,y,t) \ll 1$.
2. $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \ll 1$ (small slopes).
3. Motion is purely vertical
4. Tension is constant in magnitude, that is $|\mathbf{F}_T| = T_0$.

From physics, the force is given by

$$\mathbf{F}_T = T_0 \hat{\mathbf{t}} \times \hat{\mathbf{n}}.$$

Hence, the vertical component of the force is given by

$$\hat{\mathbf{k}} \cdot \mathbf{F}_T = T_0(\hat{\mathbf{t}} \times \hat{\mathbf{n}}) \cdot \hat{\mathbf{k}}.$$

Recall Newton's 2$^{\text{nd}}$ law states $\mathbf{F} = m\mathbf{a}$. We have the acceleration at a point as $\frac{\partial^2 u}{\partial t^2}$. To find $m\mathbf{a}$ for a patch, we must find this at a point and integrate over area A. We have

$$ma = \iint\limits_A \rho_0 \frac{\partial^2 u}{\partial t^2} \, \mathrm{d}A.$$

Next, we look at the sum at the force over the perimeter. Recall the identity $(\hat{\mathbf{t}} \times \hat{\mathbf{n}}) \cdot \hat{\mathbf{k}} = (\hat{\mathbf{n}} \times \hat{\mathbf{k}}) \cdot \hat{\mathbf{t}}$. We sub this into our equation for the vertical component of the force and obtain

$$\hat{\mathbf{k}} \cdot \mathbf{F}_T = T_0(\hat{\mathbf{t}} \times \hat{\mathbf{n}}) \cdot \hat{\mathbf{k}} = T_0(\hat{\mathbf{n}} \times \hat{\mathbf{k}}) \cdot \hat{\mathbf{t}}.$$

We must integrate the force over the perimeter to find the net force on the patch.

$$\oint\limits_{\partial A} T_0(\hat{\mathbf{n}} \times \hat{\mathbf{k}}) \cdot \hat{\mathbf{t}} \, \mathrm{d}s = \oint\limits_{\partial A} T_0(\hat{\mathbf{t}} \times \hat{\mathbf{n}}) \cdot \hat{\mathbf{k}} \, \mathrm{d}s.$$

Newton's 2$^{\text{nd}}$ law yields

$$\iint\limits_A \rho_0 \frac{\partial^2 u}{\partial t^2} \, \mathrm{d}A = \oint\limits_{\partial A} T_0(\hat{\mathbf{t}} \times \hat{\mathbf{n}}) \cdot \hat{\mathbf{k}} \, \mathrm{d}s.$$

This is also known as the global version of Newton's law.

Recall 1.5.1 (Stokes' Theorem)**:** As a consequence of Stokes' theorem we can write

$$\oint\limits_{\partial A} \mathbf{F} \cdot \hat{\mathbf{t}} \, \mathrm{d}s = \iint\limits_A (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}} \, \mathrm{d}A. \qquad \triangleleft$$

So, by Stokes theorem we rewrite line integrals as a double integral and get

$$\oint_{\partial A} T_0(\hat{\mathbf{n}} \times \hat{\mathbf{k}}) \cdot \hat{\mathbf{t}}\, \mathrm{d}s = \iint_A T_0[\nabla \times (\hat{\mathbf{n}} \times \hat{\mathbf{k}})] \cdot \hat{\mathbf{n}}\, \mathrm{d}A.$$

If we sub into Newton's 2^{nd} law and combine the two integrals we get

$$\iint_A \left[\rho_0 \frac{\partial^2 u}{\partial t^2} - T_0[\nabla \times (\hat{\mathbf{n}} \times \hat{\mathbf{k}})] \cdot \hat{\mathbf{n}}\right] \mathrm{d}A = 0.$$

Since this is true for any A, we deduce

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0(\nabla \times (\hat{\mathbf{n}} \times \hat{\mathbf{k}})) \cdot \hat{\mathbf{n}}. \tag{1.5.1}$$

This is another version of the wave equation. To get a simple PDE, we rewrite $\hat{\mathbf{n}}$ in terms of u. Recall if the surface is $z = u(x, y)$ and if we have small slopes, then we can find the unit normal as

$$\hat{\mathbf{n}} = \frac{\left(-\frac{\partial u}{\partial x}, -\frac{\partial u}{\partial y}, 1\right)}{\left\|\left(-\frac{\partial u}{\partial x}, -\frac{\partial u}{\partial y}, 1\right)\right\|} = \frac{\left(-\frac{\partial u}{\partial x}, -\frac{\partial u}{\partial y}, 1\right)}{\sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2}} \approx \left(-\frac{\partial u}{\partial x}, -\frac{\partial u}{\partial y}, 1\right).$$

We also have

$$\begin{aligned}
\hat{\mathbf{n}} \times \hat{\mathbf{k}} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -\frac{\partial u}{\partial x} & -\frac{\partial u}{\partial y} & 1 \\ 0 & 0 & 1 \end{vmatrix} = \left(-\frac{\partial u}{\partial y}, -\frac{\partial u}{\partial x}, 0\right), \\
\nabla \times (\hat{\mathbf{n}} \times \hat{\mathbf{k}}) &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{\partial u}{\partial x} & -\frac{\partial u}{\partial y} & 0 \end{vmatrix} = \left(0, 0, \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right).
\end{aligned}$$

Hence we have

$$[\nabla \times (\hat{\mathbf{n}} \times \hat{\mathbf{k}})] \cdot \hat{\mathbf{n}} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \nabla_H^2 u,$$

where H subscript denotes horizontal. We sub this in to (1.5.1) and obtain

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \nabla_H^2 u.$$

We let $c^2 \underset{\text{def}}{=} T_0/\rho_0$ and obtain the 2D wave equation as

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \nabla_H^2 u = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right). \qquad \text{(2D wave equation)}}$$

Chapter 2 – Classification of 2^{nd}-Order PDEs

Recall we previously we found the Diffusion Equation, Wave Equation and the Laplace's Equation as

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} \qquad \text{(Diffusion Equation)}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \qquad \text{(Wave Equation)}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{(Laplace's Equation)}$$

We will classify PDEs in 3 types: **parabolic**, **hyperbolic**, **elliptic**.

2.1 General 2^{nd}-Order Linear PDEs

The most general 2^{nd}-order linear PDE that we can write down is,

$$A\frac{\partial^2 U}{\partial x^2} + 2B\frac{\partial^2 U}{\partial x \partial y} + C\frac{\partial^2 U}{\partial y^2} + D\frac{\partial U}{\partial x} + E\frac{\partial U}{\partial y} + FU = G.$$

We must specify A, B, C, D, E, F, G that is our PDE. To classify, we focus on 2^{nd}-order terms

$$A\frac{\partial^2 U}{\partial x^2} + 2B\frac{\partial^2 U}{\partial x \partial y} + C\frac{\partial^2 U}{\partial y^2} = A\left(\frac{\partial}{\partial x}\right)^2 U + 2B\left(\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial y}\right)U + C\left(\frac{\partial}{\partial y}\right)^2 U.$$

This is a quadratic form in terms of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. Compare this with

$$Ax^2 + 2Bxy + Cy^2 = 0.$$

We can use quadratic equation to find the roots as follows:

$$\omega^{\pm} = \frac{-2B \pm \sqrt{4B^2 - 4AC}}{2A} = \frac{-B \pm \sqrt{B^2 - AC}}{A}.$$

Since there are 3 classifications of quadratic forms based on the sign of the discriminant, then there are 3 classes of PDEs.

$$B^2 - AC = \begin{cases} \Delta > 0 \text{ hyperbolic} \\ \Delta = 0 \text{ parabolic} \\ \Delta < 0 \text{ elliptic} \end{cases}$$

Example 2.1.1 (Wave Equation)**:** Take $y = t$. We get

$$c^2\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0.$$

So we have $A = c^2, B = 0, C = -1$. Since we have $B^2 - AC = 0 - c^2(-1) = c^2 > 0$ then the wave equation is **hyperbolic**. ◁

Example 2.1.2 (Diffusion Equation)**:** We have

$$D\frac{\partial^2 u}{\partial x^2}.$$

In this case we have $A = D$ and $B = 0 = C$. So $B^2 - AC = 0$. Hence the diffusion equation is **parabolic**. ◁

Example 2.1.3 (Laplace's Equation)**:** We have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

So we have $A = 1, B = 0, C = 1$, which gives us $B^2 - AC = 0 - 1(1) = -1 < 0$. Hence the Laplace's equation is **elliptic**. ◁

2.2 Reduction to Standard Form

2.2.1 Wave Equation

The wave equation can be written as

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} &= 0\\ \left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right)u = 0.\end{aligned}$$

We are going to factor differential operators as if they are are algebraic terms. We have

$$\boxed{\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u = 0.}$$

This equation (which resembles to advection) is hard to solve. So we will rewrite the equation for x, t to ζ, η to reduce the equation to a simpler form we can solve. Define

$$\left.\begin{aligned}\zeta &= x - ct\\ \eta &= x + ct\end{aligned}\right\} \begin{aligned} x &= \frac{\zeta + \eta}{2}\\ y &= \frac{\eta - \zeta}{2c}\end{aligned}$$

We define the variables ζ and η as ***characteristic variables***. By change of variables we have

$$\frac{\partial}{\partial \zeta}u = \frac{\partial u}{\partial \zeta} = \frac{\partial x}{\partial \zeta}\frac{\partial u}{\partial x} + \frac{\partial t}{\partial \zeta}\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial u}{\partial x} - \frac{1}{2c}\frac{\partial u}{\partial t} = -\frac{1}{2c}\left[\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right]u.$$
$$\frac{\partial}{\partial \eta}u = \frac{\partial u}{\partial \eta} = \frac{\partial x}{\partial \eta}\frac{\partial u}{\partial x} + \frac{\partial t}{\partial \eta}\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{2c}\frac{\partial u}{\partial t} = \frac{1}{2c}\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u.$$

Hence we get

$$-2c\frac{\partial}{\partial \zeta} = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) \quad \text{and} \quad 2c\frac{\partial}{\partial \eta} = \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right).$$

Hence, we get the wave equation as

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u = 0 \implies \left(-2c\frac{\partial}{\partial \zeta}\right)\left(2c\frac{\partial}{\partial \eta}\right)u = 0$$
$$\implies \frac{\partial^2 u}{\partial \zeta \partial \eta} = 0. \qquad \text{(wave equation in characteristic form)}$$

We integrate this w.r.t η. We get

$$\frac{\partial u}{\partial \zeta} = \alpha'(\zeta).$$

We now integrate w.r.t ζ and get

$$u(\zeta, \eta) = \alpha(\zeta) + \beta(\eta).$$

We rewrite in terms of x and t and obtain

$$\boxed{u(x,t) = \alpha(x - ct) + \beta(x + ct) \qquad \text{(d'Alembert's Solution)}}$$

Question: What if we want to impose ICs? Consider the ICs

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = g(x). \tag{2.2.1}$$

We plug in the solution into first IC. We get

$$u(x,0) = \alpha(x) + \beta(x) = f(x).$$

Similarly for second IC, we get

$$\frac{\partial u}{\partial t}(x,0) = -c\alpha'(x) + c\beta'(x) = g(x).$$

Wave Equation (continued)

Recall that last lecture when we imposed the ICs given in (2.2.1) on d'Alembert's solution and obtained

$$\frac{\partial u}{\partial t}(x,0) = -c\alpha'(x) + c\beta'(x) = g(x). \tag{2.2.2}$$

We divide this expression by c and integrate to obtain

$$-\alpha(x) + \beta(x) = \frac{1}{c}\int_0^x g(s), \mathrm{d}s. \tag{2.2.3}$$

We add (2.2.2) with (2.2.3) and sub (2.2.2) in (2.2.3) to obtain

$$\beta = \frac{1}{2}f + \frac{1}{2}\int_0^x g(s)\,\mathrm{d}s,$$
$$\alpha = \frac{1}{2}f - \frac{1}{2}\int_0^x g(s)\,\mathrm{d}s.$$

We sub this into d'Alembert's solution and obtain

$$u(x,t) = \frac{1}{2}f(x-ct) - \frac{1}{2}\int_0^{x-ct} g(s)\,\mathrm{d}s + \frac{1}{2}f(x+c) + \frac{1}{2}\int_0^{x+c} g(s)\,\mathrm{d}s.$$

In summary, we get

$$\boxed{u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s.}$$

2.3 Formulation of IVPs and BVPs

Definition 2.3.1: If a problem

1. has a solution,
2. the solution is unique and,
3. the solution depends continuously on ICs and BCs.

then we say the problem is a **well-posed problem**. Otherwise, we say the problem is an **ill-posed problem**. ◁

Example 2.3.2: An example of an ill-posed problem is

$$\text{(PDE)} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{where} \qquad -\infty < x < \infty, y > 0,$$
$$\text{(BCs)} \quad u(x,0) = 0 \text{ and } \frac{\partial u}{\partial x}(x,0) = \frac{\sin(nx)}{n}.$$

The solution is

$$u(x,y) = \frac{\sinh(nx)\sinh(nx)}{n^2}.$$

Case 1: In the limit as $n \to \infty$, the only solution to the PDE and the BCs is $u(x,y) = 0$.
Case 2: Limit at the solution $n \to \infty$ is $u(x,y) \to \infty$ (since $\sinh x = \frac{e^x - e^{-x}}{2}$)

Since small variations in BCs yield huge changes in solution then the problem is ill-posed. ◁

Chapter 4 – IBVPs in Bounded Domains

4.1 Introduction

4.1.1 Governing Equations (PDEs)

Recall we obtain the n-dimensional conservation law in (1.2.2). We now consider it in the form

$$\rho(x)\frac{\partial u}{\partial t} + \nabla \cdot \boldsymbol{\phi} = f. \tag{4.1.1}$$

Here we have added $\rho(x)$ to generalize the conservation law slightly. We assume the following:

1. **Fick's law:** $\boldsymbol{\phi} = -p(\mathbf{x})\nabla u$.
2. **Newton's law of cooling:** $f = -q(\mathbf{x})u + \rho(\mathbf{x})F$.

We sub our assumptions into (4.1.1) and obtain

$$\rho(\mathbf{x})\frac{\partial u}{\partial t} - \nabla \cdot (p(\mathbf{x})\nabla u) = -q(\mathbf{x})u + \rho(\mathbf{x})F$$

Which gives us

$$\rho(\mathbf{x})\frac{\partial u}{\partial t} - \nabla \cdot (p(\mathbf{x})\nabla u) + q(\mathbf{x})u = \rho(\mathbf{x})F.$$

In 1-D we have

$$\rho(x)\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(p(x)\frac{\partial u}{\partial x}\right) + q(x)u = \rho(x)F.$$

Notation 4.1.1: We define the operator $\mathcal{L}$ as

$$\begin{aligned} \mathcal{L}[u] &= -\frac{\partial}{\partial x}\left(p(x)\frac{\partial u}{\partial x}\right) + q(x)u && \text{(1D)} \\ \text{or } &= -\nabla \cdot (p(\mathbf{x})\nabla u) + q(\mathbf{x})u. && \text{(n-dim)} \end{aligned}$$

We will show this operator satisfies the eigenvalue relations. ◁

By using this notation, our PDE becomes

$$\rho\frac{\partial u}{\partial t} + \mathcal{L}[u] = \rho F.$$

Example 4.1.2: In 1-D, pick $\rho(x) = 1, p(x) = D, q(x) = 0, F(x) = 0$. We get the diffusion equation,

$$\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} = 0. \qquad \text{(Diffusion Equation)}$$

In 2D, pick $\rho(\mathbf{x}) = 0, p(\mathbf{x}) = 1, q(\mathbf{x}) = 0, F(\mathbf{x}) = 0$. We get

$$\nabla^2 u = 0,$$

which is the Laplace's equation. ◁

4.1.2 Boundary Conditions

General BCs we impose 3D are

$$\alpha(\mathbf{x})\, u\Big|_{\partial V} + \beta(\mathbf{x})\,\frac{\partial u}{\partial n}\Big|_{\partial V} = B(\mathbf{x}, t).$$

Remark 4.1.3: We define $\dfrac{\partial u}{\partial n} \equiv \hat{\mathbf{n}} \cdot \nabla u$ as **the exterior normal derivative** on the boundary.
If $\beta \equiv 0$ then we have a *Dirichlet BC*.
If $\alpha \equiv 0$ then we have a *Neumann BC*.

In 1-D this becomes (for $x = 0$ and $x = L$)

$$\begin{aligned} \alpha_1 u(0,t) - \beta_1 \frac{\partial u}{\partial x}(0,t) &= B_1(t), \\ \alpha_2 u(L,t) + \beta_2 \frac{\partial u}{\partial x}(L,t) &= B_2(t). \end{aligned}$$

It is similar in higher dimensions. ◁

4.2 Separation of Variables

Hyperbolic case: We use the same $\mathcal{L}$ operator as before, which we defined in (4.1.1), we get

$$\rho(x)\frac{\partial^2 u}{\partial t^2} + \mathcal{L}[u] = \rho F. \qquad \text{(Hyperbolic PDE)}$$

$$\left.\begin{aligned} &\alpha_1 u(0,t) - \beta_1 \frac{\partial u}{\partial x}(0,t) = 0. \\ &\alpha_2 u(L,t) + \beta_2 \frac{\partial u}{\partial x}(L,t) = 0. \end{aligned}\right\}\text{BCs}$$

$$\left.\begin{aligned} &u(x,0) = f(x), \\ &\frac{\partial u}{\partial t}(x,0) = g(x). \end{aligned}\right\}\text{ICs}$$

<u>Special case:</u> When we have $\rho(x) = 1, p(x) = c^2, q(x) = 0$ and $F = 0$ we get

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0. \qquad \text{(Wave Equation)}$$

Parabolic case: The PDE from last lecture was parabolic. We use the $\mathcal{L}$ same operator which defined in (4.1.1). We had

$$\rho(x)\frac{\partial u}{\partial t} + \mathcal{L}[u] = \rho F. \qquad \text{(Parabolic PDE)}$$

$$\left.\begin{aligned} &\alpha_1 u(0,t) - \beta_1 \frac{\partial u}{\partial x}(0,t) = 0. \\ &\alpha_2 u(L,t) + \beta_2 \frac{\partial u}{\partial x}(L,t) = 0. \end{aligned}\right\} \text{BCs}$$

$$u(x,0) = f(x). \quad \}\text{IC}$$

<u>Special case:</u> When we have $\rho(x) = 1, p(x) = D, q(x) = 0$ and $F = 0$ we get

$$\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} = 0. \qquad \text{(Diffusion Equation)}$$

Elliptic case: We use the same $\mathcal{L}$ operator as before, we get

$$-\rho(x)\frac{\partial^2 u}{\partial y^2} + \mathcal{L}[u] = \rho F, \qquad \text{(Elliptic PDE)}$$

$$\left.\begin{aligned} &\alpha_1 u(0,y) - \beta_1 \frac{\partial u}{\partial x}(0,y) = 0, \\ &\alpha_2 u(L,y) + \beta_2 \frac{\partial u}{\partial x}(L,y) = 0. \end{aligned}\right\} \text{BCs}$$

We need two conditions ("ICs") in the y direction.

Special case: When we have $\rho(x) = 1, p(x) = 1, q(x) = 0$ and $F = 0$ we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \qquad \text{(Laplace's Equation)}$$

4.2.1 Separating the Variables

For the hyperbolic problem, assume the solution $u(x,t)$ can be written as a product of a function of space and a function of time in the following way,

$$u(x,t) = M(x)N(t).$$

1. We sub into the (homogeneous) PDE in (Hyperbolic PDE) (so $F \equiv 0$)

$$\begin{aligned} \rho \frac{\partial^2 u}{\partial t^2} + \mathcal{L}[u] &= 0 \\ \rho \frac{\partial^2}{\partial t^2}[MN] + \mathcal{L}[MN] &= 0 \\ \rho M N'' + N\mathcal{L}[M] &= 0 \end{aligned}$$

2. We divide the expression by ρMN and move 2^{nd} terms to RHS

$$\frac{N''}{N} = -\frac{\mathcal{L}[M]}{\rho M}.$$

3. Since the LHS is a function of time and the RHS is a function of space and they are equal, then the functional dependencies must cancel out, so we must have a constant. We denote this constant as $-\lambda$,

$$\boxed{\frac{N''}{N} = -\frac{\mathcal{L}[M]}{\rho M} = -\lambda.}$$

4. This gives us two ODEs,

$$\begin{aligned} N'' + \lambda N &= 0 \\ \mathcal{L}[M] &= \lambda \rho M. \end{aligned}$$

Which gives us

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}M}{\mathrm{d}x}\right) + q(x)M = \lambda \rho M.$$

We divide by ρ and we get

$$\frac{1}{\rho}\mathcal{L}[M] = \lambda M.$$

Here we refer to λ as an eigenvalue and M as an eigenfunction. Note that we have

$$\frac{1}{\rho}\mathcal{L}[M] = \frac{1}{\rho}\left[-\frac{\mathrm{d}}{\mathrm{d}x}\left(p\frac{\mathrm{d}M}{\mathrm{d}x}\right) + q(x)M\right],$$

so here $\mathcal{L}$ is a linear differential operator.

Parabolic Case: $\rho\frac{\partial u}{\partial t} + \mathcal{L}[u] = 0$. Following the same steps as before, we get almost the same thing.

$$\boxed{\begin{aligned} \frac{N'}{N} = -\frac{\mathcal{L}[M]}{\rho M} = -\lambda &\implies N' + \lambda N = 0 \\ &\implies \mathcal{L}[M] = \lambda \rho M. \end{aligned}}$$

Elliptic Case: $-\rho\frac{\partial^2 u}{\partial y^2} + \mathcal{L}[u] = 0$. Following the same steps as before, we get

$$\begin{aligned} -\frac{N''}{N} = -\frac{\mathcal{L}[M]}{\rho M} = -\lambda &\implies N'' - \lambda N = 0 \\ &\implies \mathcal{L}[M] = \lambda\rho M. \end{aligned}$$

In three cases the ODE for M is identical. Note that we can also separate the boundary conditions $u(x,t) = M(x)N(t)$. We have

$$\begin{aligned} \alpha_1 u(0,t) - \beta_1\frac{\partial u}{\partial x}(0,t) &= 0 \\ \alpha_1 M(0)N - \beta_1 M'(0)N &= 0 \\ [\alpha_1 M(0) - \beta_1 M'(0)]N &= 0 \end{aligned}$$

In order to get non-trivial solutions, we require

$$\left.\begin{aligned} {}_1 M(0) - \beta_1 M'(0) = 0, \\ \alpha_1 M(L) + \beta_2 M'(L) = 0, \\ \mathcal{L}[M] = \lambda\rho M. \end{aligned}\right\} \begin{array}{l} \text{Boundary Value Problem} \\ \text{(Eigenvalue Problem)} \end{array}$$

<u>Special case:</u> When we have $\beta_1 = 0 = \beta_2 = 0, \rho(x) = 1, p(x) = 1, q(x) = 0$ we get

$$\begin{aligned} -\frac{\mathrm{d}^2 M}{\mathrm{d}x^2} &= \lambda M \\ M(0) = 0 &= M[L], \end{aligned} \qquad \begin{array}{l} \text{an example} \\ \text{of a BVP} \end{array}$$

In order to get a non-trivial solution, we have three cases to consider.

① $\lambda = 0$, so the solutions are $M = a + bx$,

$$\left.\begin{aligned} &\text{BC 1}: M(0) = a = 0 \\ &\text{BC 2}: M(L) = bL = 0 \implies b = 0 \end{aligned}\right\} M = 0.$$

② $\lambda < 0$, so the solutions are $M = a\sinh\left(\sqrt{\lambda}\,x\right) + b\cosh\left(\sqrt{\lambda}\,b\right)$,

$$\left.\begin{aligned} &\text{BC 1}: M(0) = b = 0 \\ &\text{BC 2}: M(L) = a\sinh\left(\sqrt{-\lambda}\,L\right) = 0 \implies a = 0 \end{aligned}\right\} M = 0.$$

③ $\lambda > 0$... (this will be completed)

4.2.2 Self-Adjoint Operators

In the 3 different cases of PDEs and BCs we found the same boundary value problem (BVP) which is in the form

$$\text{BVP} \quad \begin{cases} \mathcal{L}[M] = \lambda\rho M, & \text{ODE} \\ \alpha_1 M(0) - \beta_1 M'(0) = 0 & \text{BC1}, \\ \alpha_1 M(L) + \beta_2 M'(L) = 0 & \text{BC2}. \end{cases}$$

The 2[nd] ODE depends on the particular case

$$N'' + \lambda N = 0 \qquad \text{(hyperbolic)}$$
$$N' + \lambda N = 0 \qquad \text{(parabolic)}$$
$$N'' - \lambda N = 0 \qquad \text{(elliptic)}$$

Special Case: For the case $\rho = 1, p = 1, q = 0$ we have

$$\text{BVP} \begin{cases} -\dfrac{\mathrm{d}^2 M}{\mathrm{d}x^2} = \lambda M, \\ M(0) = 0, \\ M(L) = 0. \end{cases}$$

Last time we showed that if $\lambda \leq 0$ we only have the trivial solution

If $\lambda > 0$ the general solution of 2[nd] order ODE is

$$M = c_1 \sin\left(\sqrt{\lambda}\, x\right) + c_2 \cos\left(\sqrt{\lambda}\, x\right).$$

When we impose the boundary conditions we get

$$\text{BC1:} \quad M(0) = c_2 = 0.$$
$$\text{BC2:} \quad M(L) = c_1 \sin\left(\sqrt{\lambda}\, L\right) = 0.$$

To get non-trivial solutions, we need $c_1 \neq 0$ and $\sqrt{\lambda}\, L = \sqrt{\lambda_n}\, L = n\pi$ where $n = 1, \ldots$. Hence we deduce

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$

where λ_n are the eigenvalues of the system. The corresponding eigenfunction is

$$M_n = \sin\left(\frac{n\pi x}{L}\right).$$

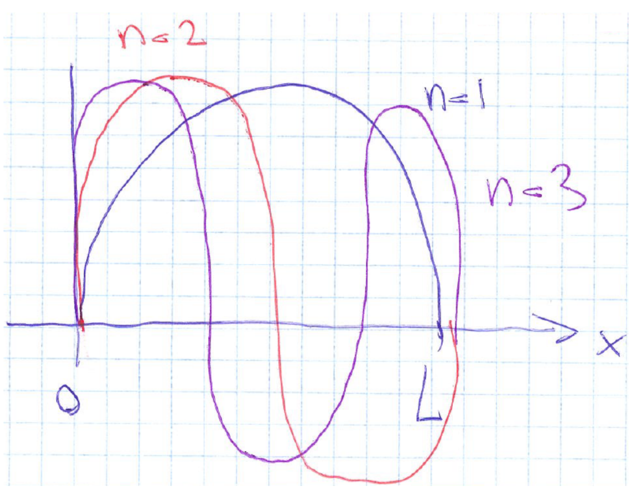


Figure 4.2.1: Eigenfunctions for $n = 1, 2, 3$.

Note that in general, the general eigenfunction is in the form

$$M_n = c_1 \sin\left(\frac{n\pi x}{L}\right).$$

4.2.3 Properties of BVPs in General

Definition 4.2.1: The **inner product of functions** $f(x), g(x)$ with *weight* $\rho(x) > 0$ is

$$(f, g) \underset{\text{def}}{=} \int_0^L \rho(x) f(x) g(x) \, \mathrm{d}x.$$

Recall that **weight of a function**, $\rho(x)$, is the term that appears in front of the term with the time derivative. Here the functions f, g are well-defined (and $\rho(x) > 0$)in the interval $(0, L)$. In 3-dimensions, we have

$$(f, g) \underset{\text{def}}{=} \iiint\limits_V \rho f g \, \mathrm{d}V. \quad \triangleleft$$

Remark 4.2.2: So, f, g are **orthogonal** with respect to the weight function $\rho(x)$ if $(f, g) = 0$. ◁

Definition 4.2.3: The **norm of functions** with respect to the inner product of functions is

$$\|f\|_\rho^2 = \|f\|^2 = (f, f) \equiv \int_0^L \rho(x) f^2(x) \, \mathrm{d}x. \quad \triangleleft$$

Definition 4.2.4: An operator $\mathcal{L}$ is said to be **self-adjoint** if

$$(w, \mathcal{L}[u]) = (\mathcal{L}[w], u). \quad \triangleleft$$

Example 4.2.5: We want to show that the operator $\frac{1}{\rho}\mathcal{L}$ we defined in Notation 4.1.1 is self-adjoint. Recall that we have

$$\mathcal{L}[u] = -\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}u}{\mathrm{d}x}\right) + qu.$$

We have

$$\begin{aligned}
\left(w, \frac{1}{\rho}\mathcal{L}[u]\right) - \left(\frac{1}{\rho}\mathcal{L}[w], u\right) &= \int_0^L \rho w \frac{1}{p}\big(-(pu')' + qu\big) \, \mathrm{d}x - \int_0^L \rho \frac{1}{\rho}\big(-(pw')' + qw\big) u \, \mathrm{d}x \\
&= \int_0^L -w(pu')' + quw + u(pw')' - quw \, \mathrm{d}x \\
&= \int_0^L -(pwu')' + pu'w' + (puw')' - pu'w' \, \mathrm{d}x \\
&= -\int_0^L \frac{\mathrm{d}}{\mathrm{d}x}(pwu' - puw') \, \mathrm{d}x \\
&= -\Big[p(wu' - uw')\Big]_0^L.
\end{aligned}$$

Assuming we have the boundary conditions $\begin{aligned} \alpha_1 u(0) &= \beta_1 u'(0), \\ \alpha_2 u(L) &= -\beta_2 u'(L), \end{aligned}$ and $\begin{aligned} \alpha_1 w(0) &= \beta_1 w'(0), \\ \alpha_2 w(L) &= -\beta_2 w'(L), \end{aligned}$ we get

$$\begin{aligned}
\left(w, \frac{1}{\rho}\mathcal{L}[u]\right) - \left(\frac{1}{\rho}\mathcal{L}[w], u\right) &= -\left[p\left(-\frac{\beta_2}{\alpha_2}u'w' + \frac{\beta_2}{\alpha_2}u'w'\right)\right]_{x=L} + \left[p\left(\frac{\beta_1}{\alpha_1}w'u' - \frac{\beta_1}{\alpha_1}u'w'\right)\right]_{x=0} \\
&= 0.
\end{aligned}$$

Hence $\left(w, \frac{1}{\rho}\mathcal{L}[u]\right) = \left(\frac{1}{\rho}\mathcal{L}[w], u\right)$. Hence $\frac{1}{\rho}\mathcal{L}$ is self-adjoint with BCs. ◁

4.2.4 Positivity of the $\mathcal{L}$ Operator

Definition 4.2.6: An operator T is said to be **positive** if

$$(u, T[u]) \geq 0. \qquad \triangleleft$$

Proposition 4.2.7: The operator $\frac{1}{\rho}\mathcal{L}$ is positive.

Proof: We have

$$\begin{aligned}\left(u, \frac{1}{\rho}\mathcal{L}[u]\right) &= \int_0^L \rho u \frac{1}{\rho}\left[-\frac{\mathrm{d}}{\mathrm{d}x}\left(p\frac{\mathrm{d}u}{\mathrm{d}x}\right) + qu\right]\mathrm{d}x \\ &= \int_0^L p\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 + qu^2\,\mathrm{d}x - \left[pu\frac{\mathrm{d}u}{\mathrm{d}x}\right]_0^L.\end{aligned}$$

Since by boundary conditions we have

$$\alpha_1 u(0) = \beta_1 u'(0) \quad \text{and} \quad \alpha_2 u(L) = -\beta_2 u'(L).$$

Hence we get

$$\left(u, \frac{1}{\rho}\mathcal{L}[u]\right) = \int_0^L p\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 + qu^2\,\mathrm{d}x + \left[\frac{\beta_2}{\alpha_2}p\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2\right]_{x=L} + \left[\frac{\beta_1}{\alpha_1}p\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2\right]_{x=0}$$

Since $p > 0, q \geq 0$ and since $\alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$ then we have

$$\left(u, \frac{1}{\rho}\mathcal{L}[u]\right) \geq 0$$

as required. □

4.3 Eigenfunction Expansions

4.3.1 Orthogonality of Eigenfunctions

Recall the operator $\frac{1}{\rho}\mathcal{L}$ is self-adjoint, i.e.

$$\left(w, \frac{1}{\rho}\mathcal{L}[u]\right) - \left(\frac{1}{\rho}\mathcal{L}[w], u\right) = 0.$$

So, if we pick $u = M_k$ and $w = M_j$, where M_j, M_k are eigenfunctions, then we have

$$\left(M_j, \frac{1}{\rho}\mathcal{L}[M_k]\right) - \left(\frac{1}{\rho}\mathcal{L}[M_j], M_k\right) = 0.$$

Since M_j, M_k are eigenfunctions, then they must satisfy the eigenvalue relation, that is

$$\frac{1}{\rho}\mathcal{L}[M_k] = \lambda_k M_k.$$

Hence we have

$$(M_j, \lambda_k M_k) - (\lambda_j M_j, M_k) = 0 \implies (\lambda_k - \lambda_j)(M_j, M_k) = 0.$$

Hence, if $\lambda_k - \lambda_j \neq 0$, then $(M_j, M_k) = 0$. In other words, eigenfunctions corresponding to distinct eigenvalues are orthogonal.

Eigenvalues are Non-negative

Since $\frac{1}{\rho}\mathcal{L}$ is positive, then

$$\left(M_k, \frac{1}{\rho}\mathcal{L}[M_k]\right) \geq 0.$$

Hence,

$$(M_k, \lambda M_k) \geq 0 \implies \lambda_k (M_k, M_k) = \lambda_k \|M_k\|^2 \geq 0.$$

Since the eigenfunctions are non-trivial (non-zero), then $\|M_k\|^2 > 0$. Hence

$$\lambda_k \geq 0.$$

Hence, all eigenvalues of $\mathcal{L}$ operator with BCs are non-negative.

Recall the 3 different cases and their solutions which we considered

Hyperbolic:	$N_k'' + \lambda_k N_k = 0$	which has the solution	$N_k = a_k \cos\left(\sqrt{\lambda_k}\, t\right) + b_k \sin\left(\sqrt{\lambda_k}\, t\right)$,
Parabolic:	$N_k' + \lambda_k N_k = 0$		$N_k = a_k \exp(-\lambda_k t)$,
Elliptic:	$N_k'' - \lambda_k N_k = 0$		$N_k = a_k \cosh\left(\sqrt{\lambda_k}\, t\right) + b_k \sinh\left(\sqrt{\lambda_k}\, t\right)$.

Remark 4.3.1: We use the separation of variables which we assumed $u(x,t) = M(x)N(t)$. In each case, the following is a solution

$$u_k(x,t) = M_k(x) N_k(t).$$

This solves the PDEs and BCs but in general this does not satisfy the ICs. Even though u_k is as solution, we need a general solution for all ICs. By *linear superposition*, since PDEs and BCs are linear and homogeneous, then the sum of two solutions is still a solution. The most general solution is a superposition over all the eigenfunctions, which is of the form

$$u(x,t) = \sum_{k=1}^{\infty} u_k(x,t) = \sum_{k=1}^{\infty} M_k(x) N_k(t). \qquad \triangleleft$$

Hyperbolic Case: Consider the initial conditions

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = g(x).$$

We have

$$u(x,t) = \sum_{k=1}^{\infty} M_k(x) \left[a_k \cos\left(\sqrt{\lambda_k}\, t\right) + b_k \sin\left(\sqrt{\lambda_k}\, t\right)\right]$$

We need to find a_k, b_k that satisfy the given ICs. For first IC, we have

$$u(x,0) = \sum_{k=1}^{\infty} a_k M_k(x) = f(x).$$

For the second IC we get

$$\frac{\partial u}{\partial t}(x,0) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\, a_k M_k(x) = g(x).$$

These equations are called *generalized Fourier series* for functions $f(x), g(x)$ and the constants a_k, b_k determine the *generalized Fourier coefficients*.

To determine a_k, b_k we project onto M_j. We have

$$(f, M_j) = \left(\sum_{k=1}^{\infty} a_k M_k, M_j \right) = \sum_{k=1}^{\infty} a_k (M_k, M_j) = a_j (M_j, M_j) = a_j = \frac{(f, M_j)}{\|M_j\|^2}.$$

Similarly,

$$b_j = \frac{(g, M_j)}{\sqrt{\lambda_j}(M_j, M_j)}$$

Note that for the case where $\|M_k\| = 1$ and $\|M_j\| = 1$ we get

$$a_k = (f, M_j) \quad \text{and} \quad b_k = \frac{(g, M_j)}{\sqrt{\lambda_k}}.$$

Remark: We start the lecture by watching a short segment of a video featuring a vibrating string. This allows us to see the behavior of Eigenfunctions on a spring: https://www.youtube.com/watch?v=BSIw5SgUirg ◁

Recall 4.3.2: In the 3 cases (hyperbolic, parabolic and elliptic), we have

$$\text{BVP} \quad \begin{cases} \mathcal{L}[M] = \lambda \rho M, & \text{ODE} \\ \alpha_1 M(0) - \beta_1 M'(0) = 0 & \text{BC1,} \\ \alpha_1 M(L) + \beta_2 M'(L) = 0 & \text{BC2.} \end{cases}$$

If $p > 0, q \geq 0, \rho > 0$ and $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ then $\frac{1}{\rho}\mathcal{L}$ is a positive operator and only has positive eigenvalues (non-negative). ◁

Parabolic Case: $N_n' + \lambda_n N_n = 0$. The solution is $N_n = a_n e^{-\lambda_n t}$. The general solution is

$$u(x,t) = \sum_{n=1}^{\infty} M_n a_n(x) e^{-\lambda_n t} \quad \text{with} \quad \text{IC } u(x,0) = f(x).$$

We evaluate the solution at $t = 0$. We have

$$u(x,0) = \sum_{n=1}^{\infty} a_n M_n(x) = f(x).$$

We project on $M_j(x)$, we get

$$(f, M_j) = \left(\sum_{n=1}^{\infty} a_n M_n, M_j\right) = \sum_{n=1}^{\infty} a_n (M_n, M_j) = a_j (M_j, M_j).$$

Hence, we have

$$a_j = \frac{(f, M_j)}{(M_j, M_j)}.$$

Special Case: Consider the diffusion equation with

$$\begin{aligned} \frac{\partial u}{\partial t} &= D\frac{\partial^2 u}{\partial x^2}, && \text{(PDE)} \\ u(0,t) &= 0 = u(L,t), && \text{(BCs)} \\ u(x,0) &= f(x). && \text{(IC)} \end{aligned}$$

1. Separate variables: $u(x,t) = M(x)N(t)$.

   a. Sub into PDE.
   $$MN' = DM''N.$$

   b. Divide by DMN.
   $$\frac{N'}{DN} = \frac{M''}{M} = -\lambda.$$

   c. Obtain the ODEs for M and N.
   $$\begin{aligned} M'' + \lambda M &= 0, \\ N'' + \lambda DN &= 0. \end{aligned}$$

   d. Separate the BCs and combine with the obtained ODEs.
   $$\begin{aligned} u(0,t) = M(0)N = 0 &\implies M(0) = 0. \\ u(L,t) = M(L)N = 0 &\implies M(L) = 0. \end{aligned}$$

   We get
   $$M'' + \lambda M = 0 \quad \text{with} \quad M(0) = 0 = M(L).$$

   From before, the only case that yields non-trivial solutions is $\lambda > 0$.

   Aside: Suppose, for contradiction, $\lambda < 0$. We have
   $$\begin{aligned} M &= a\cosh\left(\sqrt{-\lambda}\,x\right) + b\sinh\left(\sqrt{-\lambda}\,x\right) \\ \text{BC1: } M(0) &= a = 0. \\ \text{BC2: } M(L) &= b\sinh\left(\sqrt{-\lambda}\,L\right) = 0. \end{aligned}$$

If BC2 is zero, then either $b = 0$ or $\sinh\left(\sqrt{-\lambda}L\right)$. If we are looking for non-trivial solutions, then $b \neq 0$. Then, $\sinh\left(\sqrt{-\lambda}L\right) = 0$. This is only true if $L = 0$ or $\sqrt{-\lambda} = 0$. Since $L \neq 0$ then $\sqrt{-\lambda} = 0$. Then $\lambda = 0$. This is a contradicts with the assumption that $\lambda < 0$. ◁

2 Since $\lambda \geq 0$, then the solution to BVP is

$$M = a\cos\left(\sqrt{\lambda}\,x\right) + b\sin\left(\sqrt{\lambda}\,x\right).$$

Hence we get

$$\begin{aligned} \text{BC1: } M(0) &= a = 0, \\ \text{BC2: } M(L) &= b\sin\left(\sqrt{\lambda}L\right) = 0. \end{aligned}$$

Hence,

$$\begin{aligned} \sqrt{\lambda}L &= n\pi \quad \text{where} \quad n = 1, 2, \ldots \\ \lambda_n &= \left(\frac{n\pi}{L}\right)^2 \\ M_n &= \sin\left(n\frac{\pi x}{L}\right). \end{aligned}$$

3 For $N(t)$ we have

$$N_n' + \lambda_n D N_n = 0 N_n = a_k \exp(-\lambda_n) = a_n \exp\left(-\left(\frac{n\pi}{L}Dt\right)^2 Dt\right).$$

4 Hence, we have the general solution as

$$u(x,t) = \sum_{n=1}^{\infty} M_n(x)N_n(t) = \sum_{n=1}^{\infty} \sin\left(n\frac{\pi x}{L}\right) a_n \exp\left(-\left(n\frac{\pi}{L}\right)^2 Dt\right).$$

If $n = 1$, the eigenfunction decays like $\exp\left(-\frac{\pi}{L}\right)^2 Dt$.

If $n = 2$, the eigenfunction decays like $\exp\left(-\frac{2\pi}{L}\right)^2 Dt$.

The shorter the wavelength, the faster the eigenfunction is going to decay. If t is large but finite, then we expect the solution to be dominated by the first term ($n = 1$).

$$u(x,t) \to \sin\left(\frac{\pi x}{L}\right) a_1 \exp\left(-\left(\frac{\pi}{L}\right)^2 Dt\right).$$

Recall 4.3.3: We have $\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$. Given length scale L and diffusion rate D, time scale of decay is $T = L^2/D$. Note that this is clear if we simply look at the exponential term in sum. ◁

Definition 4.3.4: We denote the time it takes for each mode to decay as ***e*-folding time**. In this case, the e-folding time is

$$\left(\frac{L}{n\pi}\right)^2 \frac{1}{D}.$$

◁

4.4 Sturm-Liouville Problems and Fourier Series

Recall 4.4.1: We defined our BVP as

$$\mathcal{L}[u] \underset{\text{def}}{=} -\frac{\mathrm{d}}{\mathrm{d}x}\left(p\frac{\mathrm{d}u}{\mathrm{d}x}\right) + qu = \lambda\rho u$$

on a domain $x \in [0, L]$ with BCs

$$\alpha_1 u(0) - \beta\frac{\mathrm{d}u}{\mathrm{d}x}(0) = 0,$$
$$\alpha_1 u(L) + \beta\frac{\mathrm{d}u}{\mathrm{d}x}(L) = 0. \qquad \triangleleft$$

Definition 4.4.2: If a problem is of the form

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(p\frac{\mathrm{d}u}{\mathrm{d}x}\right) + qu = \lambda\rho u \equiv -(p(x)u')' + q(x)u = \lambda\rho(x)u$$

and if $p, \rho > 0$ and $q \geq 0$ and if $p, \rho, q, \frac{\mathrm{d}p}{\mathrm{d}x}$ are continuous on the interval $[0, L]$ then we call this problem a **Sturm-Lioville problem**. In addition, if we also have

$$\alpha_i \geq 0, \quad \beta_i \geq 0 \quad \text{and} \quad \alpha_i + \beta_i > 0 \text{ for } i = 1, 2$$

then we call this problem a *regular Sturm-Lioville problem.* $\triangleleft$

Definition 4.4.3:

1. We define the **Hermitian inner product** of two functions $\varphi(x)$ and $\psi(x)$ as

$$(\varphi(x), \psi(x)) \underset{\text{def}}{=} \int_0^L \rho(x)\varphi(x)\overline{\psi}(x)\,\mathrm{d}x,$$

   with respect to the weighing function $\rho(x) > 0$. Hermitian inner product can be complex.

2. For a complex number $\mathbb{C} \ni z = a + ib$ where $a, b \in \mathbb{R}$, we define the **complex conjugate** of z as $\overline{z} = \overline{a + ib} = a - ib$.

3. We define norm of $\varphi(x)$ with respect to Hermitian inner product as

$$\|\varphi\|^2 = (\varphi, \varphi) \underset{\text{def}}{=} \int_0^L \rho(x)\varphi(x)\overline{\varphi(x)}\,\mathrm{d}x = \int_0^L \rho|\varphi|^2\,\mathrm{d}x \geq 0 \text{ with } \|\varphi\|^2 = 0 \iff \varphi \equiv 0.$$

   This norm is also referred as the *two-norm.*

4. If norm of a function is finite, then such function is said to be **square integrable**.

5. If $\|\varphi\| = 1$ then we say $\varphi(x)$ is **normalized**.

6. We say two functions φ, ψ are orthogonal if $(\varphi, \psi) = 0$

7. We refer to a set of functions $\{\varphi_k(x)\}$ for $k = 1, \ldots$ as an **orthogonal set** if $(\varphi_i, \varphi_j) = 0$ for distinct $i, j = 1, \ldots$.

8. An orthogonal set where each function is normalized is called an **orthonormal set**.

9 We define the Fourier coefficients of $\varphi(x)$ with respect to the orthonormal set $\{\varphi_k(x)\}$ where $k = 1, 2, \ldots$ as

$$(\varphi, \varphi_k).$$

10 We define the **Fourier series** of $\varphi(x)$ with respect to the orthonormal set the orthonormal set $\{\varphi_k(x)\}$ where $k = 1, 2, \ldots$ as

$$\varphi(x) = \sum_{k=1}^{\infty} (\varphi, \varphi_k)\varphi_k(x). \qquad \triangleleft$$

4.4.1 Convergence

Consider the N^{th} partial sum of a Fourier series

$$\psi_N = \sum_{k=1}^{N} (\varphi, \varphi_k)\varphi_k(x).$$

We can determine whether the Fourier series converges or not by checking if the above limit exists as $N \to \infty$. One way to do this is to consider the difference between the function, $\varphi(x)$, and its N^{th} partial sum $\psi_N(x)$. Consider the following

$$\begin{aligned}
\|\varphi - \psi_N\|^2 = \left\| \varphi - \sum_{k=1}^{N} (\varphi, \varphi_k)\varphi_k \right\|^2 &= \left(\varphi - \sum_{k=1}^{N} (\varphi, \varphi_k)\varphi_k, \varphi - \sum_{k=1}^{N} (\varphi, \varphi_k)\varphi_k \right) \\
&= (\varphi, \varphi) + \left(\varphi, -\sum_{k=1}^{N} (\varphi, \varphi_k)\varphi_k \right) + \left(-\sum_{k=1}^{N} (\varphi, \varphi_k)\varphi_k, \varphi \right) \\
&\quad + \left(\sum_{k=1}^{N} (\varphi, \varphi_k)\varphi_k, \sum_{\ell=1}^{N} (\varphi, \varphi_\ell)\varphi_\ell \right) \\
&= \|\varphi\|^2 - 2\sum_{k=1}^{N} (\varphi, \varphi_k)^2 + \sum_{k=1}^{N} (\varphi, \varphi_k)^2 \\
&= \|\varphi\|^2 - \sum_{k=1}^{N} (\varphi, \varphi_k)^2 \geq 0. \qquad (4.4.1)
\end{aligned}$$

The inequality in (4.4.1) is known as the **Bessel's inequality**.

Definition 4.4.4: A sequence of square integrable functions $\{\psi_N(x)\}$ where $N = 1, 2, \ldots$ is said to **converge to a function** $\varphi(x)$ **in the mean** if

$$\lim_{N \to \infty} \|\varphi - \psi_N\| = 0.$$

This is also referred to as **mean square convergence**. If the equality

$$\sum_{k=1}^{\infty} (\varphi, \varphi_k)^2 = \|\varphi\|^2 \qquad (4.4.2)$$

holds, then

$$\lim_{N\to\infty} \|\varphi - \psi_N\|^2 = \lim_{N\to\infty} \left\| \varphi(x) - \sum_{k=1}^{N} (\varphi, \varphi_k)\varphi_k \right\|^2 = 0,$$

which means Fourier series converges in the mean square sense. The equality in (4.4.2) is known as **Parseval's equality**. ◁

Definition 4.4.5: A set of square integrable functions is called **complete** if for any square integrable function $\varphi(x)$, the Fourier series of $\varphi(x)$ converges in the mean. ◁

4.4.2 Properties of the $\mathcal{L}$ Operator

The $\mathcal{L}$ operator (Sturm-Lioville operator), which is given by,

$$\mathcal{L}[u] = -\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}v}{\mathrm{d}x}\right) + q(x)v = \rho(x)\lambda v,$$

with boundary conditions

$$\begin{aligned}\alpha_1 u(0) - \beta_1 \frac{\mathrm{d}u}{\mathrm{d}x}(0) = 0,\\ \alpha_2 u(L) + \beta_1 \frac{\mathrm{d}u}{\mathrm{d}x}(L) = 0\end{aligned}$$

with $\rho, p > 0$ and $q, \alpha_1, \beta_1 \geq 0$ with $\alpha_i + \beta_i > 0$ for $i = 1, 2$ has the following properties.

P1 **Eigenfunctions Corresponding to Different Eigenvalues are Orthogonal**

In order to show the orthogonality, we first show the self-adjointness. For this, we need to show the following:

$$\left(\frac{1}{\rho}\mathcal{L}[u], v\right) = \left(u, \frac{1}{\rho}\mathcal{L}[v]\right)$$

P1.a **Eigenfunctions are self-adjoint.**

To simply the calculations, we assume homogeneous Dirichlet BCs $u(0) = 0 = u(L)$. We have

$$\begin{aligned}\left(u, \frac{1}{\rho}\mathcal{L}[v]\right) = \left(\frac{1}{\rho}\mathcal{L}[u], v\right) &\iff \left(u, \frac{1}{\rho}\mathcal{L}[v]\right) - \left(\frac{1}{\rho}\mathcal{L}[u], v\right) = 0\\ &\iff \int_0^L \rho u \frac{1}{\rho}\big[-(p\overline{v}')' + q\overline{v}\big]\,\mathrm{d}x - \int_0^L \rho\overline{v}\frac{1}{\rho}\big[-(pu')' + qu\big]\,\mathrm{d}x = 0.\end{aligned}$$

By integration by parts we have

$$\left(u, \frac{1}{\rho}\mathcal{L}[v]\right) = \left(\frac{1}{\rho}\mathcal{L}[u], v\right) \iff \int_0^L pu'\overline{v}' - pu'\overline{v}'\,\mathrm{d}x = 0.$$

Since this is always equal to zero, then the operator $\frac{1}{\rho}\mathcal{L}$ is self-adjoint (even if we allow for complex functions).

P1.b **Eigenfunctions corresponding to distinct eigenvalues are orthogonal.**

Since

$$\frac{1}{\rho}\mathcal{L}[u] = \lambda_1 u \quad \text{and} \quad \frac{1}{\rho}\mathcal{L}[v] = \lambda_2 v,$$

after substituting we obtain

$$(u, \lambda_2 v) - (\lambda_1 u, v) = 0.$$

If eigenvalues are real (which we will show they are always real), then we can factor out and obtain

$$(\lambda_2 - \lambda_1)(u, v) = 0.$$

If eigenvalues are distinct, then $\lambda_2 - \lambda_1 \neq 0$. Then we must have $(u, v) = 0$. Hence eigenfunctions are orthogonal.

P2 **Eigenvalues are real and non-negative. Furthermore, eigenfunctions may be chosen to be real valued**

P2.a **Eigenvalues are real**.

Assume (λ, v) is an eigenpair. Then they must satisfy the eigenvalue relation, that is $\frac{1}{\rho}\mathcal{L}[v] = \lambda v$. Since $\rho(x)$ and $\mathcal{L}$ are real we compute the complex conjugate and get

$$\frac{1}{\rho}\mathcal{L}[\overline{v}] = \overline{\lambda}\overline{v}.$$

Observe that $(\overline{\lambda}, \overline{v})$ are also an eigenpair. Since $\frac{1}{\rho}\mathcal{L}$ is self-adjoint, then we must have

$$\left(v, \frac{1}{\rho}\mathcal{L}[v]\right) - \left(\frac{1}{\rho}\mathcal{L}[v], v\right) = 0$$

Hence, when we substitute the eigenvalue problem we get

$$(v, \lambda v) - (\lambda v, v) = \overline{\lambda}(v, v) - \lambda(v, v) = (\overline{\lambda} - \lambda)(v, v) = (\overline{\lambda} - \lambda)\|v\|^2 = 0.$$

For non-trivial solutions, we have $v \neq 0$. Hence we must have $\overline{\lambda} = \lambda \in \mathbb{R}$.

P2.b **Eigenvalues are non-negative**.

We now show that the eigenvalues are non-negative. We have

$$\left(v, \frac{1}{\rho}\mathcal{L}[v]\right) \equiv \int_0^L \rho v \frac{1}{\rho}\left[-(p\overline{v}')' + q\overline{v}\right] \mathrm{d}x = \int_0^L p v' \overline{v}' + q v \overline{v} \, \mathrm{d}x.$$

Here we assumed Dirichlet BCs and used integration by parts. Hence

$$\left(v, \frac{1}{\rho}\mathcal{L}[v]\right) \geq 0.$$

When we plug in the eigenvalue relation, we get

$$\left(v, \frac{1}{\rho}\mathcal{L}[v]\right) = (v, \lambda v) = \lambda\|v\|^2 \geq 0.$$

When we solve for λ we get

$$\lambda = \frac{\left(v, \frac{1}{\rho}\mathcal{L}[v]\right)}{\|v\|^2} \geq 0.$$

Hence the eigenvalue is non-negative.

We will show that we can take eigenfunctions to be real valued in the next lecture.

Properties of the $\mathcal{L}$ Operator (continued)

Last time we showed the following properties for the Sturm-Liouville operator

P1 Eigenfunctions corresponding to different eigenvalues are orthogonal (we showed this by first showing self-adjointness).

P2 Eigenvalues are real and non-negative (we already showed these as **P2.a** and **P2.b**), furthermore, eigenfunctions may be chosen to be real valued (we still need to show this).

P2.c **Eigenfunctions may be chosen to be real valued**.

Let $v \in \mathbb{C}$ be an eigenfunction of the system with the associated eigenvalue λ. Then $v = v_R + iv_I$ where $\mathbb{R} \ni v_R = \text{Re}\{v\}$ and $\mathbb{R} \ni v_I = \text{Im}\{v\}$. Since $\lambda \in \mathbb{R}$ and since v must satisfy the eigenvalue relations, then we must have

$$\frac{1}{\rho}\mathcal{L}[v] = \lambda v \implies \frac{1}{\rho}\mathcal{L}[v_R + iv_I] = \lambda(v_R + iV_i).$$

By linearity we have

$$\frac{1}{\rho}\mathcal{L}[v_R] + i\frac{1}{\rho}\mathcal{L}[v_I] = \lambda v_R + i\lambda v_I.$$

Since the real and imaginary parts must vanish, we must have

$$\left.\begin{aligned} \frac{1}{\rho}\mathcal{L}[v_R] &= \lambda v_R \\ \frac{1}{\rho}\mathcal{L}[v_I] &= \lambda v_I. \end{aligned}\right\} \quad \begin{aligned} &\text{these are eigenrelations for } \frac{1}{\rho}\mathcal{L}, \\ &\text{and if we solve each of these we get a real eigenfunction} \end{aligned}$$

Hence, each eigenfunction can be taken as real valued.

P3 Each Eigenvalue is Simple

In other words, each eigenvalue has a multiplicity of one.

P4 There Exists a Countably Infinite Number of Eigenvalues Having a Limit of Infinity

In other words, the set of eigenvalues can be ordered as follows:

$$0 \leq \lambda_1 < \lambda_2 < \dots \quad \text{with} \quad \lambda_k \to \infty \text{ as } k \to \infty.$$

Definition 4.4.6: The set of eigenvalues of an operator is called the **spectrum of the operator**. ◁

P5 The Set of Eigenfunctions $\{v_k(x)\}_{k=1,2,\dots}$ Forms a *Complete* Orthonormal Set of Square Integrable Functions on $0 < x < L$

The eigenfunction expansion (Fourier series) of $v(x)$ converges in the mean

$$v(x) = \sum_{k=1}^{\infty} (v, v_k) v_k.$$

If $v(x)$ is continuous, then it can be shown that the series converges uniformly to $v(x)$ in the interval question.

4.4.3 Examples

4.4.3.1 5 Steps for Solving PDEs with Separation of Vars. and Eigenfunction Expansions

Remark 4.4.7: We use the following steps to analytically solve a PDE with separation of variables.

1. **Separating the Variables:** Assume the solution is a multiplication of functions each with a single variable and obtain separate ODEs for each.
2. **Solving for the BVP and One of the Separated Variables:** We try to impose homogeneous BCs and usually start with solving for the spatial solution.
3. **Solving for the Other Separated Variable:** We solve for the remaining solutions by imposing the remaining BCs.
4. **Forming the General Solution:** We use the principle of superposition to obtain the general solution.
5. **Imposing ICs:** We impose the ICs on the general solution (usually to obtain the Fourier coefficients). ◁

4.4.3.2 Fourier Sine Series

We start with $\rho(x) = 1$. We have the following BVP.

$$\left.\begin{aligned} &-\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = \lambda v \\ &v(0) = 0 = v(L) \quad \text{where} \quad 0 < x < L. \end{aligned}\right\} \text{BVP}$$

Previously, we found the eigenvalues as $\lambda_n = \left(n\frac{\pi}{L}\right)^2$ where $n = 1, 2, \ldots$. The orthogonal eigenfunctions are

$$v_n(x) = \sin\left(n\frac{\pi}{L}x\right).$$

To normalize, we want $\|v_n\|^2 = 1$. So we find $\|v_n\|^2 = (v_n, v_n)$.

$$\|v_n\|^2 = (v_n, v_n) = \int_0^L \sin^2\left(n\frac{\pi}{L}x\right) = \frac{1}{2}\int_0^L 1 - \cos\left(2n\frac{\pi}{L}x\right)\mathrm{d}x = \frac{L}{2}.$$

The orthonormal series is

$$\hat{v}_n(x) = \sqrt{\frac{2}{L}}\sin\left(n\frac{\pi}{L}x\right) \quad \text{where} \quad n = 1, 2, \ldots$$

4.4.3.3 Fourier Cosine Series

We start with $\rho(x) = 1$. We have the following BVP.

$$\left.\begin{aligned} &-\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = \lambda v \\ &\frac{\mathrm{d}v}{\mathrm{d}x}(0) = 0 = \frac{\mathrm{d}v}{\mathrm{d}x}(L) \quad \text{where} \quad 0 < x < L \end{aligned}\right\} \text{BVP}$$

We found the eigenvalues and eigenfunctions

$$\lambda_n = \left(n\frac{\pi}{L}\right)^2 \quad \text{and} \quad \cos\left(n\frac{\pi}{L}x\right) \text{ where } n = 0, 1, 2, \ldots$$

To normalize, we consider two cases.

$$\begin{aligned} n &= 1, 2, \ldots & &\Longrightarrow & \|v_n\|^2 &= \frac{L}{2}, \\ n &= 0 & &\Longrightarrow & \|v_0\|^2 &= L \end{aligned}$$

Orthonormal functions are

$$v_n(x) = \begin{cases} \dfrac{1}{\sqrt{L}} & \text{for } n = 0, \\ \sqrt{\dfrac{2}{L}}\cos\left(n\dfrac{\pi}{L}x\right) & \text{for } n = 1, 2, \ldots \end{cases}$$

4.4.3.4 Fourier Series

We start with $\rho(x) = 1$. We have the following BVP.

$$\left.\begin{aligned} &-\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = \lambda v \\ &v(0) = v(L) \\ &\frac{\mathrm{d}v}{\mathrm{d}x}(0) = \frac{\mathrm{d}v}{\mathrm{d}x}(L) \quad \text{where} \quad 0 < x < L \end{aligned}\right\} \text{BVP}$$

Since the general solution is a sum of two linearly independent functions, we get

$$v(x) = a\cos\left(\sqrt{\lambda}\,x\right) + b\sin\left(\sqrt{\lambda}\,x\right).$$

To ensure the solution is L-periodic, we must pick λ so that the trig functions fit in the domain.

$$\sqrt{\lambda}\,L = n2\pi \quad \text{or} \quad \lambda_n = \left(2n\frac{\pi}{L}\right)^2.$$

So, our orthogonal eigenfunctions

$$\boxed{1, \quad \cos\left(2n\frac{\pi}{L}x\right), \quad \sin\left(2n\frac{\pi}{L}x\right) \text{ where } n = 1, 2, \ldots}$$

To find orthonormal basis, we compute the norm of each function.

$$\begin{aligned} \hat{v}_0(x) &= \frac{1}{\sqrt{L}} \\ \hat{v}_n(x) &= \sqrt{\frac{2}{L}}\cos\left(2n\frac{\pi}{L}x\right) \\ \hat{u}_n(x) &= \sqrt{\frac{2}{L}}\sin\left(2n\frac{\pi}{L}x\right) \quad \text{where} \quad n = 1, 2, \ldots \end{aligned}$$

Recall from Fourier analysis that these form a complete basis and can reproduce functions with Fourier series. Given $v(x)$,

$$\boxed{v(x) = (v, \hat{v}_0)\hat{v}_0 + \sum_{k=1}^{\infty}(v, \hat{v}_k)\hat{v}_k + (v, \hat{u}_k)\hat{u}_k.}$$

4.4.3.5 Bessel Series

The Bessel's equation is given by

$$x^2 v'' + xv' + (x^2 - n^2)v = 0 \quad \text{where} \quad n \in \mathbb{Z}_+.$$

When we make the substitution $x \to \sqrt{\lambda}\,z$ and $v(x) \to v(z)$, we get

$$z^2 v'' + zv' + (\lambda z^2 - n^2)v = 0,$$

which is equivalent to

$$\lambda z^2 v = -z^2 v'' - z v' + n^2 v = 0,$$

which is not necessarily in Sturm-Liouville form. We divide the above expression by z. We have

$$-zv'' - v' + \frac{n^2}{z} v = \lambda z v,$$

equivalently,

$$-\frac{\mathrm{d}}{\mathrm{d}z}\left(z \frac{\mathrm{d}v}{\mathrm{d}z} \right) + \frac{n^2}{z} v = \lambda z v.$$

This looks like our Sturm-Liouville equation with

$$\rho(z) = z, \quad q(z) = \frac{n^2}{z}, \quad \rho(z) = z.$$

If our domain is $0 < z < L$, then,

$$\text{as } z \to 0, \text{ we have } \begin{aligned} p &\to 0, \\ \rho &\to 0, \\ q &\to \infty. \end{aligned}$$

In order to close the problem, we need to impose certain boundary conditions.

We assume that $v(0)$ is bounded and $v(L) = 0$.

To show the eigenfunctions are orthogonal, we must adapt our theory to this BVP.

Self-Adjointness

We want to show $\left(u, \frac{1}{\rho}\mathcal{L}[w]\right) = \left(\frac{1}{\rho}\mathcal{L}[u], w\right)$. We have

$$\begin{aligned}
\left(u, \frac{1}{\rho}\mathcal{L}[w]\right) - \left(\frac{1}{\rho}\mathcal{L}[u], w\right) &= \int_0^L zu\frac{1}{z}\left[-(zw')' + \frac{n^2}{z}w\right] \mathrm{d}z - \int_0^L z\frac{1}{z}\left[-(zu')' + \frac{n^2}{z}u\right] w \, \mathrm{d}z \\
&= \int_0^L -u(zw')' + w(zu')' \, \mathrm{d}z \\
&= \Big[-zuw' + zwu' \Big]_0^L + \int_0^L zu'w' - zu'w' \, \mathrm{d}z \qquad (\dagger) \\
&= [-Lu(L)w'(L) + Lw(L)u'(L)] - [-0 \cdot u(0)w'(0) + 0 \cdot w(0)u'(0)] \\
&= 0,
\end{aligned}$$

where in step ($\dagger$) we used integration by parts. Hence, the operator is self-adjoint.

Orthogonality

Let (λ_i, u_i) and (λ_j, u_j) be two eigenpairs that solve the system where $\lambda_i \neq \lambda_j$. Since the operator is self-adjoint, then

$$\left(u_i, \frac{1}{\rho}\mathcal{L}[u_j]\right) - \left(\frac{1}{\rho}\mathcal{L}[u_i], u_j\right) = 0.$$

When we sub in the eigenvalue relation, we obtain

$$(u_i, \lambda_j u_j) - (\lambda_i u_i, u_j) = (\lambda_j - \lambda_i)(u_i, u_j) = 0.$$

Since $\lambda_i \neq \lambda_j$, then we must have $(u_i, u_j) = 0$. Hence, the eigenfunctions corresponding to distinct eigenvalues are orthogonal.

General Solution

In AMATH 351, we solve Bessel's equation using power series. We obtain two linearly independent solutions

$$v(z) = c_1 J_n(\sqrt{\lambda}\, z) + c_2 Y_n(\sqrt{\lambda}\, z).$$

J_n is the Bessel function of order n of the 1^{st} kind and *Y_n is the Bessel function of order n of the 2^{nd} kind.*

Remark 4.4.8: For all n, as $z \to 0$, $Y_n(z) \to \infty$. For solutions to be bounded at $z = 0$, we need $c_2 = 0$. Hence, we get

$$\boxed{v(z) = J_n(\sqrt{\lambda}\, z).}$$

To find eigenvalues, we must impose 2^{nd} boundary conditions where

$$J_n(\sqrt{\lambda}\, L) = 0.$$

This equation determines eigenvalues. ◁

Bessel Series (continued)

Last time we showed the general solution (viewed as an eigenvalue problem) is

$$v(z) = c_1 J_n(\sqrt{\lambda}\, z) + c_2 Y_n(\sqrt{\lambda}\, z)$$

with the boundary conditions

$$v(0) \text{ is bounded} \quad \text{and} \quad v(L) = 0,$$

we got $c_2 = 0$ and obtained

$$v(z) = J_n(\sqrt{\lambda}\, z) = 0$$

Aside: This is analogous to $\sin\left(\sqrt{\lambda}L\right)$. Define α_{kn} to be the k-th root of $J_n(z)$. With this, we can compute the eigenvalues to be

$$\sqrt{\lambda}L = \alpha_{kn} \implies \lambda_{kn} = \left(\frac{\alpha_{kn}}{L}\right)^2 \quad \text{where} \quad k = 1, 2, \ldots,$$

with the corresponding eigenfunctions

$$v_{kn}(z) = J_n\left(\frac{\alpha_{kn}}{L}z\right). \qquad \triangleleft$$

Normalization

Since we have

$$\|v_{kn}\|^2 = \int_0^L zJ_n^2(\sqrt{\lambda_{kn}}z)\,\mathrm{d}z = \frac{L^2}{2}J_{n+1}^2(\sqrt{\lambda_{kn}}L),$$

we get

$$\hat{v}_{kn}(z) = \frac{\sqrt{2}}{L}\frac{J_n(\alpha_{kn}\frac{z}{L})}{J_{n+1}(\alpha_{kn})} \quad \text{with} \quad k = 1, 2, \ldots$$

4.4.3.6 Fourier-Bessel Series

For $v(z)$ defined on $0 < z < L$ that satisfies the same BCs hen

$$v(z) = \sum_{k=1}^{\infty}(v, \hat{v}_{kn})\hat{v}_{kn}(z).$$

4.4.3.7 Wave Equation

We have the wave equation with the boundary conditions (clamped down) and initial conditions as

$$\left.\begin{aligned} \frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \quad \text{where} \quad 0 < x < L \end{aligned}\right\} \text{PDE}$$

$$\left.\begin{aligned} u(0,t) = 0 = u(L,t), \quad \text{where} \quad t > 0, \end{aligned}\right\} \text{BCs}$$

$$\left.\begin{aligned} u(x,0) &= f(x) \\ \frac{\partial u}{\partial t}(x,0) &= g(x), \quad \text{where} \quad 0 < x < L. \end{aligned}\right\} \text{ICs}$$

1 **Separating the Variables:** We assume $u(x,t) = M(x)N(t)$ and plug it into the PDE. After simplifying, we get

$$MN'' = c^2M''N \implies \frac{N''}{c^2N} = \frac{M''}{M} = -\lambda.$$

We obtain the ODEs

$$\begin{aligned} N'' + \lambda c^2 N &= 0 \\ M'' + \lambda M &= 0. \end{aligned}$$

We separate the BCs and obtain

$$\begin{aligned} M(0)N = 0 &\implies M(0) = 0, \\ M(L)N = 0 &\implies M(L) = 0. \end{aligned}$$

Hence we get the BVP for spatial component as

$$\left.\begin{aligned} -M'' &= \lambda M \\ M(0) &= 0 = M(L) \end{aligned}\right\} \text{BVP}$$

② **Solving the BVP for Spatial Variable:** We have the general solution as

$$M = c_1 \sin\left(\sqrt{\lambda}\, x\right) + c_2 \cos\left(\sqrt{\lambda}\, x\right).$$

With the BCs, we get $M(0) = 0 \implies c_2 = 0$. Hence we have

$$M = c_1 \sin\left(\sqrt{\lambda}\, x\right).$$

We also have $M(L) = c_1 \sin\left(\sqrt{\lambda}\, L\right) = 0$, which gives us

$$\sqrt{\lambda}\, L = n\pi \implies \lambda_n = \left(n\frac{\pi}{L}\right)^2.$$

Hence we obtain

$$\begin{aligned} M_n(x) &= \sin\left(n\frac{\pi}{L}x\right) \\ \implies \hat{M}_n(x) &= \sqrt{\frac{2}{L}} \sin\left(n\frac{\pi}{L}x\right). \end{aligned}$$

③ **Solving for the Temporal Equation:** For the temporal component, $N(t)$, we have

$$N'' + \lambda^2 c^2 N = 0 \implies N'' + \left(n\frac{\pi}{L}\right)^2 c^2 N = 0.$$

This has the solution

$$N(t) = a_n \sin\left(n\frac{\pi}{L}ct\right) + b_n \cos\left(n\frac{\pi}{L}ct\right).$$

④ **Forming the General Solution to PDE:** By superposition of the all solutions, we get

$$u(x,t) = \sum_{n=1}^{\infty} N_n(t) M_n(x) = \sum_{n=1}^{\infty} \left[a_n \sin\left(n\frac{\pi}{L}t\right) + b_n \cos\left(n\frac{\pi}{L}t\right)\right] \sqrt{\frac{2}{L}} \sin\left(n\frac{\pi}{L}x\right). \tag{4.4.3}$$

⑤ **Imposing ICs:** We project the ICs onto eigenfunction M_j. From the ICs we get

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \hat{M}_n(x) \implies b_n = (f, \hat{M}_n).$$

We also have

$$\frac{\partial u}{\partial t}(x,0) = g(x) = \sum_{n=1}^{\infty} a_n \left(n\frac{\pi}{L}c\right)\hat{M}_n(x) \implies a_n = \left(\frac{L}{n\pi c}\right)(g, \hat{M}_n).$$

Wave Equation (continued)

Recall 4.4.9: We found the solution for wave equation in (4.4.3), where

$$a_n = \frac{L}{n\pi c}(g, \hat{M}_n(x)) \quad \text{and} \quad b_n = (f, \hat{M}_n(x)).$$

Note that we have the frequency of the oscillation as

$$\omega_n = \frac{n\pi c}{L}.$$

◁

Energetics of Wave Equation: The PDE can be written as

$$\rho\frac{\partial^2 u}{\partial t^2} = T\frac{\partial^2 u}{\partial x^2}.$$

To find how the energy changes in time we multiply by $\frac{\partial u}{\partial t}$ and integrate over domain. We have

$$\int_0^L \left[\rho\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2}\right] \mathrm{d}x = \int_0^L \left[T\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2}\right] \mathrm{d}x \iff \int_0^L \frac{\rho}{2}\frac{\partial}{\partial t}\left[\left(\frac{\partial u}{\partial t}\right)^2\right] \mathrm{d}x = \int_0^L T\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\right) - \frac{\partial^2 u}{\partial t \partial x}\frac{\partial u}{\partial x}\right] \mathrm{d}x.$$

The BCs that we imposed are

$$u(0,t) = 0 = u(L,t).$$

Since $\frac{\partial u}{\partial t}(0,t) = 0 = \frac{\partial u}{\partial t}(L,t)$, we have the first term on RHS as

$$\int_0^L T\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\right)\right] \mathrm{d}x = T\left[\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\right]_0^L = 0.$$

We have the second term of RHS as

$$-T\int_0^L \frac{\partial^2 u}{\partial t \partial x}\frac{\partial u}{\partial x} \mathrm{d}x = -\frac{T}{2}\int_0^L \frac{\partial}{\partial t}\left(\left(\frac{\partial u}{\partial x}\right)^2\right) \mathrm{d}x.$$

We combine the two non-zero terms and obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \frac{\rho}{2}\left(\frac{\partial u}{\partial t}\right)^2 \mathrm{d}t + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \frac{T}{2}\left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x = 0 \implies \frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \left[\frac{\rho}{2}\left(\frac{\partial u}{\partial t}\right)^2 + \frac{T}{2}\left(\frac{\partial u}{\partial x}\right)^2\right] \mathrm{d}x.$$

We denote the ***kinetic energy density*** as the term

$$\frac{\rho}{2}\left(\frac{\partial u}{\partial t}\right)^2,$$

and the ***total kinetic energy*** as

$$\int_0^L A\frac{\rho}{2}\left(\frac{\partial u}{\partial t}\right)^2 \mathrm{d}x, \tag{4.4.4}$$

where A is the cross sectional area. Following the argument from before, we define the ***potential energy density*** as

$$\frac{T}{2}\left(\frac{\partial u}{\partial x}\right)^2,$$

and the total potential energy as

$$\int_0^L A\frac{T}{2}\left(\frac{u}{x}\right)^2 \mathrm{d}x. \tag{4.4.5}$$

The Equations (4.4.4) and (4.4.5) state that the sum of KE and PE (total energy) is conserved. These equations express the ***conservation of energy***.

Total Energy of the String: We have

$$\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} \big[a_n \cos(\omega_n t) + b_n \sin(\omega_n t)\big]\hat{M}_n(x) \\
\frac{\partial u}{\partial t}(x,t) &= \sum_{n=1}^{\infty} \omega_n \big[-a_n \sin(\omega_n t) + b_n \cos(\omega_n t)\big]\hat{M}_n(x) \\
\frac{\partial u}{\partial x}(x,t) &= \sum_{n=1}^{\infty} \big[a_n \cos(\omega_n t) + b_n \sin(\omega_n t)\big]\frac{\mathrm{d}\hat{M}_n}{\mathrm{d}x}(x),
\end{aligned}$$

where

$$\frac{\mathrm{d}\hat{M}_n}{\mathrm{d}x} = n\frac{\pi}{L}\sqrt{\frac{2}{L}}\cos\left(n\frac{\pi}{L}x\right).$$

After some algebra we find

$$\boxed{\int_0^L \frac{\rho}{2}\left(\frac{\partial u}{\partial t}\right)^2 + \frac{T}{2}\left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x = \sum_{n=1}^{\infty}\left(n\frac{\pi}{L}\right)^2\frac{T}{2}(a_n^2+b_n^2),}$$

which is closely related with the Parseval's theorem. The total energy (which is an integral) is equal to a sum of the square of all Fourier coefficients. In particular, the eigenfunction n has

$$\left(n\frac{\pi}{L}\right)\frac{T}{2}\big[a_n^2+b_n^2\big].$$

amount of energy which is conserved for all time.

4.4.3.8 Diffusion Equation

We have the diffusion equation as

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial t^2}$$

on the domain $0 < x < L$ and $t > 0$. We impose the Neumann BCs,

$$\frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(L,t) \quad \text{where} \quad t > 0$$

and the initial condition

$$u(x,0) = f(x) \quad \text{on the domain} \quad 0 < x < L.$$

Conservation of Mass: If the total amount of dye is conserved, then

$$\frac{\partial}{\partial t}\int_0^L u \, \mathrm{d}x = 0.$$

To determine if the total mass is conserved, we integrate the PDE over the domain. We have

$$\int_0^L \frac{\partial u}{\partial t}\, \mathrm{d}x = \int_0^L D\frac{\partial^2 u}{\partial x^2}\, \mathrm{d}x \iff \frac{\mathrm{d}}{\mathrm{d}t}\int_0^L u \, \mathrm{d}x = \int_0^L D\frac{\partial^2 u}{\partial x^2}\, \mathrm{d}x = \int_0^L \frac{\partial}{\partial x}\left(D\frac{\partial u}{\partial x}\right) \mathrm{d}x.$$

On RHS, by FTC we have

$$\left[D\frac{\partial u}{\partial x}\right]_0^L,$$

which is zero due to Neumann BCs. Hence, the total mass is conserved. Due to this nature, Neumann BCs are called *insulating boundary conditions*. When we have homogeneous Dirichlet BCs, (which states the solution itself is zero at the boundary), the mass reaches zero given enough time. Hence, the mass is not conserved under Dirichlet BCs.

Exercise 4.4.10: Read section 4.4.2 in the course notes about finding a solution to diffusion equation. Looking at solutions to diffusion equation with either Dirichlet or Neumann BCs, why does one conserve total mass and not the other? $\triangleleft$

Remark: Assignment #5 is skipped, assignment #6 will be posted later this week. $\triangleleft$

4.4.3.9 Laplace's Equation on a Rectangle

We have the Laplace's equation as

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{on the boundary} \quad \begin{array}{l} 0 < x < L_x, \\ 0 < y < L_y. \end{array}$$

The boundary conditions we pick are

$$\begin{aligned} &u(x,0) = f(x) \quad \text{and} \quad u(x,L_y) = g(x), \\ &u(0,y) = 0 \quad \text{and} \quad u(L_x,y) = 0. \end{aligned}$$

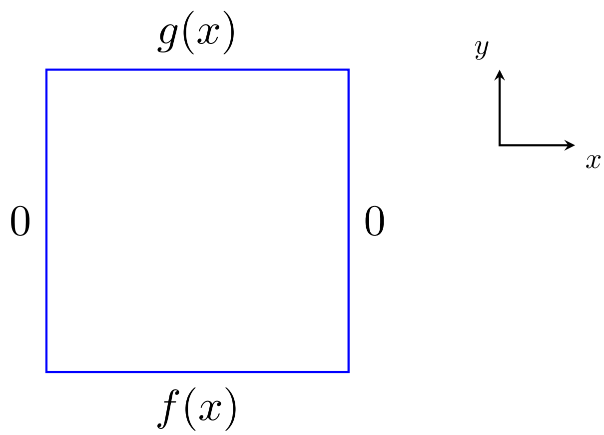


Figure 4.4.1: Laplace's equation on a rectangle.

We use the 5 steps we discussed in Remark 4.4.7 to solve the problem as follows.

1 **Separation of Variables:** We assume the solution is of the form $u(x, y) = M(x)N(y)$ and substitute this into PDE and simplify. We obtain

$$M''N + MN'' = 0 \implies \frac{M''}{M} + \frac{N''}{N} = 0 \implies \frac{N''}{N} = -\frac{M''}{M} = \lambda.$$

We obtain two ODEs,

$$N'' - \lambda N = 0 \quad \text{and} \quad M'' + \lambda M = 0.$$

We separate the homogeneous BCs. When we try

$$u(x, 0) = f(x) \implies M(x)N(0) = f(x),$$

we don't get homogeneous BC on M. We cannot build a complete set of eigenfunctions. We can only separate homogeneous BCs.

$$\begin{aligned} u(0.y) &= M(0)N(y) = 0 \implies M(0) = 0, \\ u(L_x, y) &= M(L)N(y) = 0 \implies M(L_x) = 0. \end{aligned}$$

Hence, we get our BVP as

$$\begin{aligned} &M'' + \lambda M = 0, \\ &M(0) = 0 = M(L_x). \end{aligned}$$

2 **Solving the BVP:** The ODE we want to solve is

$$M'' + \lambda M = 0 \quad \text{with} \quad M(0) = 0 = M(L_x),$$

which has the solution

$$\lambda_n = \left(n\frac{\pi}{L_x}\right)^2 \quad \text{with} \quad \hat{M}_n = \sqrt{\frac{2}{L_x}} \sin\left(n\frac{\pi}{L_x}x\right).$$

③ **Solving for $N(y)$:** The equation looks like

$$N_n'' - \lambda_n N_n = 0.$$

After we plug in the eigenvalues we obtain

$$N_n'' - \left(n\frac{\pi}{L_x}\right) N_n = 0.$$

The solution can be written in different ways

$$N_n(y) = \widetilde{a}_n \exp\left(n\frac{\pi}{L_x}y\right) + \widetilde{b}_n \exp\left(-n\frac{\pi}{L_x}y\right),$$

or, equivalently,

$$N_n(y) = a_n \sinh\left(n\frac{\pi}{L_x}y\right) + b_n \cosh\left(-n\frac{\pi}{L_x}y\right).$$

As long as the two solutions are linearly independent and solve the problem, we can rewrite it in various ways.

④ **Forming the General Solution:** A solution is

$$u_n(x,y) = M_n(x)N_n(y) = \sqrt{\frac{2}{L_x}}\sin\left(n\frac{\pi}{L_x}x\right)\left[a_n \sinh\left(n\frac{\pi}{L_x}y\right) + b_n \cosh\left(n\frac{\pi}{L_x}y\right)\right].$$

The general solution is

$$u(x,y) = \sum_{n=1}^{\infty} \hat{M}_n(x)N_n(y) = \sum_{n=1}^{\infty} \hat{M}_n(x)\left[a_n \sinh\left(n\frac{\pi}{L_x}y\right) + b_n \cosh\left(n\frac{\pi}{L_x}y\right)\right].$$

⑤ **Finding the Fourier Coefficients:** We have the first boundary condition as

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} \hat{M}_n(x) b_n \implies b_n = (f, \hat{M}_n)$$

For the second boundary condition, we have

$$u(x,L_y) = g(x) = \sum_{n=1}^{\infty} \hat{M}_n(x)\left[a_n \sinh\left(n\frac{\pi}{L_x}L_y\right) + b_n \cosh\left(n\frac{\pi}{L_x}L_y\right)\right].$$

We take the inner product with $\hat{M}_k$. We get

$$(g, \hat{M}_k) = a_k \sinh\left(k\frac{\pi}{L_x}L_y\right) + b_k \cosh\left(k\frac{\pi}{L_x}L_y\right).$$

Which gives us

$$a_k = \frac{(g, \hat{M}_k) - b_k \cosh\left(k\frac{\pi}{L_x}L_y\right)}{\sinh\left(k\frac{\pi}{L_x}L_y\right)}.$$

Remark 4.4.11: What do we do if we have non-zero BCs on all four sides of the rectangle?

By linearity we can add the solutions for the homogeneous BCs to get the solution for suitable BC as follows.

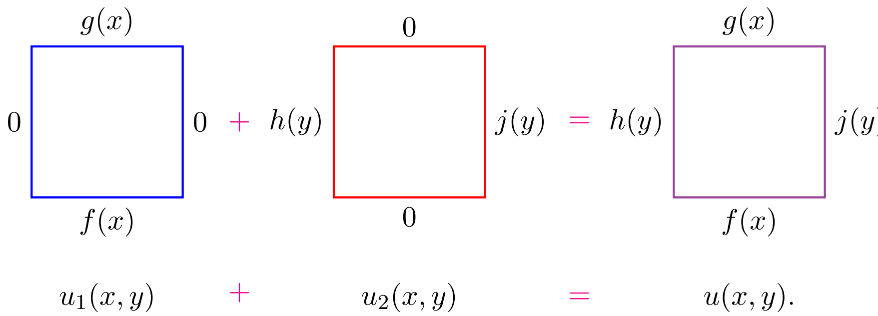


Figure 4.4.2: Using linearity of Laplace's equation.

Note that if $\nabla^2 u_1 = 0 = \nabla^2 u_2$, then $\nabla^2(u_1 + u_2) = 0$. Since Laplace's equation is linear, then we can build a solution to the general problem by solving two simpler problems. ◁

4.4.3.10 Laplace's Equation on a Circle

Recall that previously we wrote the Laplace's equation in Cartesian coordinates as

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

We start solving Laplace's equation in circle.

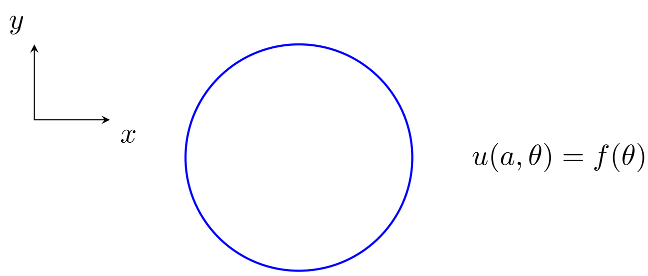


Figure 4.4.3: Laplace's equation on a circle.

We will impose the BCs on the perimeter of the circle. Laplace's equation in polar coordinates is

$$\nabla u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \quad \text{on the domain} \quad \begin{array}{l} 0 \leq \theta < 2\pi, \\ 0 \leq r \leq a, \end{array}$$

with the *explicit boundary condition*,

$$u(a, \theta) = f(\theta).$$

We have the *implicit boundary conditions*

$$u(0, \theta) \text{ is bounded},$$

with the *periodic boundary conditions*

$$\begin{aligned} u(r,0) &= u(r,2\pi), \\ \frac{\partial u}{\partial \theta}(r,0) &= \frac{\partial u}{\partial \theta}(r,2\pi). \end{aligned}$$

We use the 5 steps we discussed in Remark 4.4.7 to solve the problem as follows.

① **Separation of Variables:** We assume the solution is of the form $u(r,\theta) = R(r)\Theta(\theta)$. We substitute this into the PDE and simplify by multiplying the expression by $\frac{R\Theta}{r^2}$ to get

$$\begin{aligned} \Theta\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R}{\mathrm{d}r}\right) + \frac{1}{r^2}\Theta'' R = 0 &\implies \frac{r(rR')'}{R} + \frac{\Theta''}{\Theta} = 0 \\ &\implies \frac{r(rR')'}{R} = -\frac{\Theta''}{\Theta}. \end{aligned}$$

We recall that we specified our BVPs as $u(a,\theta) = f(\theta)$ in Figure 4.4.3. The BVPs in the θ direction. This is because we have periodic BCs which are not Sturm-Liouville which yield a complete set of eigenfunctions. Also, since the Dirichlet BC at $r = a$ is inhomogeneous that gives problems in BVP.

We obtain the two ODEs as

$$\begin{aligned} & r(rR)' - \lambda R = 0 \\ \implies & r^2R'' + rR' - \lambda R = 0, \end{aligned} \quad \text{and} \quad \Theta'' + \lambda\Theta = 0.$$

We separate the BCs in θ direction as follows.

$$\begin{aligned} \Theta(0)R &= \Theta(2\pi)R \implies \Theta(0) = \Theta(2\pi), \\ \Theta'(0)R &= \Theta'(2\pi)R \implies \Theta'(0) = \Theta'(2\pi). \end{aligned}$$

② **Solving the BVP:** We solve the BVP as follows.

$$\begin{aligned} \Theta'' + \lambda\Theta &= 0, \\ \Theta(0) &= \Theta(2\pi), \\ \Theta'(0) &= \Theta'(2\pi). \end{aligned}$$

This has the solution as

$$\lambda_n = n^2 \quad \text{and} \quad \Theta_n = A_n\cos(n\theta) + B_n\sin(n\theta) \text{ where } n = 0, 1, \ldots$$

③ **Solving for $R(r)$:** We need to solve

$$r^2R'' + rR' - n^2R = 0.$$

This ODE is of the Euler type. We find our solution with $R = r^\alpha$ (that is, we assume R is a polynomial). After the substitution we get

$$(\alpha)(\alpha-1)r^{\alpha+2} + \alpha r^{\alpha+1} - n^2r^\alpha = 0 \implies [\alpha^2 - \alpha + \alpha - n^2]r^\alpha = 0.$$

Since this is true for all , then we must have

$$[\alpha^2 - \alpha + \alpha - n^2] = 0 \implies \alpha = \pm n.$$

Hence, we obtain the solution to this ODE as $R = D_n r^n + E_n r^{-n}$ where D_n, E_n are constants. Since we have

$$u(0, \theta) < \infty \implies R(0) < \infty.$$

Hence we must have $E_n = 0$. Note that if $n = 0$ (then $\lambda = 0$) we get

$$r(rR')' = 0 \implies (rR')' = 0.$$

We integrate this to obtain

$$rR' = D_n \implies R' = \frac{D_n}{r}.$$

We integrate again and obtain

$$R(0) = D_0 \ln r + \text{ constant}.$$

Note that this solution has two linearly independent components, $\ln r$ and 1. Since R_n must be bounded at $r = 0$, we must drop $\ln r$. This forces us to have D_0. Then

$$R_0 = \text{constant}.$$

We denote this constant to be D_0. Which gives us $R_0 = D_0$. Hence, for all $n = 0, 1, \ldots$ we get

$$R_n(r) = D_n r^n.$$

4 **Forming the General Solution:** We have the general solution as

$$u(r, \theta) = \sum_{n=0}^{\infty} R_n(r)\Theta_n(\theta) = \sum_{n=0}^{\infty} D_n r^n [A_n \cos(n\theta) + B_n \sin(n\theta)].$$

After combining what we found previously (merging D_n to A_n, B_n), we find

$$\sum_{n=0}^{\infty} r^n [A_n \cos(n\theta) + B_n \sin(n\theta)].$$

5 **Finding the Fourier Coefficients:** We have the BC as

$$u(a, \theta) = f(\theta) = \sum_{n=0}^{\infty} a^n [A_n \cos(n\theta) + B_n \sin(n\theta)],$$

which gives us the Fourier series decomposition for $f(\theta)$. Since $\sin(n\theta)$ and $\cos(n\theta)$ are orthogonal, we can compute A_n and B_n using trig formulas. We get

$$A_n = \frac{(f, \cos(n\theta))}{a^n \|\cos(n\theta)\|^2} \quad \text{and} \quad B_n = \frac{(f, \sin(n\theta))}{a^n \|\sin(n\theta)\|^2}.$$

4.4.3.11 Vibrating Membrane (circular)

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u = c^2 \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right],$$

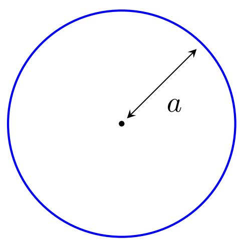
with BC

$$u(a, \theta, t) = 0,$$

and with the ICs

$$\begin{aligned} u(r, \theta, 0) \quad &= f(r, \theta), \\ \frac{\partial u}{\partial t}(r, \theta, 0) &= g(r, \theta). \end{aligned}$$

Note that the boundary condition implies



$$u(a, \theta, t) = 0.$$

Figure 4.4.4: Boundary conditions on vibrating circular membrane.

We use the 5 steps we discussed in Remark 4.4.7 to solve the problem as follows.

1 **Separation of Variables:** We assume the solution is of the form

$$u(r, \theta, t) = R(r)\Theta(\theta)T(t)$$

and sub this into the PDE and divide the expression by $c^2 R\Theta T$. We obtain

$$R\Theta T'' = c^2 \left[\Theta T \frac{1}{r}(rR')' + \frac{1}{r^2}\Theta'' RT \right] \implies \frac{T''}{c^2 T} = \frac{1}{R}\frac{1}{r}(rR')' + \frac{1}{r^2}\frac{\Theta''}{\Theta} = -\lambda.$$

We get one of the ODEs as

$$T'' + \lambda c^2 T = 0.$$

For the second ODE we have

$$\frac{1}{rR}(rR')' + \frac{1}{r^2}\frac{\Theta''}{\Theta} = -\lambda \implies \frac{r}{R}(rR')' + r^2\lambda = -\frac{\Theta''}{\Theta} = \mu.$$

We pick the RHS as μ since this yields the eigenfunctions in θ. When we separate periodic BCs (which are not stated in the beginning but they are implicitly there), we get

$$\left.\begin{aligned} &\Theta'' + \mu\Theta = 0, \\ &\Theta(0) = \Theta(2\pi), \\ &\Theta'(0) = \Theta'(2\pi). \end{aligned}\right\} \text{(BVP 1)}$$

Hence, we get the last ODE as

$$\underbrace{r(rR')' + r^2\lambda R - \mu R = 0 \implies r^2R'' + rR' + (r^2\lambda - \mu)R = 0, \quad \text{with the BCs} \quad \begin{array}{c} R(a) = 0, \\ R(0) < \infty. \end{array}}_{\text{BVP 2}}$$

Note that BCs imply drum is clamped down at radius a and the center of the drum has a finite amplitude. There are two eigenvalues to find: λ and μ.

2 Solving the BVPs: We solved BVP 1 previously. For $n = 0, 1, \ldots$, we have

$$\mu_n = n^2 \quad \text{and} \quad \Theta_n = A_n \cos(n\theta) + B_n \sin(n\theta).$$

For BVP 2, we have the ODE as the Bessel's equation (the solutions are the Bessel functions). Hence, the general solution is

$$R(r) = D_n J_n(\sqrt{\lambda}\, r) + E_n Y_n(\sqrt{\lambda}\, r).$$

Since by BC 1 we require $R(0) < \infty$, then we need to have $E_n = 0$. By BC 2 we have $R(a) = 0 = D_n J_n(\sqrt{\lambda}\, a)$.

Recall 4.4.12: $\alpha_{n,m}$ is the m^{th} zero of the n^{th} order Bessel function of 1^{st} kind. ◁

We need $\sqrt{\lambda}\, a = \alpha_{n,m}$ for $m = 1, 2, \ldots$. Hence

$$\lambda_{nm} = \frac{\alpha_{n,m}^2}{a^2} \quad \text{for} \quad \begin{array}{c} n = 0, 1, \ldots, \\ m = 1, 2, \ldots. \end{array}$$

This gives us

$$\boxed{R_{nm} = J_n \frac{(\alpha_{n,m} r)}{a}.}$$

Solving for $T(t)$: We need to solve

$$T'' + c^2 \lambda_{n,m} T = 0.$$

The equation looks identical to the 1-D case expect with the λ. The solution is

$$\boxed{T_{nm} = F_{nm} \cos\Big(c\sqrt{\lambda_{nm} t}\Big) + G_{nm} \sin\Big(c\sqrt{\lambda_{nm}}\, t\Big).}$$

The eigenvalue determines the frequency of oscillation which is set by the geometry of the problem.

Forming the General Solution: We have the general solution as

$$u(r, \theta, t) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \Big\{ J_n\Big(\alpha_{n,m} \frac{r}{a}\Big) [A_{nm} \cos(n\theta) + B_{nm} \sin(n\theta)] \cdot \Big[F_{nm} \cos\Big(c\frac{\alpha_{nm}}{a} t\Big) + G_{nm} \sin\Big(c\frac{\alpha_{nm}}{a} t\Big)\Big]\Big\}.$$

Finding the Generalized Fourier Coefficients: Given any initial conditions $f(r,\theta)$ and $g(r,\theta)$, we can compute the coefficients which is a Fourier-Bessel series and it is exhaustive to compute.

Special Case: There are some modes (eigenfunctions) that look like

$$J_n(\sqrt{\lambda_{n,m}}\,r)\cos(n\theta).$$

Definition 4.4.13: A ***node*** is a line where the eigenfunction is always zero. ◁

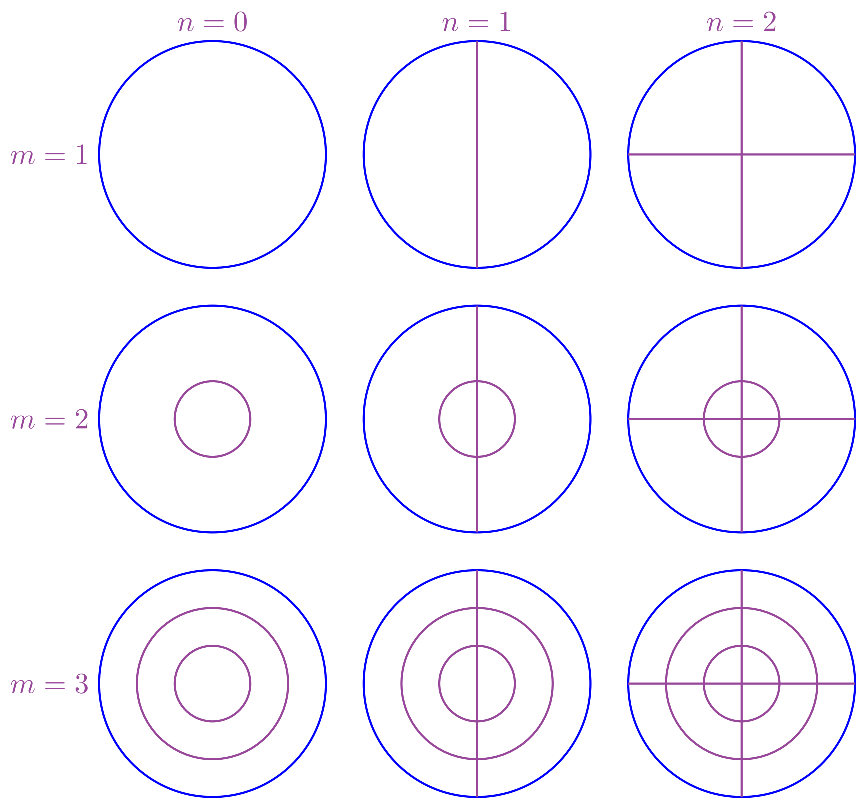


Figure 4.4.5: Nodes of modes.

4.4.3.12 Diffusion on a Circle

We have our PDE as

$$\frac{\partial u}{\partial t} = D\nabla^2 u = D\left[\frac{1}{r}\frac{\partial 1}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right],$$

which is on the physical domain

$$0 \le \theta < 2\pi,$$
$$0 \le r < a,$$

and has boundary conditions

$$\begin{aligned}
&u(a,\theta,t) = 0,\\
&u(0,\theta,t) \leq \infty,\\
&u(r,0,t) = u(r,2\pi,t),\\
&\frac{\partial u}{\partial \theta}(r,0,t) = \frac{\partial u}{\partial \theta}(r,2\pi,t),
\end{aligned}$$

and the initial condition

$$u(r,\theta,0) = f(r,\theta).$$

Exercise 4.4.14: Repeat the procedures in Remark 4.4.7 and find the solution as

$$u(r,\theta,t) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{J_n\left(\frac{\alpha_{n,m}r}{a}\right)[A_{nm}\cos(n\theta) + B_{nm}\sin(n\theta)]\cdot\exp\left(-D\frac{\alpha_{n.m}^2}{a^2}t\right)\right\}.$$ ◁

4.4.3.13 Schrödinger Equation

From classical mechanics, Newton's 2$^{\text{nd}}$ law yields, under a potential function $V(x)$

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = F = -\frac{\partial V}{\partial x}.$$

In quantum mechanics, we consider the (complex) wave function, $\Psi(x,t)$. This is useful in that $|\Psi(x,t)|^2\,\mathrm{d}x$ denotes the probability of finding a particle between x and $x+\mathrm{d}x$ at time t. Schrödinger equation is described as below.

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V(x)\Psi.$$

Motivation: Suppose the wave function is of the form

$$\Psi(x,t) = Ae^{i(kx-\omega t)}.$$

We observe that

$$\begin{aligned}
\frac{\partial \Psi}{\partial x} &= ik\Psi,\\
\frac{\partial \Psi}{\partial t} &= -i\omega\Psi.
\end{aligned}$$

de Broglie's Assumptions: We assume k is related to the momentum and ω is related to the energy.

$$k = \frac{p}{\hbar} \quad \text{and} \quad \omega = \frac{E}{\hbar}.$$

Using this, we get

$$\begin{aligned} p\Psi &= \hbar k \Psi = -i\hbar \frac{\partial \Psi}{\partial x}, \\ E\Psi &= \hbar \omega \Psi = i\hbar \frac{\partial \Psi}{\partial t}. \end{aligned}$$

Recall 4.4.15: In classical mechanics, we have the total energy of a particle as

$$E = \frac{1}{2}mv^2 + V(x) + \frac{1}{2}\frac{p^2}{m} + V(x),$$

where p is the momentum. ◁

Remark 4.4.16: In quantum mechanics, we begin with the above expression and rewrite p and E in terms of our partial derivatives.

$$E\Psi = \frac{1}{2m}p^2\Psi + V\Psi.$$

We substitute our assumptions, we get

$$\boxed{\begin{aligned} i\hbar\frac{\partial \Psi}{\partial t} &= \frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial x}\right)^2 \Psi + V\Psi, \\ ik\frac{\partial \Psi}{\partial t} &= -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi + V(x)\Psi, \end{aligned}}$$

which is called the time dependent Schrödinger equation. ◁

We use the steps we discussed in Remark 4.4.7 to solve the problem as follows.

Separating the Variables: We assume the solution is in th form $\Psi = M(x)N(t)$. We sub this into PDE and simply the expression by dividing it by MN. We obtain

$$i\hbar MN' = -\frac{\hbar^2}{2m}M''N + VMN \implies i\hbar\frac{N'}{N} = -\frac{\hbar^2}{2m}\frac{M''}{M} + V(x) = E.$$

Here the convention is to denote the non-negative constant eigenvalue as E because it is related to the energy.

Temporal Equation: We have the ODE for the temporal equation as

$$i\hbar N' = EN.$$

This has solutions

$$N(t) = N_0 e^{-i\frac{E}{\hbar}t}.$$

This gives rise to oscillatory behavior.

Spatial Equation: We have the ODE for the spatial equation as

$$-\frac{\hbar^2}{2m}M'' + V(x)M = EM.$$

This is called the time independent Schrödinger equation. This ODE is of the Sturm-Liouville form with

$$\begin{aligned} p(x) &= \frac{\hbar^2}{2m} > 0, \\ q(x) &= V(x) \geq 0, \\ \rho(x) &= 1, \\ E &\text{ is the eigenvalue.} \end{aligned}$$

Given "nice" BCs, we can deduce that we have a countable infinite number of eigenvalues or energies E_n where each has a corresponding eigenfunction, denoted as $M_n(x)$ for $n = 1, 2, \ldots$.

Forming the General Solution: We have the general solution as

$$\boxed{\Psi(x,t) = \sum_{n=1}^{\infty} M_n(x) \exp\left(-i\frac{E_n}{\hbar}t\right).}$$

4.4.3.14 Spherical Wave Equation

In spherical coordinates, we define the position in terms of r, θ, ϕ. Recall we have the wave equation in cartesian coordinates as

$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u.$$

In spherical coordinates, we get

$$c^2\nabla^2 u = c^2\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\theta\frac{\partial u}{\partial\phi}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 u}{\partial\theta^2}\right\}.$$

We use the steps we discussed in Remark 4.4.7 to solve the problem as follows.

1 **Separation of Variables:** We assume the solution is of the form $u(\mathbf{x}, t) = M(\mathbf{x})N(t)$. We sub in this equation into the PDE and simply the expression by dividing it by c^2MN. We obtain

$$MN'' = c^2\nabla^2 MN \implies \frac{N''}{c^2N} = \frac{\nabla^2 M}{M} = -\lambda.$$

We get the ODE for the temporal equation as

$$\boxed{N'' + c^2\lambda N = 0.}$$

For the spatial equation we have

$$-\nabla^2 M = \lambda M.$$

Here λ is the 3-D eigenvalue and M is the 3-D eigenfunction. We need to separate this into ODEs. We assume $M(\mathbf{x}) = R(r)\Phi(\phi)\Theta(\theta)$.

4.5 Inhomogeneous PDEs

Consider the inhomogeneous PDEs where $t > 0$ and ICs,

$$\rho\frac{\partial^2 u}{\partial t^2} + \mathcal{L}[u] = g(x,t) \quad \text{with} \quad u(x,0) = 0 \text{ and } \frac{\partial u}{\partial t}(x,0) = 0, \qquad \text{(Hyperbolic)}$$

$$\rho\frac{\partial u}{\partial t} + \mathcal{L}[u] = g(x,t) \quad \text{with} \quad u(x,0) = 0. \qquad \text{(Parabolic)}$$

Method for Solving Inhomogeneous PDEs: The equation we want to solve is $u(x,t)$. We introduce a temporary variable $v(x,t;\tau)$. We get the PDEs and initial conditions as

$$\rho\frac{\partial^2 v}{\partial t^2} + \mathcal{L}[v] = 0, \quad t > \tau, \quad \text{with} \quad v(x,\tau;\tau) = 0 \text{ and } \frac{\partial v}{\partial t}(x,\tau;\tau) = \frac{g(x,z)}{\rho(x)}, \qquad \text{(Hyperbolic)}$$

$$\rho\frac{\partial v}{\partial t} + \mathcal{L}[v] = 0, \quad t > \tau, \quad \text{with} \quad v(x,\tau;\tau) = \frac{g(x,z)}{\rho(x)}. \qquad \text{(Parabolic)}$$

We recover our actual solution using

$$u(x,t) = \int_0^t v(x,t;\tau)\,\mathrm{d}\tau.$$

4.5.0.1 Leibniz's Principle

Suppose we have the time derivative of the integral as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{g(t)}^{f(t)} h(s,t)\,\mathrm{d}s.$$

With change of variables (exercise) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{g(t)}^{f(t)} h(s,t)\,\mathrm{d}s = \frac{\mathrm{d}f}{\mathrm{d}t}h(f(t),t) - \frac{\mathrm{d}g}{\mathrm{d}t}h(g(t),t) + \int_{g(t)}^{f(t)} \frac{\partial h}{\partial t}(s,t)\,\mathrm{d}s.$$

We need the above result to compute partial derivatives of u in our given PDEs. We have

$$\frac{\partial u}{\partial t} = 1\cdot v(x,t;t) - 0 + \int_0^t \frac{\partial v}{\partial t}(x,t;\tau)\,\mathrm{d}z = v(x,t;t) + \int_0^t \frac{\partial v}{\partial t}(x,t;\tau)\,\mathrm{d}z,$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}[v(x,t;\tau)] + \frac{\partial v}{\partial t}(x,t;t) + \int_0^t \frac{\partial^2 v}{\partial t^2}(x,t;\tau)\,\mathrm{d}\tau.$$

Parabolic Case: We substitute the 1st expression into the LHS of the PDE. We want to show we get $g(x,t)$ on the RHS. We have

$$\begin{aligned}
\rho\frac{\partial}{\partial ut} + \mathcal{L}[u] &= \rho\left[\underbrace{v(x,t;t)}_{g(x,t)/\rho(x)} + \int_0^t \frac{\partial v}{\partial t}(x,t;\tau)\,\mathrm{d}\tau\right] + \mathcal{L}\left[\int_0^t v(x,t;\tau)\,\mathrm{d}\tau\right] \\
&= g(x,t) + \int_0^t \left[\underbrace{\rho\,\frac{\partial v}{\partial t}(x,t;\tau) + \mathcal{L}(v)}_{=0}\right]\mathrm{d}\tau \\
&= g(x,t).
\end{aligned}$$

as required. Hence, we have the means of computing $u(x,t)$.

Hyperbolic Case: some line here fix

$$\begin{aligned}
\rho\frac{\partial^2 u}{\partial t^2} + \mathcal{L}[u] &= \rho\left[\frac{\partial}{\partial t}[\underbrace{v(x,t;t)}_{=0}] + \underbrace{\frac{\partial v}{\partial t}(x,t;\tau)}_{g(x,t)/\rho(x)} + \int_0^t \frac{\partial^2 v}{\partial t^2}(x,t;\tau)\,\mathrm{d}z\right] + \mathcal{L}\left[\int_0^t v(x,t;\tau)\,\mathrm{d}\tau\right] \\
&= g(x,t) + \int_0^t \underbrace{\rho\frac{\partial^2 v}{\partial t^2} + \mathcal{L}[v]}_{=0}\,\mathrm{d}z \\
&= g(x,t)
\end{aligned}$$

as expected.

4.5.0.2 Inhomogeneous Wave Equation

We have the inhomogeneous wave equation as

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = g(x,t) \quad \text{on} \quad -\infty < x < \infty, \text{ and } t > 0,$$

with ICs

$$u(x,0) = 0, \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = 0 \quad \text{on} \quad -\infty < x < \infty.$$

4.5.1 Duhamel's Principle

We define the corresponding system for $v(x,t;\tau)$. We have the PDE as

$$\frac{\partial^2 v}{\partial t^2} - c^2\frac{\partial^2 v}{\partial x^2} = 0 \quad \text{on} \quad -\infty < x < \infty, \text{ and } t > \tau,$$

with ICs

$$v(x,\tau;\tau) = 0, \quad \text{and} \quad \frac{\partial v}{\partial t}(x,\tau;\tau) = g(x,\tau) \quad \text{on} \quad -\infty < x < \infty.$$

We can apply d'Alembert's solution where the IC is at $t = \tau$ instead of $t = 0$. We have

$$v(x,t;\tau) = \frac{1}{2c}\int_{x-c(t-\tau)}^{x+c(t-\tau)} g(s,\tau)\,\mathrm{d}s.$$

Hence, we can obtain the solution as

$$u(x,t) = \int_0^t v(x,t;\tau)\,\mathrm{d}\tau = \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} g(s,\tau)\,\mathrm{d}s\,\mathrm{d}\tau.$$

Remark 4.5.1: Suppose we have non-zero ICs as

$$u(x,0) = F(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = G(x).$$

We have the solution as

$$u(x,t) = \frac{1}{2}[F(x+ct) + F(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)\,\mathrm{d}s + \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} g(s,\tau)\,\mathrm{d}s\,\mathrm{d}\tau. \qquad \triangleleft$$

Remark 4.5.2: Read section 4.5.2 in course notes for inhomogeneous diffusion equation. $\triangleleft$

4.6 Eigenfunction Expansions

We went over 3 classes of PDEs (hyperbolic, parabolic and elliptic). We had

$$\rho\frac{\partial^2 u}{\partial t^2} + \mathcal{L}[u] = \rho F \tag{Hyperbolic}$$

$$\rho\frac{\partial u}{\partial t} + \mathcal{L}[u] = \rho F \tag{Parabolic}$$

$$-\rho\frac{\partial^2 u}{\partial y^2} + \mathcal{L}[u] = \rho F \tag{Elliptic}$$

Each of these have the same eigenvalue problem.

$$\mathcal{L}[M_k] = \lambda_k \rho M_k, \quad \text{for} \quad k = 1, 2, \ldots,$$

with some BCs. We have found for Sturm-Liouville problems, the M_k's are orthogonal. Therefore, we can consider an orthonormal basis, M_k, that is complete.

Motivation: Previously, in the homogeneous PDEs, we used separation of variables and found a series solution. The series summed up over all the orthonormal eigenfunctions.

An inhomogeneous PDE does not allow for separation of variables in the same way. However, since the eigenfunctions are complete, we can look for a series solution and determine how this can be satisfied. Note that this uses the same eigenfunctions. The idea is the following:

1. Consider the homogeneous PDE and compute the eigenfunctions.

$$M_k(x), \lambda_k \quad \text{where} \quad k = 1, 2, \ldots$$

2. Assume a series solution

$$u(x,t) = \sum_{k=1}^{\infty} N_k(t) M_k(x).$$

3. Assume a series for the forcing

$$F(x,t) = \sum_{k=1}^{\infty} F_k(t) M_k(x).$$

This works because eigenfunctions arise from the physics and geometry of the problem. Other bases will not turn out to be as convenient.

Hyperbolic Case: We have the hyperbolic problem as

$$\rho \frac{\partial^2 u}{\partial t^2} + \mathcal{L}[u] = \rho F.$$

We will assume the following.

1. The eigenfunctions are of the form $M_k(x)$ for $k = 1, 2, \ldots$.
2. The solution $u(x,t)$ can be written as a series in $M_k(x)$.
3. $F(x,t)$ can be written as a series in $M_k(x)$.

We project the PDE on $M_\ell(x)$ (in other words multiply the PDE by $M_\ell(x)$ and $\int_0^L \mathrm{d}x$). We have

$$\int_0^L \rho \frac{\partial^2 u}{\partial t^2} M_\ell , \mathrm{d}x + \int_0^L \rho \frac{1}{\rho} \mathcal{L}[u] M_\ell(x) \, \mathrm{d}x = \int_0^L \rho F(x,t) M_\ell(x) \, \mathrm{d}x.$$

Equivalently, we can also write this in terms of the inner product. We have

$$\left(\frac{\partial^2 u}{\partial t^2}, M_\ell \right) + \left(\frac{1}{\rho} \mathcal{L}[u], M_\ell \right) = (F, M_\ell).$$

For the first term, we factor our the time derivative. For the second term, we use self-adjointness and then the eigenvalue relations. For the third term, we rewrite it using Fourier coefficients of F. We have

$$\begin{aligned} u(x,t) &= \sum_{k=1}^{\infty} N_k(t) M_k(x) \implies (u, M_\ell) = N_\ell, \\ F(x,t) &= \sum_{k=1}^{\infty} F_k(t) M_k(x) \implies (F, M_\ell) = F_\ell. \end{aligned}$$

This yields

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}t^2}(u, M_\ell) + \left(u, \frac{1}{\rho}\mathcal{L}[M_\ell]\right) &= (F, M_\ell) \\ \implies \frac{\mathrm{d}^2 N_\ell}{\mathrm{d}t^2} + \lambda_\ell N_\ell &= F_\ell \quad \text{for} \quad \ell = 1, 2, \ldots. \end{aligned}$$

Note that when we have no forcing, then we get

$$\frac{\mathrm{d}^2 N_\ell}{\mathrm{d}t^2} + \lambda_\ell N_\ell \equiv 0,$$

which was the equation for temporal equation as expected. To find ICs of N_ℓ's, we consider the initial conditions.

$$\begin{aligned} u(x, 0) &= \sum_{k=1}^{\infty} N_k(0) M_k(x) \quad = f(x), \\ \frac{\partial u}{\partial t}(x, 0) &= \sum_{k=1}^{\infty} \frac{\mathrm{d}N_k}{\mathrm{d}t}(0) M_k(x) = g(x). \end{aligned}$$

We project each equation on M_ℓ. We have

$$\begin{aligned} N_\ell(0) &= (f, M_\ell), \\ \frac{\mathrm{d}N_\ell}{\mathrm{d}t}(0) &= (g, M_\ell), \\ \frac{\mathrm{d}^2 N_\ell}{\mathrm{d}t^2} + \lambda_\ell N_\ell &= F_\ell. \end{aligned} \tag{4.6.1}$$

Some of the techniques we can use this to solve this are

- Laplace transforms (we use this method in Remark 4.6.3),
- Green's functions,
- Variation of parameters.

We get the solution as

$$\boxed{N_k(t) = N_k(0)\cos\left(\sqrt{\lambda_k}\, t\right) + \frac{N_k'(0)}{\sqrt{\lambda_k}}\sin\left(\sqrt{\lambda_k}\, t\right) + \frac{1}{\sqrt{\lambda_k}}\int_0^t F_k(s)\sin\left(\sqrt{\lambda_k}\,(t-s)\right)\mathrm{d}s.}$$

Parabolic Case: When we go through the same approach, we find

$$\begin{aligned} \frac{\mathrm{d}N_k}{\mathrm{d}t} + \lambda_k N_k &= F_k \quad \text{for} \quad k = 1, 2, \ldots, \\ \text{with } N_k(0) &= (f, M_k). \end{aligned}$$

This is a linear DE and we can solve it by introducing an integrating factor. We let $\mu = e^{\lambda_k t}$ and multiply the ODE with μ. We have

$$e^{\lambda_k t}\frac{\mathrm{d}N_k}{\mathrm{d}t} + \lambda_k e^{\lambda_k t} N_k = e^{\lambda_k t} F_k.$$

After integrating we get

$$\int_0^t \frac{\mathrm{d}}{\mathrm{d}t}(e^{\lambda_k t} N_k)\,\mathrm{d}t = \int_0^t e^{\lambda_k t} F_k\,\mathrm{d}t \implies e^{\lambda_k t} N_k(t) - N_k(0) = \int_0^t e^{\lambda_k s} F_k(s)\,\mathrm{d}s.$$

Hence we have

$$N_k(t) = e^{-\lambda_k t} N_k(0) + \int_0^t e^{-\lambda_k (t-s)} F_k(s)\,\mathrm{d}s.$$

Elliptic Case: Similarly, for elliptic case we the below expression (we also have the BCs),

$$-\frac{\mathrm{d}^2 N_k}{\mathrm{d}y^2} + \lambda_k N_k = F_k \quad \text{for} \quad k = 1, 2, \dots.$$

4.6.1 Laplace Transforms

Definition 4.6.1: The ***Laplace transform*** of a function is defined as

$$\mathscr{L}[y] \underset{\text{def}}{=} \int_0^\infty e^{-st} y(t)\,\mathrm{d}t.$$

Note that we use $\mathcal{L}$ (mathcal L) to denote the Sturm-Liouville operator in Notation 4.1.1 and use $\mathscr{L}$ (curvy L) to denote the Laplace transform that we introduced in Definition 4.6.1. ◁

Remark 4.6.2: The Laplace transform has the following properties.

1. Differentiation:

$$\mathscr{L}[y'] = s\mathscr{L}[y] - y[0].$$

In general, for a function f we have

$$\mathscr{L}[f^{(n)}] = s^n \mathscr{L}[f] - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - s f^{(n-2)}(0) - s^0 f^{(n-1)}(0).$$

2. Convolution:

$$\mathscr{L}^{-1}[\mathscr{L}[f]\mathscr{L}[g]] = f * g,$$

where $(f * g)$ is the convolution of two functions given by

$$(f * g)(t) = \int_0^t f(t-\tau) g(\tau)\,\mathrm{d}\tau.$$

3. Laplace transform of exponential:

$$\mathscr{L}[e^{\alpha t}] = \frac{1}{s-a}.$$

4. Laplace transform of trigonometric functions:

$$\mathscr{L}[\cos(\omega t)] = \frac{s}{s^2+\omega^2},$$
$$\mathscr{L}[\sin(\omega t)] = \frac{\omega}{s^2+\omega^2}.$$

⑤ Laplace transform of hyperbolic trigonometric functions:

$$\mathscr{L}[\cosh(\omega t)] = \frac{s}{s^2 - \omega^2},$$
$$\mathscr{L}[\sinh(\omega t)] = \frac{\omega}{s^2 - \omega^2}.$$

The proofs for these properties are not in the scope of this course. We will take these properties as given. That being said the proofs are straight forward. They use the linearity of Laplace transform. ◁

Remark 4.6.3: We will solve the hyperbolic ODE for $N_k(t)$ we found in (4.6.1) by using Laplace transform. Recall that we had

$$N_k(0) = (f, M_k),$$
$$\frac{\mathrm{d}N_k}{\mathrm{d}t}(0) = (g, M_k),$$
$$\frac{\mathrm{d}^2 N_k}{\mathrm{d}t^2} + \lambda_k N_k = F_k, \quad \text{for} \quad k = 1, 2, \ldots.$$

Note that we can also solve this by using variation of parameters. We take the Laplace transform of the ODE and use its linearity. We have

$$\mathscr{L}\left[\frac{\mathrm{d}^2 N_k}{\mathrm{d}t^2} + \lambda_k N_k\right] = \mathscr{L}[F_k],$$

$$\mathscr{L}\left[\frac{\mathrm{d}^2 N_k}{\mathrm{d}t^2}\right] + \mathscr{L}[\lambda_k N_k] = \mathscr{L}[F_k].$$

By the differentiation property for Laplace transform we have

$$\mathscr{L}\left[\frac{\mathrm{d}^2 N_k}{\mathrm{d}t^2}\right] = s^2 \mathscr{L}[N_k] - sN_k(0) - N_k'(0) + \lambda_k[N_k] = \mathscr{L}[F_k].$$

We first solve for $\mathscr{L}[N_k]$. We have

$$\mathscr{L}[N_k] = \frac{s}{s^2 + \lambda_k} N_k(0) + \frac{1}{s^2 + \lambda_k} N_k'(0) + \frac{1}{s^2 + \lambda_k} \mathscr{L}[F_k].$$

Since we have

$$\mathscr{L}\left[\cos\left(\sqrt{\lambda_k}\, t\right)\right] = \frac{s}{s^2 + \lambda_k}, \quad \text{and} \quad \mathscr{L}\left[\frac{1}{\sqrt{\lambda_k}\sin\left(\sqrt{\lambda_k}\, t\right)}\right] = \frac{1}{s^2 + \lambda_k},$$

we obtain

$$\mathscr{L}[N_k] = \mathscr{L}[\cos\left(\sqrt{\lambda_k}\, t\right)] N_k(0) + \mathscr{L}\left[\frac{\sin\left(\sqrt{\lambda_k}\, t\right)}{\sqrt{\lambda_k}}\right] N_k'(0) + \mathscr{L}\left[\frac{\sin\left(\sqrt{\lambda_k}\, t\right)}{\sqrt{\lambda_k}}\right] \mathscr{L}[F_k].$$

We now take the inverse Laplace transform. We obtain

$$\boxed{N_k(t) = \cos\left(\sqrt{\lambda_k}\, t\right) N_k(0) + \frac{\sin\left(\sqrt{\lambda_k}\, t\right)}{\sqrt{\lambda_k}} N_k'(0) + \left(\frac{\sin\left(\sqrt{\lambda_k}\, t\right)}{\sqrt{\lambda_k}} * F_k(t)\right).}$$

This is equivalent to

$$N_k(t) = N_k(0)\cos\left(\sqrt{\lambda_k}\,t\right) + N_k'(0)\frac{\sin\left(\sqrt{\lambda_k}\,t\right)}{\sqrt{\lambda_k}} + \int_0^t \frac{\sin\left(\sqrt{\lambda_k}\,(t-\tau)\right)}{\sqrt{\lambda_k}} F_k(\tau)\,\mathrm{d}\tau.$$

We used the convolution property above. ◁

4.6.2 Resonance

Consider a general hyperbolic PDE.

$$\rho(x)\frac{\partial^2 u}{\partial t^2} + \mathcal{L}[u] = \rho F(x,t).$$

Here $\mathcal{L}$ is the Sturm-Liouville operator. Pick $F(x,t) = M_i(x)\sin(\omega t)$ where M_i is an eigenfunction and ω is a real frequency. To make things simple, we suppose

$$\begin{aligned} f(x) = 0 \quad &\implies \quad N_k(0) = 0, \\ g(x) = 0 \quad &\implies \quad N_k'(0) = 0. \end{aligned}$$

Given our previous result, we have

$$N_k(t) = \int_0^t \frac{\sin\left(\sqrt{\lambda_k}\,(t-\tau)\right)}{\sqrt{\lambda_k}} F_k(\tau)\,\mathrm{d}\tau \quad \text{for} \quad k = 1, 2, \ldots.$$

We compute $F_k(t)$ by projecting it onto M_k. We have

$$F_k = (F, M_k) = (M_i \sin(\omega t), M_k) = \sin(\omega t)(M_i, M_k) = \sin(\omega t)\delta_{ik}(M_k, M_k).$$

Assuming M_k is normalized, we have

$$F_k = \begin{cases} 0 & \text{if } k \neq i, \\ \sin(\omega t) & \text{if } k = i. \end{cases}$$

Hence, we have

$$N_k(t) = \begin{cases} 0 & \text{if } k \neq i, \\ \int_0^t \frac{\sin\left(\sqrt{\lambda}_i (t-\tau)\right)}{\sqrt{\lambda_i}} \sin(\omega\tau)\,\mathrm{d}\tau & \text{if } k = i. \end{cases}$$

Recall that $2\sin A \sin B = \cos(A-B) - \cos(A+B)$ and obtain

$$\begin{aligned} N_i &= \frac{1}{\sqrt{\lambda_i}} \int_0^t \sin\left(\sqrt{\lambda_i}\,(t-\tau)\right)\sin(\omega\tau)\,\mathrm{d}\tau = \frac{1}{2\sqrt{\lambda_i}} \int_0^t \cos[\omega\tau - \sqrt{\lambda_i}\,(t-\tau)] - \cos[\omega\tau + \sqrt{\lambda_i}\,(t-\tau)]\,\mathrm{d}\tau \\ &= \frac{1}{2\sqrt{\lambda_i}} \left[\frac{\sin[(\omega+\sqrt{\lambda_i})\tau - \sqrt{\lambda_i}\,t]}{\omega+\sqrt{\lambda_i}} - \frac{\sin[(\omega-\sqrt{\lambda_i})\tau + \sqrt{\lambda_i}\,t]}{\omega-\sqrt{\lambda_i}}\right]_0^t \\ &= \frac{1}{2\sqrt{\lambda_i}} \left[\frac{\sin(\omega t) + \sin\left(\sqrt{\lambda_i}\,t\right)}{\omega+\sqrt{\lambda_i}} - \frac{\sin(\omega t) - \sin\left(\sqrt{\lambda_i}\,t\right)}{\omega-\sqrt{\lambda_i}}\right] \\ &= \frac{1}{\sqrt{\lambda_i}} \frac{\omega\sin\left(\sqrt{\lambda_i}\,t\right) - \sqrt{\lambda_i}\sin\left(\sqrt{\lambda_i}\,t\right)}{(\omega^2 - \lambda_i)}. \end{aligned}$$

Hence, we obtain our solution as

$$u(x,t) = \frac{(\omega \sin(\sqrt{\lambda_i}\,t) - \sqrt{\lambda_i}\sin(\sqrt{\lambda_i}\,t))}{\sqrt{\lambda_i}(\omega^2 - \lambda_i)} M_i(x).$$

Following the calculation from last time, the solution to the forced hyperbolic problem is

$$N_k(t) = \delta_{ik} N_i \quad \text{where} \quad N_i = \frac{(\omega \sin(\sqrt{\lambda_i}\,t) - \sqrt{\lambda_i}\sin(\omega t))}{\sqrt{\lambda_i}(\omega^2 - \lambda_i)},$$

The complete solution is

$$u(x,t) = \frac{(\omega \sin(\sqrt{\lambda_i}\,t) - \sqrt{\lambda_i}\sin(\omega t))}{\sqrt{\lambda_i}(\omega^2 - \lambda_i)} M_i(x).$$

The two frequencies that arise in the solution are ω (forcing frequency) and $\sqrt{\lambda_i}$ (natural frequency). The resonant case occurs where $\omega \to \sqrt{\lambda_i}$. We cannot plug this in since we get $\frac{0}{0}$ but we can take the limit as $\omega \to \sqrt{\lambda_i}$. We have

$$\begin{aligned}
\lim_{\omega \to \sqrt{\lambda_i}} u(x,t) &= \lim_{\omega \to \sqrt{\lambda_i}} \frac{(\omega \sin(\sqrt{\lambda_i}\,t) - \sqrt{\lambda_i}\sin(\omega t))}{\sqrt{\lambda_i}(\omega^2 - \lambda_i)} M_i(x) \\
&\underset{(\star)}{=} \lim_{\omega \to \sqrt{\lambda_i}} \frac{(\sin(\sqrt{\lambda_i}\,t) - \sqrt{\lambda_i}\,t\cos(\omega t))}{\sqrt{\lambda_i}\,2\omega} M_i \\
&= \frac{(\sin(\sqrt{\lambda_i}\,t) - \sqrt{\lambda_i}\,t\cos(\lambda_i t))}{2\lambda_i} M_i.
\end{aligned}$$

Where in $(\star)$ we used L'Hopital's rule.

Remark 4.6.4: The solution increases linearly with time. As $t \to \infty$ we get $u \to \infty$. This never happens because the linear approximation in derivations the wave equation (or other hyperbolic equations) break down when u is sufficiently large. This is the solution to the resonant problem. $\triangleleft$

4.6.3 Inhomogeneous Boundary Conditions

Suppose we have mixed (Robin) BCs that are inhomogeneous where

$$\begin{aligned}
\alpha_1 V(0,t) - \beta_1 \frac{\partial V}{\partial x}(0,t) &= g_1(t), \\
\alpha_2 V(L,t) + \beta_2 \frac{\partial V}{\partial x}(L,t) &= g_2(t).
\end{aligned}$$

The presence of non-zero g_1 and g_2 prevent us from using separation of variables on this problem. The idea is to decompose our solution into the sum of two terms, one of which solves the inhomogeneous BCs. We decompose our solution as

$$u(x,t) = W(x,t) + V(x,t).$$

We will pick $V(x,t)$ such that it satisfies the inhomogeneous BCs and it is a linear function of x.

Example 4.6.5: Consider the PDE

$$\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} = g(x,t) \text{ on the domain } \begin{array}{l} 0 < x < L, \\ t > 0, \end{array}$$

with the inhomogeneous Dirichlet BCs and IC

$$\begin{array}{lll} \text{BCs:} & \begin{array}{c} u(0,t) = g_1(t), \\ u(L,t) = g_2(t), \end{array} & \text{on the domain } t > 0, \\ \text{IC:} & u(x,0) = f(x) & \text{on the domain } 0 < x < L. \end{array}$$

We decompose the solution and obtain

$$u(x,t) = W(x,t) + V(x,t).$$

For this case we pick

$$\boxed{V(x,t) = \frac{x}{L}g_2(t) + \frac{L-x}{L}g_1(t).}$$

What determines W?

Since V is a linear function of x, then we have $\frac{\partial^2 V}{\partial x^2} = 0$. When we substitute this into PDE we get

$$\frac{\partial}{\partial t}(W+V) - D\frac{\partial^2}{\partial x^2}(W+V) = g(x,t) \implies \frac{\partial W}{\partial t} - D\frac{\partial^2 W}{\partial x^2} = g(x,t) - \frac{\partial V}{\partial t} \equiv \widetilde{g}(x,t).$$

When we sub in IC we get

$$u(x,0) = W(x,0) + V(x,0) = f(x) \implies W(x,0) = f(x) - V(x,0) \equiv \widetilde{f}(x).$$

We we sub into BCs we get

$$\begin{aligned} u(0,t) = W(0,t) + V(0,t) = g_1(t) &\implies W(0,t) + g_1(t) = g_1(t) \implies W(0,t) = 0, \\ u(L,t) = W(L,t) + V(L,t) = g_2(t) &\implies W(L,t) + g_2(t) = g_2(t) \implies W(L,t) = 0. \end{aligned}$$

We can now solve for $W(x,t)$ using Duhamel's principle or eigenfunction expansions. Given $W(x,t)$ and $V(x,t)$, we can find $u(x,t)$. ◁

Chapter 5 – Fourier Transform Methods

We consider the same PDEs as in previous chapter but on infinite domain, that is $-\infty < x < \infty$. Previously we found solutions of the form

$$u(x,t) = \sum_{n=1}^{\infty} N_n(t) M_n(x).$$

If we have $\rho(x) = 1, p(x) = 1, q(x) = 0$, we found the eigenfunctions were the Fourier basis. We can use Euler's formula to write the functions as

$$e^{ikx} = \cos(kx) + i\sin(kx) = \operatorname{cis}(kx).$$

In the bounded case, the eigenvalues are set by the eigenfunctions that "fit" in the domain. if the domain is infinite, then all the trigonometric functions "fit" in the domain. We must integrate over all values of k. We could try

$$u(x,t) = \int_{-\infty}^{\infty} N_k(t) e^{ikx} \, \mathrm{d}k.$$

This will be replaced by Fourier transforms.

Definition 5.0.1: Let $\mathbb{F}$ be an arbitrary field with $\operatorname{char}(\mathbb{F}) \neq 2$. ***Fourier transform*** of a function $f : \mathbb{F} \to \mathbb{F}$ is defined as follows.

$$\mathcal{F}[f(x)] = F(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda x} f(x) \, \mathrm{d}x. \tag{5.0.1}$$

The inverse Fourier transform is

$$\mathcal{F}^{-1}[F(\lambda)] = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\lambda x} F(\lambda) \, \mathrm{d}\lambda. \tag{5.0.2}$$

Here F is a function on the transform space and f is a function on the physical space (if we take $\mathbb{F} = \mathbb{R}$). We will always consider the field of real numbers $\mathbb{R}^n$, generally when $n = 1$. ◁

Remark 5.0.2: When we substitute (5.0.1) into (5.0.2) we get

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\lambda(x-s)} f(s) \, \mathrm{d}s \, \mathrm{d}\lambda. \quad ◁$$

5.0.1 Dirac Delta Function

Definition 5.0.3: ***Dirac delta function*** (generalized function) is defined as

$$\delta(x-s) \underset{\text{def}}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda(x-s)} \, \mathrm{d}\lambda. \quad ◁$$

Remark 5.0.4: Using this definition we obtain

$$f(x) = \int_{-\infty}^{\infty} \delta(x-s) f(s) \, \mathrm{d}s. \quad ◁$$

The Dirac delta "plucks out" the value of the integrand where the argument of delta function is zero.

Remark 5.0.5: Using the definition of the Fourier transform, we can find the FT of the Dirac delta function. We have

$$\mathcal{F}[\delta(x-x_0)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x}\delta(x-x_0)\,\mathrm{d}x = \frac{1}{2\pi}e^{i\lambda x_0}.$$

Using the IFT, we obtain

$$\delta(x-x_0) = \mathcal{F}^{-1}\left[\frac{1}{2\pi}e^{i\lambda x_0}\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda(x-x_0)}\,\mathrm{d}\lambda. \qquad \triangleleft$$

Motivation: Consider the following.

$$\delta_N(x) = \begin{cases} N & \text{if } |x| \le \frac{1}{2N}, \\ 0 & \text{if } |x| > \frac{1}{2N}. \end{cases}$$

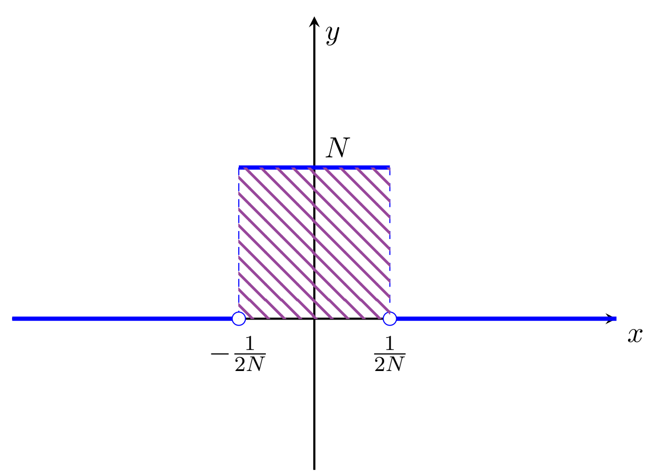


Figure 5.0.1: The $\delta_N(x)$ function. Note that the shaded region in purple has always area of 1.

We will see that $\delta(x) = \lim_{N\to\infty} \delta_N(x)$. The delta function has the property that

$$\int_{-\infty}^{\infty} \delta(x)\,\mathrm{d}x = 1.$$

Consider the FT of $\delta_N(x)$. We have

$$\begin{aligned}\mathcal{F}[\delta_N(x)] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\lambda x}\delta_N(x)\,\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}}\int_{-1/2N}^{1/2N} e^{i\lambda x} N\,\mathrm{d}x \\ &= \frac{N}{\sqrt{2\pi}}\left[\frac{e^{i\lambda x}}{i\lambda}\right]\Bigg|_{-1/2N}^{1/2N} \\ &= \frac{2N}{\sqrt{2\pi}\,2i\lambda}\left[e^{i\lambda/2N} - e^{-i\lambda/2N}\right] \\ &= \frac{1}{\sqrt{2\pi}}\frac{\sin(\lambda/2N)}{(\lambda/2N)}. \end{aligned} \tag{5.0.3}$$

The function in (5.0.3) is called the ***sinc function***. Hence we get

$$\lim_{N\to\infty}\mathcal{F}[\delta_N(x)] = \mathcal{F}[\delta(x)] = \lim_{N\to\infty}\sqrt{2\pi}\frac{\sin(\lambda/2N)}{\lambda/2N} = \frac{1}{\sqrt{2\pi}}.$$

Similarly by using IFT we get

$$\begin{aligned}\delta(x) = \lim_{N\to\infty}\delta_N(x) &= \lim_{N\to\infty}\mathcal{F}^{-1}\left[\frac{1}{\sqrt{2\pi}}\frac{\sin(\lambda/2N)}{\lambda/2N}\right] \\ &= \lim_{N\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\lambda x}\frac{1}{\sqrt{2\pi}}\frac{\sin(\lambda/2N)}{\lambda/2N}\,\mathrm{d}\lambda \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda x}\,\mathrm{d}\lambda.\end{aligned}$$

Remark 5.0.6: Let $\mathcal{F}(f(x)) = F(\lambda)$ and $\mathcal{F}(g(x)) = G(\lambda)$. The Fourier transform has the following properties.

1. The inverse of $F(\lambda)G(\lambda)$ is the normalized convolution of f and g.

$$\begin{aligned}\mathcal{F}^{-1}[F(\lambda)G(\lambda)] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\lambda x}F(\lambda)G(\lambda)\,\mathrm{d}\lambda \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\lambda x}e^{i\lambda s}f(s)G(\lambda)\,\mathrm{d}s\,\mathrm{d}\lambda \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\left[\int_{-\infty}^{\infty} e^{-i\lambda(x-s)}G(\lambda)\,\mathrm{d}\lambda\right]\mathrm{d}s \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(s)g(x-s)\,\mathrm{d}s \\ &= \frac{1}{\sqrt{2\pi}}(f*g)(x).\end{aligned}$$

2. Parseval's theorem:

$$\int_{-\infty}^{\infty}|F(\lambda)|^2\,\mathrm{d}\lambda = \int_{-\infty}^{\infty}|f(x)|^2\,\mathrm{d}x.$$

③ Differentiation formula.

$$\mathcal{F}\left[\frac{\mathrm{d}f}{\mathrm{d}x}\right] = (-i\lambda)\mathcal{F}[f],$$
$$\mathcal{F}\left[\frac{\mathrm{d}^n f}{\mathrm{d}x^n}\right] = (-i\lambda)^n\mathcal{F}[f],$$

The proofs for these properties can be obtain by using the linearity of the Fourier transform and are not included in these notes. They are presented in the course notes posted on Learn. ◁

5.0.2 Applications of Fourier Transforms to ODEs

Consider the ODE below on the domain $-\infty < x < \infty$,

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - k^2 y = -f(x).$$

We assume that as $|x|$ approaches to infinity then $y(x)$, and $y'(x)$ approach to zero. We solve this problem by first computing FT of ODE by using its linearity and differentiation properties and solving for $\mathcal{F}[y]$.

$$\begin{aligned}\mathcal{F}\left[\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right] - k^2\mathcal{F}[y] = -\mathcal{F}[f] &\iff (-i\lambda)^2\mathcal{F}[y] - k^2\mathcal{F}[y] = -\mathcal{F}[f] \\ &\iff (\lambda^2 + k^2)\mathcal{F}[y] = \mathcal{F}[f] \\ &\iff \mathcal{F}[y] = \mathcal{F}[f]\frac{1}{\lambda^2 + k^2}.\end{aligned}$$

We then IFT to obtain y as follows. Note that we have

$$\frac{1}{\lambda^2 + k^2} = \mathcal{F}\left[\frac{\sqrt{2\pi}}{2k}e^{-k|x|}\right].$$

We want to solve for y where

$$\mathcal{F}[y] = \mathcal{F}[f]\frac{1}{\lambda^2 + k^2} = \mathcal{F}[f]\mathcal{F}\left[\frac{\sqrt{2\pi}}{2k}e^{-k|x|}\right].$$

This gives us

$$\begin{aligned}\mathcal{F}[y] = \mathcal{F}[f]\mathcal{F}\left[\frac{2\pi}{2k}e^{-k|x|}\right] \iff y(x) &= \mathcal{F}^{-1}\left[\mathcal{F}[f]\mathcal{F}\left(\frac{\sqrt{2\pi}}{2k}e^{-k|x|}\right)\right] \\ &\underset{(\star)}{=} \frac{1}{\sqrt{2\pi}}\left(f * \frac{\sqrt{2\pi}}{2k}e^{-k|x|}\right)(x) \\ &= \frac{1}{2k}\left(f * e^{-k|x|}\right)(x) \\ &= \frac{1}{2k}\int_{-\infty}^{\infty} f(s)e^{-k|x-s|}\,\mathrm{d}s.\end{aligned}$$

Here in $(\star)$ we used the convolution property.

5.0.3 Diffusion Equation

We have the PDE for the diffusion equation as

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}, \text{ on the domain } \begin{matrix} -\infty < x < \infty \\ t > 0, \end{matrix}$$

with IC

$$u(x,0) = f(x), \text{ on the domain } -\infty < x < \infty.$$

Since the domain is infinite, Fourier transforms are a good approach to solve this problem. We use the following three steps to solve this problem.

1. Compute the Fourier transform of the PDE and the IC. For the PDE, we define $U(\lambda, t) = \mathcal{F}[u(x,t)]$. We have

$$\mathcal{F}\left[\frac{\partial u}{\partial t}\right] = \mathcal{F}\left[D\frac{\partial^2 u}{\partial x^2}\right].$$

Since the Fourier transform is linear, by using the derivative property we obtain

$$\frac{\partial}{\partial t}U = D(-i\lambda)^2 U = -D\lambda^2 U.$$

For the IC we get

$$\mathcal{F}[u(x,0)] = \mathcal{F}[f(x)] \iff U(\lambda,0) = F(\lambda).$$

2. Solve the ODE that we obtained by using Fourier transform. We have the ODE

$$\frac{\partial}{\partial t}U = -D\lambda^2 U.$$

This has the solutions

$$U(\lambda,t) = Ae^{-D\lambda^2 t}.$$

We impose the IC and get

$$U(\lambda,t) = F(\lambda)e^{-D\lambda^2 t}.$$

3. We take the inverse Fourier transform to get the solution in physical space. We have

$$\begin{aligned} u(x,t) = \mathcal{F}^{-1}[U(\lambda,t)] = \mathcal{F}^{-1}[Fe^{-D\lambda^2 t}] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\lambda x}F(\lambda)e^{-D\lambda^2 t}\,\mathrm{d}\lambda \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda x - D\lambda^2 t}\int_{-\infty}^{\infty} e^{i\lambda x} f(s)\,\mathrm{d}s\,\mathrm{d}\lambda \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\int_{-\infty}^{\infty} \exp\big(-i\lambda(x-s) - D\lambda^2 t\big)\,\mathrm{d}\lambda\,\mathrm{d}s. \end{aligned}$$

Denote the integral below as I.

$$I(x-s) \underset{\text{def}}{=} \int_{-\infty}^{\infty} \exp\big(-i\lambda(x-s) - D\lambda^2 t\big)\,\mathrm{d}\lambda.$$

By Euler's formula we have

$$I(x-s) = \int_{-\infty}^{\infty} [\cos(\lambda(x-s)) - i\sin(\lambda(x-s))]e^{-D\lambda^2 t}\,\mathrm{d}\lambda.$$

Since $\sin(\lambda(x-s))$ is odd, the it becomes zero in the integrand. Hence we get

$$I(x-s) = 2\int_0^\infty \cos[\lambda(x-s)]e^{-D\lambda^2 t}\,\mathrm{d}\lambda.$$

We can rewrite this as

$$I(\alpha) \underset{\text{def}}{=} 2\int_0^\infty \cos(\lambda\alpha)e^{-D\lambda^2 t}\,\mathrm{d}\lambda.$$

When we differentiate it with respect to α we obtain

$$\begin{aligned}
\frac{\mathrm{d}I}{\mathrm{d}\alpha} &= -2\int_0^\infty \lambda\sin(\lambda\alpha)e^{-D\lambda^2 t}\,\mathrm{d}\lambda \\
&= -\int_0^\infty 2\lambda e^{-D\lambda^2 t}\sin(\lambda\alpha)\,\mathrm{d}\lambda \\
&= -\left[\frac{\mathrm{d}}{\mathrm{d}t}\exp\bigl(-D\lambda^2 t\bigr)\sin(\lambda\alpha)\right]_0^\infty - \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t}\exp\bigl(-D\lambda^2 t\bigr)\alpha\cos(\lambda\alpha)\,\mathrm{d}\lambda \\
&= -\frac{\alpha}{2Dt}I.
\end{aligned}$$

It can be shown $I(0) = \sqrt{\frac{\pi}{Dt}}$. Hence we have

$$I(\alpha) = \sqrt{\frac{\pi}{Dt}}\exp\left(-\frac{\alpha^2}{4Dt}\right).$$

We substitute this into our solution. We obtain

$$\boxed{\begin{aligned}
u(x,t) &= \frac{1}{2\pi}\int_{-\infty}^\infty f(s)\sqrt{\frac{\pi}{Dt}}\exp\left(-\frac{(x-s)^2}{4Dt}\right)\mathrm{d}s \\
&= \frac{1}{\sqrt{4\pi Dt}}\int_{-\infty}^\infty \exp\left(-\frac{(x-s)^2}{4Dt}\right)f(s)\,\mathrm{d}s.
\end{aligned}}$$

We define

$$G(x-s,t) = \frac{1}{\sqrt{4\pi Dt}}\exp\left(-\frac{(x-s)^2}{4Dt}\right).$$

hence we obtain

$$\boxed{u(x,t) = \int_{-\infty}^\infty G(x-s,t)f(s)\,\mathrm{d}s.}$$

The function $G(x-s,t)$ is called the *fundamental solution to the diffusion equation* or the *heat kernel.*

Consider $f(x) = \delta(x-x_0)$. Then our solution becomes

$$\boxed{u(x,t) = \int_{-\infty}^\infty G(x-s,t)\delta(s-x_0)\,\mathrm{d}s = u(x,t) = G(x-x_0,t).}$$

5.0.4 Wave Equation

We have the PDE as

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, \text{ on the domain } \begin{array}{c} -\infty < x < \infty \\ t > 0, \end{array}$$

with ICs

$$\begin{aligned} u(x,0) &= f(x), \\ \frac{\partial u}{\partial t}(x,0) &= g(x), \end{aligned} \quad \text{on the domain } -\infty < x < \infty.$$

Since the domain is infinite, Fourier transforms are a good approach to solve this problem. We use the following three steps to solve this problem as before.

1. Compute the Fourier transform of the PDE and the IC. For the PDE, we define $U(\lambda, t) = \mathcal{F}[u(x,t)]$. We have

$$\frac{\partial^2 U}{\partial t^2} + c^2 \lambda^2 U = 0.$$

For the IC we get

$$\begin{aligned} U(\lambda, 0) &= F \underset{\text{def}}{=} \mathcal{F}[f], \\ \frac{\partial U}{\partial t}(\lambda, 0) &= G \underset{\text{def}}{=} \mathcal{F}[g]. \end{aligned}$$

2. Solve the ODE that we obtained by using Fourier transform. We have the ODE

$$\frac{\partial^2 U}{\partial t^2} + c^2 \lambda^2 U = 0.$$

The solution to complex U is

$$U(\lambda, t) = \alpha(\lambda) e^{ic\lambda t} + \beta(\lambda) e^{-ic\lambda t}.$$

We impose the ICs and get

$$U(x, 0) = \alpha + \beta = F,$$

$$\frac{\partial U}{\partial t}(\lambda, 0) = ic\lambda\alpha - ic\lambda\beta = G.$$

To solve this, we write the above as a system of linear equations. We have

$$\begin{bmatrix} 1 & 1 \\ ic\lambda & -ic\lambda \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} F \\ G \end{bmatrix} \implies \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = -\frac{1}{2ic\lambda} \begin{bmatrix} -ic\lambda & -1 \\ -ic\lambda & 1 \end{bmatrix} \begin{bmatrix} F \\ G \end{bmatrix}$$

This gives us

$$\begin{aligned} \alpha &= \frac{1}{2}F + \frac{1}{2ic\lambda}G, \\ \beta &= \frac{1}{2}F - \frac{1}{2ic\lambda}G. \end{aligned}$$

Hence, our solution for $U(\lambda, t)$ becomes

$$U(\lambda, t) = \left[\frac{1}{2}F + \frac{1}{2ic\lambda}G\right] e^{ic\lambda t} + \left[\frac{1}{2}F - \frac{1}{2ic\lambda}G\right] e^{-ic\lambda t}.$$

③ We take the inverse Fourier transform to get the solution in physical space. We have

$$
\begin{aligned}
u(x,t) = \mathcal{F}^{-1}[U(\lambda,t)] &= \mathcal{F}^{-1}\left[\left(\frac{1}{2}F + \frac{1}{2ic\lambda}G\right)e^{ic\lambda t}\right] + \mathcal{F}^{-1}\left[\left(\frac{1}{2}F - \frac{1}{2ic\lambda}G\right)e^{-ic\lambda t}\right] \\
&= \frac{1}{2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp(-i\lambda(x-ct))F(\lambda)\,\mathrm{d}\lambda + \frac{1}{2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp(-i\lambda(x+ct))F(\lambda)\,\mathrm{d}\lambda \\
&\quad + \underbrace{\frac{1}{2c}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp(-i\lambda(x-ct))\frac{G(\lambda)}{i\lambda}\,\mathrm{d}\lambda - \frac{1}{2c}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp(-i\lambda(x+ct))\frac{G(\lambda)}{i\lambda}\,\mathrm{d}\lambda}_{(\star)} \\
&= \frac{1}{2}f(x-ct) + \frac{1}{2}f(x+ct) + (\star).
\end{aligned}
$$

Define the function $g(x)$ as

$$
g(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\lambda x}G(\lambda)\,\mathrm{d}\lambda.
$$

This gives us

$$
\int_0^x g(s)\,\mathrm{d}s = -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\lambda x}\frac{G(\lambda)}{i\lambda}\,\mathrm{d}\lambda.
$$

Our solution becomes

$$
u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] - \frac{1}{2c}\int_0^{x-ct} g(s)\,\mathrm{d}s + \frac{1}{2c}\int_0^{x+ct} g(s)\,\mathrm{d}s.
$$

Hence we obtain

$$
\boxed{u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s.}
$$

which is the (d'Alembert's Solution).

Chapter 3 – Method of Characteristics

3.1 Linear First-order PDEs

Remark 3.1.1: Recall that we have the wave equation as

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right)u = 0.$$

We can factor the operator as

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) = 0.$$

If we define v such that

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)u = v,$$

we get

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)v = 0.$$

This is a system of first order couples PDEs. $\triangleleft$

We will discus methods of solving first order PDEs.

3.1.1 Basis for Method of Characteristics

Remark 3.1.2: We have the general first order linear PDE as

$$\boxed{a(x,t)\frac{\partial v}{\partial x} + b(x,t)\frac{\partial v}{\partial t} = c(x,t)v(x,t) + d(x,t),}$$

with an "initial curve" Γ on example initial condition $v(x,0) = f(x)$. $\triangleleft$

If we have a solution, say $v(x,t)$, this tells us the solution at any (x,t). Since these equations are like the wave equation, it will be useful to discuss how information propagates along curves. Suppose we have a curve parameterized with s. So we have $\Gamma = (x(s), t(s))$.

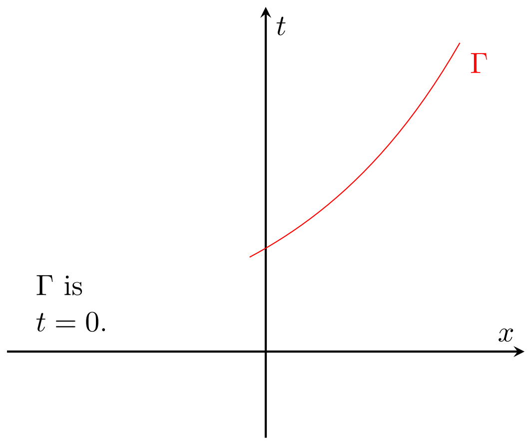


Figure 3.1.1: An example of initial curve Γ (in red).

We check how the solution changes along parameterized curve. The solution on curve is $v(x(s), t(s))$. By chain rule we have

$$\frac{\mathrm{d}v}{\mathrm{d}s} = \frac{\partial v}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}s} + \frac{\partial v}{\partial t}\frac{\mathrm{d}t}{\mathrm{d}s}.$$

We pick

$$\left.\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}s} &= a, \\ \frac{\mathrm{d}t}{\mathrm{d}s} &= b, \end{aligned}\right\} \tag{3.1.1}$$

$$\frac{\mathrm{d}v}{\mathrm{d}s} = cv + d. \tag{3.1.2}$$

The equations in (3.1.1) define the *characteristic curves*. The equation in (3.1.2) shows how the solution changes along these curves.

3.1.2 Using the Method of Characteristics

Theorem 3.1.3: Given a PDE of the form

$$a(x,t)\frac{\partial v}{\partial x} + b(x,t)\frac{\partial v}{\partial t} = c(x,t)v(x,t) + d(x,t),$$

if the functions $a(x,t), b(x,t), c(x,t)$, and $d(x,t)$ are smooth, and if the "initial curve" Γ can be parameterized as

$$\Gamma = \{(x,t) \mid x = \hat{x}(\tau), t = \hat{t}(\tau)\}, \quad \text{with} \quad v = \hat{v}(\tau) \text{ on } \Gamma,$$

then a solution $v(x,t)$ exists and it is unique.

Proof: What is a proof? This is an AMATH course. :| $\square$

Remark 3.1.4: We use the following four steps to use the method of characteristics.

1. Parameterize "initial curve" Γ with

$$x = \hat{x}(\tau), \quad t = \hat{t}(\tau), \quad \text{and} \quad v = \hat{v}(\tau).$$

2. Solve for the characteristic curves using ICs.

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}s} &= a(x,t), \\ \frac{\mathrm{d}t}{\mathrm{d}s} &= b(x,t),\end{aligned} \quad \text{with} \quad \begin{aligned} x|_{s=0} &= \hat{x}(\tau), \\ t|_{s=0} &= \hat{t}(\tau).\end{aligned}$$

   Note that the initial condition occurs at $s = 0$. This yields $x(s,\tau), t(s,\tau)$. Also note that τ is the parameter that parameterizes the initial curve Γ with bounds $-\infty < \tau < \infty$ unless the domain is explicitly defined.

3. Substitute the above to solve for v.

$$\frac{\mathrm{d}v}{\mathrm{d}s} = c(x,t)v + d(x,t), \quad \text{with} \quad v|_{s=0} = \hat{v}(\tau).$$

   This yields solutions in terms of the parameters s and τ. We get

$$v(s,\tau), \quad \text{and} \quad x = x(s,\tau), \quad \text{and} \quad t = t(s,\tau).$$

4. Invert the characteristic equations to find $s(x,t)$ and $\tau(x,t)$ in terms of x and t. Then substitute these into $v = v(s,\tau)$ and obtain

$$v(x,t) = v(s(x,t), \tau(x,t)).$$

We will use these steps to solve questions with the method of characteristics. $\triangleleft$

Example 3.1.5: Consider the PDE

$$\frac{\partial v}{\partial t} + c\frac{\partial v}{\partial x} = 0 \quad \text{on the domain} \quad \begin{aligned} -\infty < x < \infty, \\ t > 0,\end{aligned}$$

with IC

$$v(x,0) = f(x).$$

Recall that this is the advection equation introduced in (Linear Advection (Transport) Equation).

Solution: We use the four steps we introduced in Remark 3.1.4.

1. Find the parameterization of initial curve Γ.

$$x = \tau, \quad t = 0, \quad \text{and} \quad v = f(\tau).$$

   This is at $s = 0$.

2. Solve for the characteristic curves. We need to use given initial condition.

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}s} &= c, & x|_{s=0} &= \tau, \\ \frac{\mathrm{d}t}{\mathrm{d}s} &= 1, & t|_{s=0} &= 0.\end{aligned}$$

   We solve the equations above and obtain

$$\begin{aligned} x &= cs + \tau, \\ t &= s.\end{aligned}$$

③ Solve for v.

$$\frac{\mathrm{d}v}{\mathrm{d}s} = 0, \quad v|_{s=0} = f(\tau)$$
$$\implies v = f(\tau).$$

④ Getting the solution in terms of x, t.

$$s = t, \quad \text{and} \quad \tau = x - ct \implies v = f(x - ct).$$

Refer Fig. 3.4. on page 51 in official course notes for the behavior of the characteristics of linear advection equation with $c = 1$. ◁

3.1.2.1 Moving Boundary Problem Example

Consider the PDE of the form

$$\frac{\partial v}{\partial t} + 2\frac{\partial v}{\partial x} = -v,$$

with the given initial curve

$$v(x,t) = \frac{1}{1+x^2} \quad \text{on} \quad x + t = 0.$$

We have

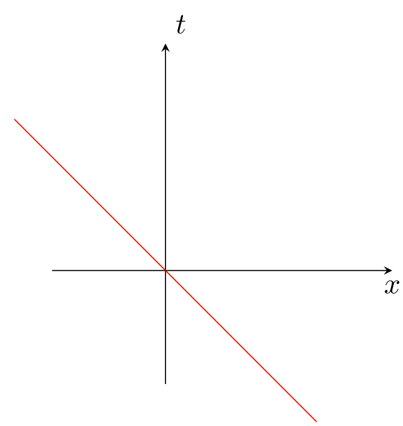


Figure 3.1.2: We have the initial curve as $x = -t$ in red.

We use the four steps we introduced in Remark 3.1.4.

① Parameterize the initial curve.

$$x = \tau, \quad t = -\tau, \quad v = \frac{1}{1+\tau^2}.$$

② Find characteristic curves.

$$\frac{\mathrm{d}x}{\mathrm{d}s} = 2, \quad x|_{s=0} = \tau \implies x = 2s + \tau,$$
$$\frac{\mathrm{d}t}{\mathrm{d}s} = 1, \quad t|_{s=0} = \tau \implies t = s - \tau.$$

③ Solution on curves (solving for the v).

$$\frac{\mathrm{d}v}{\mathrm{d}s} = -v, \quad v|_{s=0} = \frac{1}{1+\tau^2}.$$

This gives us

$$v = f(\tau)e^{-s} \implies v = \frac{1}{1+\tau^2}e^{-s}.$$

④ Invert the transformation to find the solution in terms of x and t, i.e. find $v(x,t)$. We combine the equations we found in ②. We get

$$s = \frac{x+t}{3}, \quad \text{and} \quad \tau = \frac{x-2\tau}{3}.$$

We substitute these in $v(s,\tau)$ and obtain

$$v(x,t) = \frac{\exp\left(-\frac{x+t}{3}\right)}{1+\left(\frac{x-2t}{3}\right)^2}.$$

Note that if τ is a constant, then we have $3\tau = x - 2t$. So we have

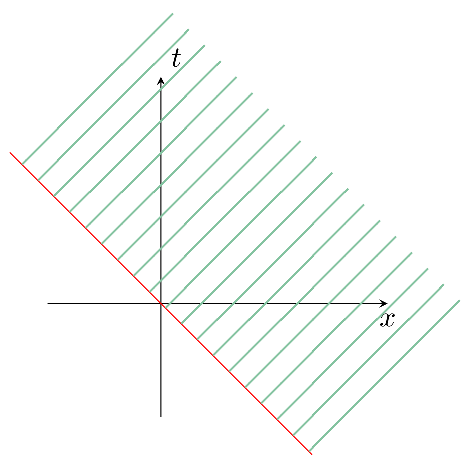


Figure 3.1.3: We obtain the solution above the initial curve (in green).

3.1.2.2 Boundary Problem Example

Consider the PDE

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0, \quad \text{on the domain} \quad \begin{matrix} x > 0, \\ t < 0, \end{matrix}$$

with the ICs

$$\begin{matrix} u(x,0) = 0 \\ u(0,t) = te^{-t}, \end{matrix} \quad \text{on} \quad \begin{matrix} x > 0, \\ t > 0. \end{matrix}$$

We have the following sketch of the problem.

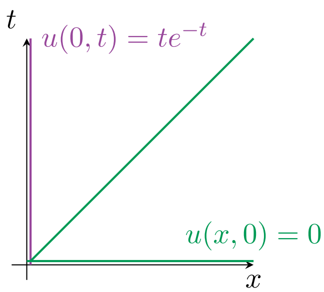


Figure 3.1.4: The problem with initial curves.

We expect the solution to look like

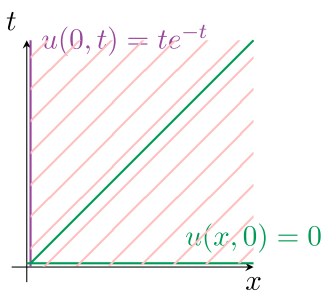


Figure 3.1.5: Expected solution (in pink).

We use the four steps we introduced in Remark 3.1.4.

1. Parameterize the initial curve.

$$x = 0, \quad t = \tau, \quad u = \tau e^{-\tau}, \quad \text{where} \quad \tau \geq 0.$$

2. Find characteristic curves.

$$\frac{\mathrm{d}x}{\mathrm{d}s} = c, \quad x|_{s=0} = \tau \implies x = cs,$$
$$\frac{\mathrm{d}t}{\mathrm{d}s} = 1, \quad t|_{s=0} = 0 \implies t = s + \tau.$$

3. Solution on curves (solving for the v).

$$\frac{\mathrm{d}u}{\mathrm{d}s} = 0, \quad u|_{s=0} = \tau e^{-\tau} \implies u = \tau e^{-\tau}.$$

4. Invert the transformation to find the solution in terms of x and t, i.e. find $v(x,t)$. We combine the equations we found in 2. We get

$$s = \frac{x}{c}, \quad \text{and} \quad \tau = t - s = t - \frac{x}{c}.$$

We substitute these in $v(s,\tau)$ and obtain

$$u(x,t) = \left(z - \frac{x}{c}\right)\exp\left(-t - \frac{x}{c}\right).$$

Note that this is the upper triangle region in Figure 3.1.5.

3.1.2.3 Wave Equation (factored)

Consider the PDE

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0,$$
$$\frac{\partial u}{\partial t} - c\frac{\partial v}{\partial x} = u,$$

with the ICs

$$v(x,0) \quad = f(x),$$
$$\frac{\partial v}{\partial t}(x,0) = g(x).$$

We first solve for $u(x,t)$. We have he PDE

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0,$$

and IC

$$u(x,0) = \frac{\partial v}{\partial t}(x,0) - c\frac{\partial v}{\partial x}(x,0) = g(x) - cf'(x).$$

We apply our previous solution and get

$$u(x,t) = g(x-ct) - cf'(x-ct).$$

We now solve for $v(x,t)$. We have the PDE

$$\frac{\partial v}{\partial t} - c\frac{\partial v}{\partial x} = g(x-ct) - cf'(x-ct),$$

with the IC

$$v(x,0) = f(x).$$

We use the four steps we introduced in Remark 3.1.4.

1. Parameterize the initial curve.

$$x = \tau, \quad t = 0, \quad v = f(\tau).$$

2. Find characteristic curves.

$$\frac{\mathrm{d}x}{\mathrm{d}s} = -c, \quad x|_{s=0} = \tau \implies x = -cs + \tau,$$
$$\frac{\mathrm{d}t}{\mathrm{d}s} = 1, \qquad t|_{s=0} = 0 \implies t = s.$$

3 Solution on curves (solving for the v).

$$\frac{\mathrm{d}v}{\mathrm{d}s} = g(x-ct) - cf'(x-ct), \quad v|_{s=0} = f(\tau).$$

From above we have $t = s$ and $x = -cs + \tau$. This gives us

$$\frac{\mathrm{d}v}{\mathrm{d}s} = g(-2cs+\tau) - cf'(-2cs+\tau), \quad v|_{s=0} = f(\tau).$$

We integrate w.r.t s from 0 to s. We get

$$\int_0^s \frac{\mathrm{d}v}{\mathrm{d}s}\,\mathrm{d}s = \int_0^s g(\tau-2cs) - cf'(\tau-2cs)\,\mathrm{d}s.$$

We substitute $\beta = \tau - 2cs$ which gives us $\mathrm{d}\beta = -2cs\,\mathrm{d}s$. We obtain

$$\begin{aligned} v(s) - v(0) = -\frac{1}{2c}\int_\tau^{\tau-2cs} g(\beta) - cf'(\beta)\,\mathrm{d}\beta &\implies v(s) = f(\tau) - \frac{1}{2c}\int_\tau^{\tau-2cs} g(\beta)\,\mathrm{d}\beta + \frac{1}{2}\int_\tau^{\tau-2cs} cf'(\beta)\,\mathrm{d}\beta \\ &\implies v(s) = f(\tau) - \frac{1}{2c}\int_\tau^{\tau-2cs} g(\beta)\,\mathrm{d}\beta + \frac{1}{2}\left[f(\beta)\right]\Big|_\tau^{\tau-2cs} \\ &\implies v(s) = f(\tau) + \frac{1}{2}f(\tau-2cs) - \frac{1}{2}f(\tau) - \frac{1}{2c}\int_\tau^{\tau-2cs} g(\beta)\,\mathrm{d}\beta. \end{aligned}$$

4 Invert the transformation to find the solution in terms of x and t, i.e. find $v(x,t)$.
We combine the equations we found in 2. We get

$$s = t, \quad \tau = x + ct.$$

This gives us

$$\begin{aligned} v(x,t) &= \frac{1}{2}f(x+ct) + \frac{1}{2}f(x-ct) - \frac{1}{2c}\int_{x+ct}^{x-ct} g(\beta)\,\mathrm{d}\beta \\ &= \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\beta)\,\mathrm{d}\beta, \end{aligned}$$

which is the d'Alembert's solution

3.1.2.4 Interpretation of the Solution

We consider two cases.

1 Initial velocity is zero. We have

$$\frac{\mathrm{d}v}{\mathrm{d}t}(x,0) = g(x) = 0 \implies v(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)].$$

This tells us the following.

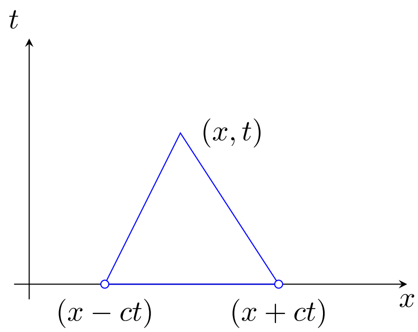


Figure 3.1.6: A plot of the two characteristic curves that transport information from the initial curve to any position (x,t).

② Initial displacement is zero. We have

$$v(x,0) = f(x) = 0 \implies v(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(\beta)\,\mathrm{d}\beta,$$

which is the average of $g(\beta)$. This tells us the following.

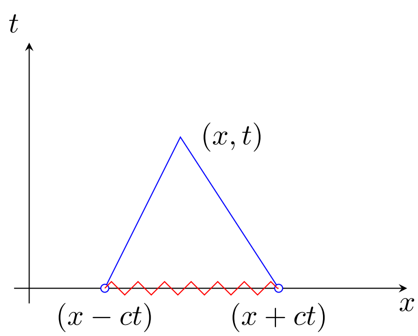


Figure 3.1.7

This behavior is different from the diffusion equation, in that case if the initial condition is $f(x) = \delta(x-x_0)$ the solution is $v = (x,t) = G(x-x_0,t)$ which is non-zero everywhere.

3.1.3 Existence and Uniqueness

Existence: For a solution to exist at (x,t) we need a characteristic curve that meets this point only once. Furthermore, we need that the "initial curve" crosses each characteristic curve once. For the initial curve to not be a characteristic curve, we need the Jacobian of the transformation from (s,τ) to (x,t) to be non-singular.

$$\left|\frac{\partial(x,t)}{\partial(s,\tau)}\right| = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial \tau} \\ \frac{\partial t}{\partial s} & \frac{\partial t}{\partial \tau} \end{vmatrix} = \begin{vmatrix} b & \frac{\partial x}{\partial \tau} \\ a & \frac{\partial t}{\partial \tau} \end{vmatrix} \neq 0,$$

on the initial curve.

Existence: As an example, consider the inhomogeneous wave equation

$$\frac{\partial^2 v}{\partial t^2} - c^2 \frac{\partial^2 v}{\partial x^2} = f(x,t), \quad \text{on the domain} \quad \begin{matrix} -\infty < x < \infty, \\ t > 0. \end{matrix}$$

with the ICs

$$v(x,0) = f(x), \quad \frac{\partial v}{\partial t}(x,0) = g(x).$$

The initial curve is $t = 0$ (this is the x-axis). The characteristic curves are $\tau = x \pm ct$.

Let v_1 and v_2 be two solutions for the problem. We want to show the solution is unique. Suppose $v = v_1 - v_2$. We claim that v solves the homogeneous system

$$\begin{aligned} \frac{\partial^2 v}{\partial t^2} - c^2 \frac{\partial^2 v}{\partial x^2} &= 0, && \text{PDE} \\ \left.\begin{aligned} v(x,0) &= 0, \\ \frac{\partial x}{\partial t}(x,0) &= 0. \end{aligned}\right\} & && \text{ICs} \end{aligned}$$

To show this is true, we find the equation for the energy. We multiply the PDE by $\frac{\partial v}{\partial t}$. We get

$$\begin{aligned} \frac{\partial v}{\partial t}\frac{\partial^2 v}{\partial t^2} - c^2 \frac{\partial v}{\partial t}\frac{\partial^2 v}{\partial x^2} = 0 &\implies \frac{\partial}{\partial t}\left(\frac{1}{2}\left(\frac{\partial v}{\partial t}\right)^2\right) - c^2 \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}\frac{\partial v}{\partial x}\right) + c^2 \frac{\partial^2 v}{\partial x \partial t}\frac{\partial v}{\partial x} = 0 \\ &\implies \frac{\partial}{\partial t}\left(\frac{1}{2}\left(\frac{\partial v}{\partial t}\right)^2\right) - c^2 \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}\frac{\partial v}{\partial x}\right) + c^2 \frac{\partial}{\partial t}\left(\frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2\right) = 0. \end{aligned}$$

To find a global expression we integrate $\int_{-\infty}^{\infty} \mathrm{d}x$. We get

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{\partial}{\partial t}\left[\frac{1}{2}\left(\frac{\partial v}{\partial t}\right)^2 + \frac{1}{2}c^2\left(\frac{\partial v}{\partial x}\right)^2\right] \mathrm{d}x = \int_{-\infty}^{\infty} c^2 \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}\frac{\partial v}{\partial x}\right) \mathrm{d}x + \cancelto{0}{\left[c^2 \frac{\partial v}{\partial t}\frac{\partial v}{\partial x}\right]\Bigg|_{-\infty}^{\infty}} \\ &\implies \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} \left(\frac{\partial v}{\partial t}\right)^2 + c^2\left(\frac{\partial v}{\partial x}\right)^2 \mathrm{d}x = 0. \end{aligned}$$

Since the integral is zero at $t = 0$, then for all time we have

$$\int_{-\infty}^{\infty} \left(\frac{\partial v}{\partial t}\right)^2 + c^2\left(\frac{\partial v}{\partial x}\right)^2 \mathrm{d}x = 0.$$

This is only possible if $\frac{\partial v}{\partial t} = 0 = \frac{\partial v}{\partial x}$. So we must have

$$v(x,t) = 0.$$

Hence we get $v_1 = v_2$ which means the solution is unique.

3.2 Quasi-linear PDEs

Consider the PDE of the form

$$a(x,t,v)\frac{\partial v}{\partial x} + b(x,t,v)\frac{\partial v}{\partial t} = c(x,t,v).$$

Method: We require the following.

1. The functions $a, b, c \in C^\infty(\mathbb{R})$. i.e. they are smooth functions.
2. Initial curve is given by Γ with

$$x = \hat{x}(\tau), \quad t = \hat{t}(\tau), \quad v = \hat{v}(\tau).$$

3. The determinant of the Jacobian is initially non-zero. We have

$$\det(A(\tau)) = \underbrace{\left|\frac{\partial(x,t)}{\partial(s,\tau)}\right|}_{(\star)} = \left.\begin{vmatrix} \frac{\partial x}{\partial \tau} & \frac{\partial x}{\partial s} \\ \frac{\partial t}{\partial \tau} & \frac{\partial t}{\partial s} \end{vmatrix}\right|_{s=0} = \frac{\partial \hat{x}}{\partial \tau} b(\hat{x}, \hat{t}, \hat{v}) - \frac{\partial \hat{t}}{\partial \tau} a(\hat{x}, \hat{t}, \hat{v}) \neq 0.$$

Here $(\star)$ is evaluated on the initial curve (at $s = 0$). If these conditions are satisfied, then there exists a unique solution for some time interval and the solution is determined by the following characteristic equations.

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}s} &= a(x,t,v), & x|_{s=0} &= \hat{x}(\tau), \\ \frac{\mathrm{d}t}{\mathrm{d}s} &= b(x,t,v), & t|_{s=0} &= \hat{t}(\tau), \\ \frac{\mathrm{d}v}{\mathrm{d}s} &= c(x,t,v), & v|_{s=0} &= \hat{v}(\tau). \end{aligned}$$

From the solutions to this system, we can invert

$$x = x(\tau, s), \quad t = t(\tau, s)$$

to find

$$\begin{gathered} \tau = \tau(x,t), \quad s = s(x,t) \\ \Longrightarrow v(x,t) = v(\tau(x,t), s(x,t)). \end{gathered}$$

3.2.0.1 Inviscid Burger's Equation

Recall that we have the inviscid Burger's equation as

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial t} = 0, \quad \text{on the domain} \quad \begin{gathered} -\infty < x < \infty, \\ t > 0. \end{gathered}$$

with the inital conditions

$$v(x,0) = f(x).$$

Consider the initial curve

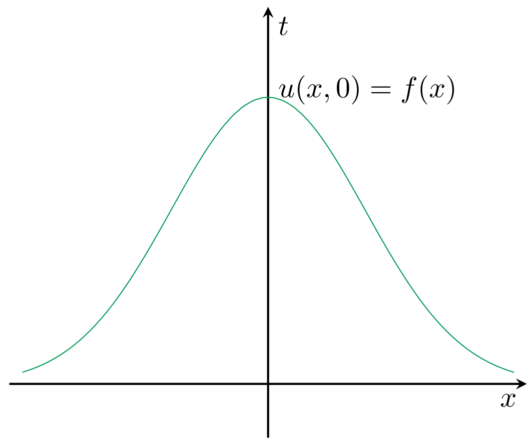


Figure 3.2.1: Initial curve.

(1) We parametrize the initial curve.

$$x = \tau, \quad t = 0, \quad v = f(\tau).$$

(2) We define the characteristic curves.

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}s} &= v, \qquad x|_{s=0} = \tau, \\ \frac{\mathrm{d}t}{\mathrm{d}s} &= 1, \qquad t|_{s=0} = 0, \\ \frac{\mathrm{d}v}{\mathrm{d}s} &= 0, \qquad v|_{s=0} = f(\tau). \end{aligned}$$

We first need to solve for t and v to obtain the solution for x. We have

$$t = s, \quad \text{and} \quad v = f(\tau).$$

This gives us

$$\frac{\mathrm{d}x}{\mathrm{d}s} = f(\tau), \quad x|_{s=0} = \tau \implies f(\tau)s + \tau.$$

Our solution is

$$\begin{aligned} v &= f(\tau) \\ &= f(x - tf(\tau)) \\ &= f(x - vt). \end{aligned}$$

A characteristic has a particular value of τ. Therefore, the solution $v = f(\tau)$ is constant along each characteristic. The characteristic curves are given by

$$\tau = x - f(\tau)t.$$

We want to know the slope in (x, t) space. To find this, we can find the derivative w.r.t time. We have

$$0 = \frac{\mathrm{d}x}{\mathrm{d}t} - f(\tau) \implies f(\tau) = \frac{\mathrm{d}x}{\mathrm{d}t}.$$

We continue our discussion on inviscid Burger's equation. Last time we had

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = 0,$$

with the IC

$$v(x,0) = f(x).$$

The solution can be formally written as

$$v(x,t) = f(x - vt)$$

Note that $f(x - vt)$ is a solution but it is implicit. In order to get the solution we need to define some characteristic variables. In terms of characteristic variables, the solution is

$$v = f(\tau),$$

where the characteristics are defined as

$$\tau = x - tf(\tau).$$

Note that this gives us

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{f(\tau)}.$$

Consider the example below.

If $x_0 < x_1$ and $f(x_0) > f(x_1)$, we expect the solution to steepen and become multivalued. This is called a ***shock***.

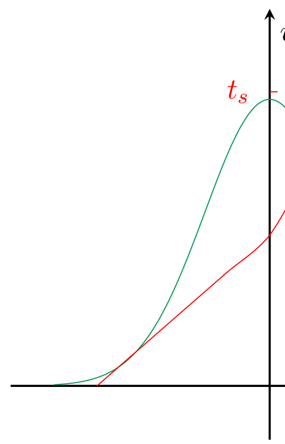
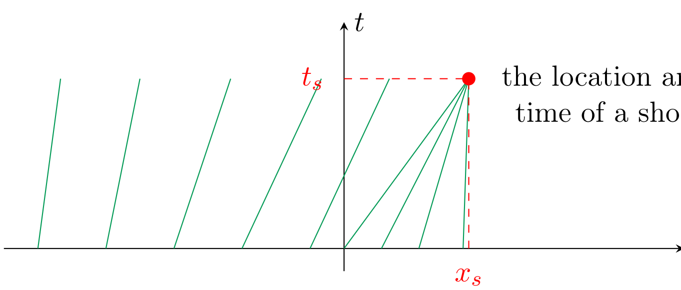


Figure 3.2.2: Shock in red.

This gives us



3.2.1 Shock Formation

A shock is where the solution is multivalued and the profile is vertical. This can be found mathematically by determining when and where $\frac{\partial v}{\partial x} \to \infty$. Our solutions are

$$v = f(\tau), \quad \text{and} \quad x = tf(\tau) + \tau.$$

First, we have

$$\frac{\partial v}{\partial x} = f'(\tau)\frac{\partial \tau}{\partial x}.$$

We now differentiate w.r.t x. We have

$$1 = tf'(\tau)\frac{\partial \tau}{\partial x} + \frac{\partial \tau}{\partial x} = (tf'(\tau) + 1)\frac{\partial \tau}{\partial x},$$

which is equivalent to

$$\frac{\partial \tau}{\partial x} = \frac{1}{tf'(\tau) + 1}.$$

We combine these two equations and obtain

$$\frac{\partial v}{\partial x} = \frac{f'(\tau)}{tf'(\tau) + 1}.$$

We want to know when this becomes infinite. Note if $f'(\tau) \geq 0$, then the denominator is never zero. Hence a shock will not occur. So we need $f'(\tau) < 0$ for some τ. By continuity, know that $f'(\tau)$ for some range of τ.

Observe that $\frac{\partial v}{\partial x} \to \infty$ as $t \to -\frac{1}{f'(\tau)}$. To find the initial shock time, we find the minimum of this function. If we denote the first shock time as t_s, then we have

$$t_s = \min_{-\infty<\tau<\infty}\left(-\frac{1}{f'(\tau)}\right) > 0.$$

If the minimum occurs at $\tau_{\min}$, then the location of the shock is

$$x_s = t_s f(\tau_{\min}) + \tau_{\min}.$$

Example 3.2.1: Suppose

$$v(x,0) = f(x) = \frac{1}{1+x^2}.$$

Note that this is maximum when $x = 0$. The solution is

$$u = \frac{1}{1+\tau^2}, \quad \text{with} \quad x = \frac{t}{f(\tau)} + \tau = \frac{t}{1+\tau^2} + \tau.$$

We want to know when and where the shock first occurs. We have

$$f'(\tau) = -\frac{2\tau}{(1+\tau^2)^2}.$$

We also have

$$t_s = \min_{\tau}\left(\frac{-1}{\frac{-2z}{(1+\tau^2)^2}}\right) = \min_{\tau}\left(\frac{(1+\tau^2)^2}{2z}\right).$$

To find the minimum, we find the extrema. We have

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{(1+\tau^2)^2}{2\tau}\right) = \frac{2(1+\tau^2)2\tau}{2\tau} - \frac{(1+\tau^2)^2}{2\tau^2} = \frac{2(1+\tau^2)2\tau^2}{2\tau^2} - \frac{(1+\tau^2)^2}{2\tau^2} = \frac{(1+\tau^2)(4\tau^2-(1-\tau^2))}{2\tau^2}$$
$$= \frac{(1+\tau^2)(-1+3\tau^2)}{2\tau^2}.$$

Since τ parametrizes the curve, then $\tau \in \mathbb{R}$ (i.e. τ is not imaginary). Then we must have $\tau_{\text{min}} = \pm 1/\sqrt{3}$. This gives us

$$t_s = \frac{(1+\tau^2)^2}{2\tau} = \frac{\left(1+\frac{1}{3}\right)^2}{\pm\frac{2}{\sqrt{3}}} = \frac{\frac{16}{9}}{\pm\frac{2}{\sqrt{3}}} = \pm\frac{8\sqrt{3}}{9}.$$

Since time cannot be negative, we take the shock time as

$$t_s = \frac{8\sqrt{3}}{9}.$$

Note that for this case $\tau_{\text{min}} = +1/\sqrt{3}$. To find the shock location, we find x_s. We have

$$x_s = t_s f(\tau_{\text{min}}) + \tau_{\text{min}} = \frac{8\sqrt{3}}{9}\frac{1}{1+\frac{1}{3}} + \frac{1}{\sqrt{3}} = \frac{2\sqrt{3}}{3} + \frac{\sqrt{3}}{3} = \sqrt{3}.$$

Hence, we find that the shock occurs at

$$(x_s, t_s) = \left(\sqrt{3}, \frac{8\sqrt{3}}{9}\right).$$ ◁

3.2.1.1 Expansion Fans

Consider solutions to the Burger's equation with the following initial condition

$$v(x,0) = \begin{cases} B & \text{if } x \le 0, \\ A & \text{if } x > 0. \end{cases}, \qquad \text{with} \qquad A > B.$$

Graphically we have

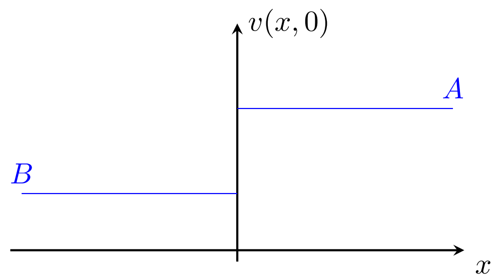


Figure 3.2.3: Graphical representation of piecewise continuous solution.

We have

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = 0.$$

The solution to the Burger's equation in general is

$$v = f(\tau), \qquad \tau = x - tf(\tau).$$

For $\tau > 0$, $f(\tau) = A$ gives us $v = A$. When we substitute this into the characteristic equation we get

$$\tau = x - tA > 0.$$

Hence, we need to have $x > tA$. Hence, we see that all of the characteristics are parallel.

For $\tau \leq 0$, we have $f(\tau) = B$ which gives us $v = B$. When we substitute this into the characteristic equation we get

$$\tau = x - tB \leq 0.$$

Hence we need to have $x \leq tB$. By this, we obtain the solution in two different pieces. We have

$$v(x,t) = \begin{cases} B & \text{if } x \leq tB, \\ A & \text{if } x > tA. \end{cases}$$

This looks like

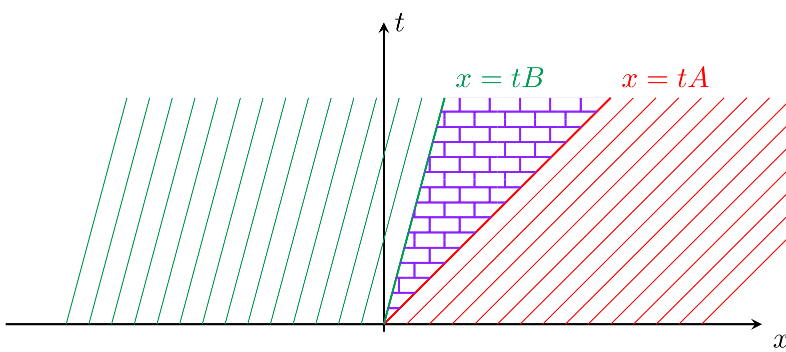


Figure 3.2.4: Piecewise continuous solution $v(x, 0)$.

We want to know how the solution looks like in the purple region. In some later time we expect to have

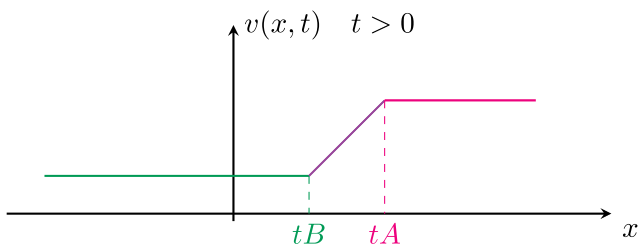


Figure 3.2.5: Solution at a later time.

We want to know how to solution behaves when $tB < x < tA$. The solution in this region is what we call an ***expansion fan*** and it is of the form

$$\phi(x,t) = \frac{x - x_0}{t - t_0}.$$

Note that this function solves the inviscid Burger's equation. We have

$$\frac{\partial \phi}{\partial x} = \frac{1}{t - t_0},$$
$$\frac{\partial \phi}{\partial t} = \frac{-(x - x_0)}{(t - t_0)^2}.$$

We substitute these into LHS and obtain

$$\frac{\partial \phi}{\partial t} + \phi \frac{\partial \phi}{\partial x} = -\frac{(x - x_0)}{(t - t_0)^2} + \frac{(x - x_0)}{(t - t_0)^2} = 0.$$

This is true for all x_0 and t_0. Hence we must pick these constants to match our solution. We do this by requiring continuity of the solution everywhere. On the right we have

$$\phi(tA, t) = A, \quad \text{and} \quad \phi(tB, t) = B.$$

These equations require

$$\frac{tA - x_0}{t - t_0} = A, \quad \text{and} \quad \frac{tB - x_0}{t - t_0} = B.$$

This gives us

$$\begin{aligned} tA - x_0 &= tA - t_0 A, \\ tB - x_0 &= tB - t_0 B, \end{aligned} \quad \text{or} \quad x_0 = t_0 A = t_0 B.$$

This requires $x_0 = 0 = t_0$, which gives us

$$\phi(x, t) = \frac{x}{t}.$$

Hence, we obtain the complete solution as

$$\boxed{v(x, t) = \begin{cases} B & \text{if } x \leq tB, \\ x/t & \text{if } tB < x \leq tA, \\ A & \text{if } x > tA. \end{cases}}$$

Note that this is consistent with our initial guess at Figure 3.2.5.

Example 3.2.2: Solve the following PDE

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial t} = -v^2, \quad \text{with} \quad v(x, 0) = g(x).$$

Solution: We use the four steps we discussed in Remark 3.1.4.

1. Parameterize the IC.

$$x = \tau, \quad \text{and} \quad t = 0, \quad \text{and} \quad v = g(\tau).$$

2. Find the characteristic equations.

$$\begin{aligned} &\text{a}: \quad \frac{\mathrm{d}t}{\mathrm{d}s} = 1 && t|_{s=0} = 0, \\ &\text{b}: \quad \frac{\mathrm{d}x}{\mathrm{d}s} = v && x|_{s=0} = \tau, \\ &\text{c}: \quad \frac{\mathrm{d}v}{\mathrm{d}s} = -v^2 && v|_{s=0} = g(\tau). \end{aligned}$$

From a, we get $t = s$. From c, we used the fact that the ODE is separable and obtain

$$\frac{\mathrm{d}v}{\mathrm{d}s} = -v^2 \implies \int \frac{\mathrm{d}v}{-v^2} = \int \mathrm{d}s \implies \frac{1}{v} = s + \alpha(\tau).$$

At $s = 0$, we have $\frac{1}{g(\tau)} = \alpha(\tau)$. Hence we obtain

$$\frac{1}{v} = s + \frac{1}{g(\tau)} = \frac{g(\tau)s + 1}{g(\tau)}.$$

Which gives us

$$v = \frac{g(\tau)}{g(\tau)s + 1}.$$

For b we have

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \frac{g(\tau)}{g(\tau)s + 1} = \frac{1}{s + \frac{1}{g(\tau)}}.$$

This is separable. We have

$$\begin{aligned} \mathrm{d}x = \frac{\mathrm{d}s}{s + \frac{1}{g(\tau)}} &\implies \int \mathrm{d}x = \int \frac{\mathrm{d}s}{s + \frac{1}{g(\tau)}} \\ &\implies x = \ln\left|s + \frac{1}{g(\tau)}\right| + \beta(\tau). \end{aligned}$$

So, at $s = 0$ we have

$$\tau = \ln|g(\tau)| + \beta(\tau) \implies \beta(\tau) = \tau + \ln|g(\tau)|.$$

We obtain

$$x = \ln\left|s + \frac{1}{g(\tau)}\right| + \tau + \ln|g(\tau)| = \ln|sg(\tau) + 1| + \tau.$$

In general, we cannot invert this equation to find a solution but if we had a particular $g(\tau)$, we could plug it in and maybe solve it. To determine if a shock occurs, we begin with $\frac{\partial v}{\partial x}$ and find if it goes to infinity. ◁

Final Exam Information

The final exam is on April 23, 2019 between 09:00–11:30 a.m in STC 0010. It includes material from the entire course, with more emphasis on material after the midterm. There will be 5-6 questions. A sample final with the formula sheet will be posted on Learn next week.

Course Summary

We give a brief summary of the course. **Here the chapter numbering follows the numbering in textbook and the order follows the order they are covered in lectures**.

Chapter 1: Modeling with PDEs

We saw that conservation law (in 1-D and 3-D) with constitutive relation gave rise to certain types of PDEs, such as the diffusion equation. Also, from Newton's 2nd Law we obtained other types of PDEs, such as the wave equation.

Chapter 2: Classification of PDEs

We saw that we can classify PDEs in three classes. We have the general form of a PDE as

$$A\frac{\partial^2 U}{\partial x^2} + 2B\frac{\partial^2 U}{\partial x \partial y} + C\frac{\partial^2 U}{\partial y^2} + D\frac{\partial U}{\partial x} + E\frac{\partial U}{\partial y} + FU = G.$$

We classified the PDEs depending on the value of $B^2 - AC$. We had

$$B^2 - AC = \begin{cases} \Delta > 0 \text{ hyperbolic} \\ \Delta = 0 \text{ parabolic} \\ \Delta < 0 \text{ elliptic} \end{cases}$$

Chapter 4: IBVPs

We considered some PDEs with BCs and ICs on bounded domains. For example, for the parabolic PDE we had

$$\rho\frac{\partial u}{\partial t} + \mathcal{L}[u] = \rho F, \quad \text{where} \quad \mathcal{L}[u] = -\frac{\partial}{\partial x}\left(\rho\frac{\partial u}{\partial x}\right) + qu.$$

If $F = 0$, we call our PDE as homogeneous, otherwise as inhomogeneous. To solve the PDE in these cases, we used the following procedures. We will use the steps introduced in Remark 4.4.7 and subsubsection 4.4.3.7 to solve for the homogeneous problem.

For the homogeneous case we used the five step procedure below.

1. Separate the variables for the PDE and the BCs. Assume the solution is of the form
$$u(x,t) = M(x)N(t).$$
2. Solve for the BVP for $M(x)$ for spatial variable by using the eigenvalue relations $\mathcal{L}[M] = \lambda\rho M$ and obtain the (solutions) eigenpairs $(\lambda_n, M_n(x))$ for $n = 1, 2, \ldots$.
3. Solve for the temporal variable $N(t)$.
4. Obtain the general solution by using the principle of superposition.
$$u(x,t) = \sum_{n=1}^{\infty} N_n(t)M_n(t).$$
5. Impose the ICs. For example for the parabolic case we have
$$u(x,0) = \sum_{n=1}^{\infty} N_n(0)M_n(x) = f(x)$$
$$\implies N_n(0) = \frac{(f, M_n)}{(M_n, M_n)}.$$

For the inhomogeneous case we used the procedure below.

1. We assume the solution is of the form
$$u(x,0) = \sum_{n=1}^{\infty} N_n(t)M_n(x) = f(x)$$
$$\implies F(x,t) = \sum_{n=1}^{\infty} F_n(t)M_n(x).$$
i.e. We started from step 4 above.
2. Substitute this into the PDE and project onto M_k.
$$\left(\frac{\partial u}{\partial t}, M_k\right) = \left(\frac{1}{\rho}\mathscr{L}[u], M_k\right) = (F, M_k),$$
by using self-adjointness property and eigenvalue relations. We get
$$\frac{\mathrm{d}}{\mathrm{d}t}N_k + \lambda_k N_k = F_k.$$
By projecting the ICs we get
$$N_k(0) = (f, M_k).$$
3. Solve for the temporal part $N_k(t)$.

Remark: Sturm-Liouville operator is self-adjoint, positive and eigenfunctions corresponding to different eigenvalues are orthogonal. ◁

Chapter 5: PDEs on Infinite Domains

We considered PDEs on infinite domains with constant coefficients. For example a PDE where
$$\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} = f(x), \quad \text{with the IC} \quad u(x,0) = g(x).$$
We used the following steps introduced in subsection 5.0.3 and subsection 5.0.4.

1. Compute the FT of PDE and IC. Define $U(\lambda, t) = \mathcal{F}[u(x,t)]$ and $F(\lambda) = \mathcal{F}[f(x)]$ and $G(\lambda) = \mathcal{F}[g(x)]$.
$$\frac{\partial U}{\partial t} + \lambda^2 DU = F(\lambda), \quad \text{and} \quad U(\lambda, 0) = G(\lambda).$$
2. Solve for $U(\lambda, t)$ using convolution and other properties.
3. Take the IFT and obtain $u(x,t) = \mathcal{F}^{-1}[U(\lambda, t)]$.

Chapter 3: Method of Characteristics

We considered the PDE

$$a(x,t,v)\frac{\partial v}{\partial t} + b(x,t,v)\frac{\partial v}{\partial x} = c(x,t,v),$$

with the an initial curve

$$v(x,0) = f(x).$$

We used the following steps introduced in Remark 3.1.4.

1. Parametrize the initial curve (at $s = 0$).
2. Find the characteristic curves.

$$\text{If quasi-linear, solve these together.}\quad \begin{cases} \left.\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}s} &= b, \qquad x|_{s=0} = \hat{x}(\tau), \\ \frac{\mathrm{d}t}{\mathrm{d}s} &= a, \qquad t|_{s=0} = \hat{t}(\tau), \end{aligned}\right\} \text{If linear, solve these first.} \\ \frac{\mathrm{d}v}{\mathrm{d}s} = c, \qquad v|_{s=0} = \hat{v}(\tau). \end{cases}$$

3. Solve for x, t, v in terms of s and τ by substituting the above.
4. Invert the characteristic equations and find s, τ in terms of x and t. Then find

$$v = (s,\tau) \equiv v(x,t) = v(s(x,t), \tau(x,t)).$$

This concludes the final lecture for AMATH 353 in Winter 2019.

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