We wish to have a notion of a parameterized SDP scheme. We will have some parameter λ, and for every value of λ, we will have an SDP (P_λ) satisfying certain properties. To make sure that λ plays nicely with the theory of SDPs, we will ensure that it lives in a space that is the intersection of a positive-semidefinite cone with an affine space (i.e., the feasible region of some SDP). Only the right-hand-side of the constraint of (P_λ) will be allowed to depend on λ, and must be an affine function of λ. The precise definition follows.

Definition 0.1. *Let Λ be the feasible region for the SDP:*

$$
\begin{aligned}
(\Lambda) \qquad\qquad & \max && 0 \\
& \textit{subject to} && \Omega(\lambda) = E \\
& && \lambda \in \mathit{Pos}(\mathcal{P})
\end{aligned}
$$

where $\mathcal{P}, \mathcal{Q}$ are complex Euclidean spaces, $\Omega : \mathcal{L}(\mathcal{P}) \to \mathcal{L}(\mathcal{Q})$ is linear and Hermitian-preserving, and $E \in \mathit{Herm}(\mathcal{Q})$. A parameterized SDP scheme (PSS) over Λ is a family of SDPs:

$$
\begin{aligned}
(P_\lambda) \qquad\qquad & \max && \langle C, X \rangle \\
& \textit{subject to} && \Psi(X) = B(\lambda) \\
& && X \in \mathit{Pos}(\mathcal{X})
\end{aligned}
$$

where $\mathcal{X}, \mathcal{Y}$ are complex Euclidean spaces, $\Psi : \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is linear and Hermitian-preserving, $C \in \mathit{Herm}(\mathcal{X})$, and $B : \Lambda \to \mathcal{L}(\mathcal{Y})$ is affine-Hermitian. That is, a PSS over Λ is given by a tuple (Φ, C, B); for each $\lambda \in \Lambda$, it defines the SDP (P_λ). Furthermore, a Slater parameterized SDP scheme (SPSS) is a PSS such that each problem (P_λ) satisfies the Slater condition, i.e., both (P_λ) and its dual have Slater points.

We quickly observe that the dual of the problem (P_λ) is:

$$
\begin{aligned}
(D_\lambda) \qquad\qquad & \min && \langle B(\lambda), Y \rangle \\
& \text{subject to} && \Psi^*(Y) \geq C \\
& && Y \in \mathrm{Herm}(\mathcal{Y})
\end{aligned}
$$

and if (P_λ) and (D_λ) both have Slater points, then feasible primal and dual solutions $X \in \mathrm{Pos}(\mathcal{X})$ and $Y \in \mathrm{Herm}(\mathcal{Y})$ are optimal if and only if they satisfy the complementary slackness condition:

$$
(CS) \qquad X \cdot [\Psi^*(Y) - C] = 0
$$

Definition 0.2. *Let $P = (\Phi, C, B)$ be a SPSS over Λ. We say a function $f : \Lambda \to \mathcal{L}(\mathcal{Y})$ is a dual oracle for P if for every $\lambda \in \Lambda$, $f(\lambda)$ is an optimal solution for (D_λ), the dual of (P_λ).*

A dual oracle is valuable in that, for each λ, it provides a means of checking the optimality of a primal solution X; one can simply check if X has the same objective value as $f(\lambda)$.

One might be interested in "solving" a PSS by finding the value of λ which maximizes the optimal value of (P_λ). It is straightforward to write this problem as a semidefinite program, in which λ simply becomes a variable. We are specifically interested in the case where we have several SPSS's, $P_k = (\Psi_k, C_k, B_k)$ for $k \in \{1, \ldots, n\}$, and wish to maximize their sum. We can think of each P_k as describing a function F_k of λ, and we want to maximize $\sum_{k=1}^{n} F_k(\lambda)$. This is easily formulated as an SDP; for each k, let $B_k(\lambda) := A_k(\lambda) + D_k$.

$$
\begin{aligned}
(P_\Sigma) \qquad\qquad & \max && \langle C_k, X_k \rangle \\
& \text{subject to} && \Psi_k(X_k) - A_k(\lambda) = D_k && \forall k \\
& && \Omega(\lambda) = E \\
& && X_k \in \mathrm{Pos}(\mathcal{X}_k) && \forall k \\
& && \lambda \in \mathrm{Pos}(\mathcal{P})
\end{aligned}
$$

The dual of this SDP is:

$$
\begin{array}{llll}
(D_\Sigma) & \min & \sum_{k=1}^{n} \langle D_k, Y_k \rangle + \langle E, Z \rangle & \\
& \text{subject to} & \Psi_k^*(Y_k) \geq C_k & \forall k \\
& & \Omega^*(Z) \geq \sum_{k=1}^{n} A_k^*(Y_k) & \\
& & Y_k \in \mathrm{Herm}(\mathcal{Y}_k) & \\
& & Z \in \mathrm{Herm}(\mathcal{Q}) &
\end{array}
$$

The complementary slackness conditions for these problems are

$$(CS1_k) \quad X_k \cdot [\Psi^*(Y_k) - C_k] = 0$$

$$(CS2) \quad \lambda \cdot \left[\Omega^*(Z) - \sum_{k=1}^{n} A_k^*(Y_k) \right] = 0$$

Now, if each problem P_k has a dual oracle f_k, then one might wonder if we can obtain some sort of dual oracle for (P_Σ). That is, can we find some sort of function f_Σ such that, if $\lambda \in \Lambda$ is optimal for (P_Σ), then $f_\Sigma(\lambda)$ is a dual solution which verifies its optimality?